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Internal Assesment Test – I-Sep 2019

Sub:	Transform calculus ,Fourier series and Numerical techniques					Code:	18MAT31
Date:	06/ 09 /2019	Duration:	90 mins	Max Marks:	50	Sem: 3	Branch: All

Question 1 is compulsory and answer any six from Question 2 to Question 8.

	Marks	OBE CO	OBE RBT
1. Express y as a Fourier series upto 2^{nd} harmonics for the given data.	[8]	CO1	L3
2. Obtain the Fourier series for the function $f(x) = x + x^2 \ in -\pi < x < \pi$.	[7]	CO1	L3
3. Obtain the Fourier series for the function $f(x)=\begin{Bmatrix} x & in\ 0<x<\pi \\ x-2\pi & in\ \pi<x<2\pi \end{Bmatrix}$. Hence deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-........$ Also sketch the graph for the function.	[7]	CO1	L3

Data for question 1:

x	0	Π/3	2Π/3	Π	4Π/3	5Π/3	2Π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

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x	0	Π/3	2Π/3	Π	4Π/3	5Π/3	2Π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

4. Obtain the Fourier series for the function $f(x) = 2x - x^2 \ in \ 0 \le x \le 2$. [7] CO1 L3

5. Obtain the sine half range Fourier series of $f(x) = \begin{Bmatrix} \frac{1}{4} - x \ in \ 0 < x < \frac{1}{2} \\ x - \frac{3}{4} \ in \ \frac{1}{2} < x < 1 \end{Bmatrix}$. [7] CO1 L3

6. Find the complex Fourier transform of $f(x) = \begin{Bmatrix} 1 \ for \ |x| \le a \\ 0 \ for |x| > a \end{Bmatrix}$.Hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$. [7] CO2 L3

7. Find the infinite Fourier cosine transform of e^{-x^2}. [7] CO2 L3

8. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$, $a > 0$. [7] CO2 L3

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①

1\. FS in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

FS upto 2nd harmonic

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{2}(a_n \cos nx + b_n \sin nx)$$

$a_0 = 2[[f(x)]]$ $a_1 = 2[[y \cos x]]$

$a_2 = 2[[y \cos 2x]]$ $b_1 = 2[[y \sin x]]$

$b_2 = 2[[y \sin 2x]]$

2M

x y $\cos x$ $\cos 2x$ $\sin x$ $\sin 2x$ $y \cos x$ $y \cos 2x$

$y \sin x$ $y \sin 2x$

x	y	$\cos x$	$\cos 2x$	$y\cos x$	$y\cos 2x$	$\sin x$	$\sin 2x$	$y\sin x$	$y\sin 2x$
0	1	1	1	0	0	0	0	0	0
$\pi/3$	1.4	0.5	−0.5	0.7	−0.7	0.866	0.866	1.2124	1.2124
$2\pi/3$	1.9	−0.5	−0.5	−0.95	−0.95	+0.866	−0.866	1.6454	−1.6454
π	1.7	−1.0	1.0	−1.7	1.7	0	0	0	0
$4\pi/3$	1.5	−0.5	−0.5	−0.75	−0.75	−0.866	0.866	−1.299	1.299
$5\pi/3$	1.2	0.5	−0.5	0.6	−0.6	−0.866	−0.866	−1.0392	−1.0392
	8.7			−1.1	−0.3			0.5196	−0.1732

3M

$$a_0 = 2\left(\frac{8.7}{6}\right) = 2.9$$

$$a_1 = 2\left(\frac{-1.1}{6}\right) = -0.3667$$

$$a_2 = 2\left(\frac{-0.3}{6}\right) = -0.1$$

$$b_1 = 2\left(\frac{0.5196}{6}\right) = 0.1732$$

$$b_2 = 2\left(\frac{-0.1732}{6}\right) = -0.0577$$

(2M)

FS

$$f(x) = \frac{2.9}{2} + \left(-0.3667\cos x + 0.1732\sin x\right) + \left(-0.1\cos 2x - 0.0577\sin 2x\right)$$

(1M)

