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USN

Internal Assesment Test - I

Sub:	Engineering Maths-IV							Code:	15MAT41
Date:	12 / 03 /2018	Duration:	90 mins	Max Marks:	50	Sem:	4	Branch:	CSE-A, EEE-A,CIV-A

NOTE: First question is compulsory. Answer any six questions from the rest.

		Marks	OBE CO	OBE RBT
1.	Given $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, $y(0)=1$, $y'(0)=0$. Evaluate $y(0.1)$ using Runge-Kutta method of order 4.	[8]	CO1	L3
2.	Using modified Euler's method find $y(0.2)$ correct to four decimal places solving the equation $\frac{dy}{dx} = x - y^2$, $y(0)=1$ taking h=0.1. Carry out two approximations in each stage.	[7]	CO1	L3
3.	Use Taylor's series method to find y at $x = 0.1$, 0.2, 0.3 considering terms upto fourth degree given that $\frac{dy}{dx} = x^2 + y^2$ and y(0)=1.	[7]	CO1	L3

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4. If $\frac{dy}{dx} = 2e^x - y$, $y(0)=2$, $y(0.1)=2.010$, $y(0.2)=2.040$ and $y(0.3)=2.090$, find $y(0.4)$ correct to four decimal places by using Adams-Bashforth predictor - corrector method. (Apply the corrector formula twice). [7] CO1 L3

5. Applying Milne's predictor and corrector formula compute $y(0.8)$ given that y satisfies the equation $y''=2\,y\,y'$ and y and y' are governed by the following values $y(0)=0$, $y(0.2)=0.2027$ $y(0.4)=0.4228$, $y(0.6)=0.6841$
$y'(0)=1$, $y'(0.2)=1.041$, $y'(0.4)=1.179$, $y'(0.6)=1.468$. Apply corrector formula twice. [7] CO1 L3

6. State and prove orthogonal property of Bessel Functions. [7] CO2 L3