Q5. Sine series of $f(x)$ in $(0, l)$ is

$$f(x) = \sum b_n \sin\left(\frac{n\pi}{l}\right)x$$

(1M)

$\underline{l=1}$

$$f(x) = \sum b_n \sin(n\pi)x$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

$$b_n = 2\int_0^1 f(x)\sin(n\pi)x\,dx$$

$l=1$

$$b_n = 2\left[\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin(n\pi)x\,dx + \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin(n\pi)x\,dx\right]$$

$$b_n = 2\left[\left(\frac{1}{4}-x\right)\frac{-\cos(n\pi)x}{n\pi} - (-1)\frac{-\sin(n\pi)x}{n^2\pi^2}\right]_0^{1/2}$$

$$+ 2\left[\left(x-\frac{3}{4}\right)\frac{-\cos(n\pi)x}{n\pi} - 1\cdot\frac{-\sin(n\pi)x}{n^2\pi^2}\right]_{1/2}^{1}$$

$$b_n = 2\left[-\frac{1}{n\pi}\left\{\left(\frac{1}{4}-x\right)\cos(n\pi)x\right\}_0^{1/2} - \frac{1}{n^2\pi^2}\left(\sin(n\pi)x\right)_0^{1/2}\right]$$

$$+ 2\left[-\frac{1}{n\pi}\left\{\left(x-3/4\right)\cos(n\pi)x\right\}_{1/2}^{1} + \frac{1}{n^2\pi^2}\left(\sin(n\pi)x\right)_{1/2}^{1}\right]$$

$$b_n = 2\left[-\frac{1}{n\pi}\left(-\frac{1}{4}\cos\frac{n\pi}{2} - \frac{1}{4}\right) - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2}\right] - \frac{2}{n\pi}\left\{\frac{1}{4}\cos n\pi + \frac{1}{4}\cos\frac{n\pi}{2}\right] + \frac{2}{n^2\pi^2}\left(-\sin\frac{n\pi}{2}\right)\right\}$$

~~$b_n = \frac{2}{4n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{4n\pi}$~~

$$b_n = \frac{2}{4n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{4n\pi}$$

$$- \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{4n\pi}\cos(n\pi)$$

$$- \frac{2}{4n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{2}{4n\pi}\left(1 - \cos n\pi\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Sine series

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{1}{2n\pi}\left(1-(-1)^n\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\}\sin(n\pi x)$$

5M

Ans

$$f(x) = \sum\left\{\frac{1}{2n\pi}\left(1-(-1)^n\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\}\sin(n\pi)x$$

1M

4. $f(x) = 2x - x^2 \quad 0 \leq x \leq 2$

FS $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x\right)$

1M

$2l = 2 \qquad l = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi)x + b_n \sin(n\pi)x \right)$$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\,dx \qquad a_0 = \int_0^2 (2x - x^2)\,dx$$

$$a_0 = \left(2\frac{x^2}{2} - \frac{x^3}{3} \right)_0^2 = \frac{4}{3} \qquad \boxed{a_0 = \frac{4}{3}}$$

1.M

~~$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos(nx)\,dx$$~~

~~$$= \int_0^2 \underset{u}{(2x - x^2)} \overset{v}{\cos(nx)}\,dx$$~~

~~$$a_n = \left[(2x - x^2) \left(\frac{\sin(n\pi x)}{\pi n} \right) - (2 - 2x) \frac{-\cos(n\pi x)}{n^2\pi^2} + (-2) \frac{-\sin(n\pi x)}{n^3\pi^3} \right]_{x=0}^{2}$$~~

~~$$a_n = \frac{2}{n^2\pi^2} \left[+2 \cos 2n\pi - 2 \cos 0 \right]$$~~

~~$$a_n = \left[(2x - x^2) \sin \right.$$~~

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \int_0^2 \underset{u}{\left(2x - x^2\right)}\ \underset{v}{\cos(n\pi)x}\,dx$$

$$= \left[(2x-x^2)\frac{\sin(n\pi)x}{n\pi} - (2-2x)\frac{-\cos(n\pi)x}{n^2\pi^2} + (-2)\frac{-\sin(n\pi)x}{n^3\pi^3}\right]_{x=0}^{2}$$

$$= \frac{1}{n^2\pi^2}\,2\left\{(1-x)\frac{\cos(n\pi)x}{n^2\pi^2}\right\}_{x=0}^{2}$$

$$= \frac{2}{n^2\pi^2}\left\{-1\cos(2n\pi) - \cos 0\right\}$$

$$= \frac{-4}{n^2\pi^2}$$

$$a_n = \frac{-4}{n^2\pi^2} \quad \forall\, n \in Z^+$$

3M

~~$b_n = \frac{}{l}$~~ $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$

$l=1$ $\quad b_n = \int_0^{2}\underset{u}{(2x - x^2)}\,\underset{v}{\sin(n\pi)x}\,dx$

$$b_n = \left[(2x-x^2)\frac{-\cos(n\pi)x}{n\pi} - (2-2x)\frac{-\sin(n\pi)x}{n^2\pi^2} + (-2)\frac{\cos(n\pi)x}{n^3\pi^3}\right]_{x=0}^{2}$$

$$b_n = \frac{-2}{n^3\pi^3}\left(\cos 2n\pi - \cos 0\right) = 0$$

(1M)

$$b_n = 0$$

FS $\quad 2x - x^2 = \frac{1}{2}\left(\frac{4}{3}\right) + \sum_{n=1}^{\infty}\frac{-4}{n^2\pi^2}\cos(n\pi)x$

$$2x - x^2 = \frac{2}{3} - \frac{4}{\pi^2}\sum\frac{\cos(n\pi)x}{n^2}$$

$$2x - x^2 = \frac{2}{3} - \frac{4}{\pi^2}\left[\frac{\cos(\pi x)}{1^2} + \frac{\cos(2\pi x)}{2^2} + \frac{\cos(3\pi)x}{3^2} + \cdots\right]$$

(1M)

2. $f(x) = x + x^2$ in $(-\pi, \pi)$

F.S $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$ (1M)

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2)\, dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$ (1M)

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos(nx)\, dx$$

$$= \frac{1}{\pi} \underbrace{\int_{-\pi}^{\pi} x \cos(nx)\, dx}_{\text{odd}} + \frac{1}{\pi} \underbrace{\int_{-\pi}^{\pi} x^2 \cos(nx)\, dx}_{\text{even}}$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos(nx)\, dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin(nx)}{n} - 2x\frac{{}^{-}\cos(nx)}{n^2} + \frac{2}{n^3}\,{}^{-}\sin(nx)\right]_{x=0}^{\pi}$$

$$a_n = \frac{2}{\pi}\,\frac{2}{n^2}\,\pi\cos(n\pi) = \frac{4(-1)^n}{n^2}$$

$$a_n = \frac{4(-1)^n}{n^2} \quad \forall\, n \in Z^+$$ (2M)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\,dx$$

odd

even

$$\frac{2}{\pi}\int_0^{\pi}\underset{u}{x}\,\underset{v}{\sin(nx)}\,dx$$

$$b_n = \frac{2}{\pi}\left[x \cdot \frac{-\cos(nx)}{n} - 1 \cdot \frac{-\sin(nx)}{n^2}\right]_{x=0}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-1}{n}\pi\cos(n\pi)\right] = \frac{-2}{n}(-1)^n$$

$$b_n = \frac{2(-1)^{n+1}}{n} \quad \forall\, n \in Z^+$$

(2M)

FS

$$x + x^2 = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)$$

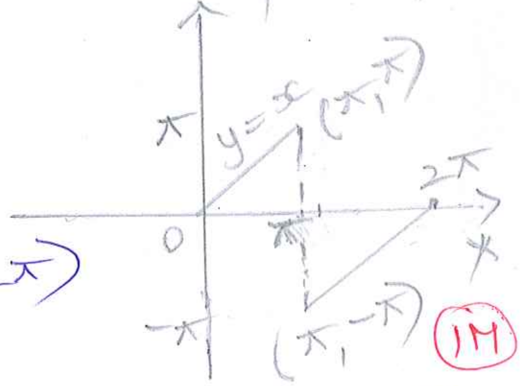
$$+ \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n}\sin(nx)$$

$$x + x^2 = \frac{\pi^2}{3} + 4\sum \frac{(-1)^n}{n^2}\cos(nx)$$

(1M)

$$+ 2\sum \frac{(-1)^{n+1}}{n}\sin(nx)$$

3. $$f(x) = \begin{cases} x & (0,\pi) \\ x-2\pi & (\pi,2\pi) \end{cases}$$



(1M)

FS

$$f(x) = \frac{a_0}{2} + \sum\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

(1M)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(x-2\pi)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(\frac{x^2}{2} - 2\pi(x)\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{1}{2}(4\pi^2-\pi^2) - 2\pi(2\pi-\pi)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{3\pi^2}{2} - 2\pi^2\right] = 0 \qquad \boxed{a_0=0}$$

(1M)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \underset{u}{x}\,\underset{v}{\cos(nx)}\,dx + \int_{\pi}^{2\pi}\underset{u}{(x-2\pi)}\,\underset{v}{\cos(nx)}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\frac{\sin(nx)}{n} - 1\cdot\frac{-\cos(nx)}{n^2}\right\}_{x=0}^{\pi} + \left\{(x-2\pi)\frac{\sin(nx)}{n} - 1\cdot\frac{-\cos(nx)}{n^2}\right\}_{x=\pi}^{2\pi}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2}(\cos n\pi - 1) + \frac{1}{n^2}(1-\cos n\pi)\right]$$

$$= 0 \qquad \boxed{a_n=0}$$

(1M)

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \underset{u}{x}\,\underset{v}{\sin(nx)}\,dx + \int_{\pi}^{2\pi} \underset{u}{(x-2\pi)}\,\underset{v}{\sin(nx)}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\,\frac{-\cos(nx)}{n} - 1.\frac{-\sin(nx)}{n^2}\right\}_{x=0}^{\pi} + \left\{(x-2\pi)\,\frac{-\cos(nx)}{n} - 1.\frac{-\sin(nx)}{n^2}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\,\pi\cos n\pi - \frac{1}{n}\left(0+\pi\cos n\pi\right)\right]$$

$$= \frac{-2\pi\cos(n\pi)}{n\pi} = \frac{2}{n}(-1)^{n+1}$$

$$b_n = \frac{2}{n}(-1)^{n+1} \quad \forall\, n \in Z^+$$

<u>FS</u>

$$f(x) = \sum \frac{2}{n}(-1)^{n+1}\sin(nx)$$

$$f(x) = 2\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right\}$$

$x = \pi/2$

$$f(\pi/2) = 2\left\{1 - \frac{1}{2}(0) + \frac{1}{3}(-1) - \frac{1}{4}(0) + \frac{1}{5}(1) - \cdots\right\}$$

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

or $$1 - \frac{1}{3} + \frac{1}{5} - \cdots\cdots = \frac{\pi}{4}$$ ③M

6. $$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{x=-a}^{a} e^{i\alpha x} dx$$ ①M

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha x}}{i\alpha}\right]_{x=-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{i\alpha}\left(e^{ia\alpha} - e^{-ia\alpha}\right)$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{i\alpha} 2i\sin(a\alpha)$$

$$\boxed{F(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(a\alpha)}{\alpha}}$$

②M

Inverse FT

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\alpha=-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\alpha=-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{\sin(a\alpha)}{\alpha} \left(\cos(\alpha x) - i\sin(\alpha x)\right) d\alpha$$

$$f(x) = \frac{1}{\pi} \int_{\alpha=-\infty}^{\infty} \frac{\sin(a\alpha)\cos(\alpha x)}{\alpha \qquad \text{even}}\, d\alpha$$

$$- \frac{i}{\pi} \int_{\alpha=-\infty}^{\infty} \frac{\sin(a\alpha)\sin(\alpha x)}{\alpha \qquad \text{odd}}\, d\alpha$$

$$f(x) = \frac{2}{\pi} \int_{\alpha=0}^{\infty} \frac{\sin(a\alpha)\cos(\alpha x)}{\alpha}\, d\alpha$$

$$\int_{\alpha=0}^{\infty} \frac{\sin(a\alpha)\cos(\alpha x)}{\alpha}\, d\alpha = \frac{\pi}{2} f(x)$$

(3M)

$\underline{x=0}$

$$\int_{\alpha=0}^{\infty} \frac{\sin(a\alpha)}{\alpha}\, d\alpha = \frac{\pi}{2} f(0)$$

for $x = 0 \in [-a, a]$.

$$\int_{\alpha=0}^{\infty} \frac{\sin(a\alpha)}{\alpha}\, d\alpha = \frac{\pi}{2}$$

(1M)

$\underline{a=1}$ $\int_0^{\infty} \frac{\sin\alpha}{\alpha}\, d\alpha = \frac{\pi}{2}$

7. To P.T $F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2}\,a} e^{-\left(\alpha^2/4a^2\right)}$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} e^{-a^2x^2} e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} e^{-(a^2x^2 - i\alpha x)}\, dx$$

$$a^2x^2 - i\alpha x = \left(ax - \frac{i\alpha}{2a}\right)^2 - \left(\frac{i\alpha}{2a}\right)^2$$

$$= \left(ax - \frac{i\alpha}{2a}\right)^2 + \frac{\alpha^2}{4a^2}$$

$$F\left(e^{-a^2x^2}\right) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} e^{-\left[\left(ax - \frac{i\alpha}{2a}\right)^2 + \frac{\alpha^2}{4a^2}\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\left(\alpha^2/4a^2\right)} \int_{x=-\infty}^{\infty} e^{-\left(ax - \frac{i\alpha}{2a}\right)^2}\, dx$$

$t = ax - \frac{i\alpha}{2a} \qquad dt = a\,dx$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} e^{-(\alpha^2/4a^2)} \int_{t=-\infty}^{\infty} e^{-t^2} \frac{1}{a}\,dt$$

$$F(\alpha) = \frac{1}{a}\frac{1}{\sqrt{2\pi}} e^{-(\alpha^2/4a^2)} \left(2\int_{t=0}^{\infty} e^{-t^2}\,dt\right)$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}} e^{-(\alpha^2/4a^2)} \Gamma\tfrac{1}{2}$$

$$F(e^{-a^2x^2}) = \frac{1}{a}\frac{1}{\sqrt{2\pi}} e^{-(\alpha^2/4a^2)} \sqrt{\pi}$$

$$= \frac{1}{a}\frac{1}{\sqrt{2}} e^{-(\alpha^2/4a^2)}$$

$$F(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-(\alpha^2/4a)}$$

6M

$$F(e^{-x^2}) = \frac{1}{\sqrt{2}} e^{-(\alpha^2/4)} \quad \text{or} \quad \frac{\sqrt{\pi}}{2} e^{-(\alpha^2/4)}$$

$$F_c(e^{-x^2}) = \frac{1}{\sqrt{2}} e^{-(\alpha^2/4)}$$

1M

8. $$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_{x=0}^{\infty} f(x) \sin(\alpha x)\, dx$$ 1M

$$= \sqrt{\frac{2}{\pi}} \int_{x=0}^{\infty} \frac{e^{-ax}}{x} \sin(\alpha x)\, dx$$

$$F_s'(\alpha) = \sqrt{\frac{2}{\pi}} \int_{x=0}^{\infty} \frac{e^{-ax}}{x} x \cos(\alpha x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{x=0}^{\infty} e^{-ax} \cos(\alpha x)\, dx$$

$$F_s'(\alpha) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{(-a)^2+\alpha^2} \left(-a\cos\alpha x + \alpha \sin(\alpha x) \right) \right]_{x=0}^{\infty}$$

$$F_s'(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{a^2+\alpha^2} (a)$$ 4M

Intg w.r.to α

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\alpha}{a}\right) + c$$ 1M

at $\alpha = 0$ $F_s(0) = 0$

$$0 = 0 + c \Rightarrow c = 0$$ 1M

Ans $$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\alpha}{a}\right)$$

or

$$\tan^{-1}\left(\frac{\alpha}{a}\right)$$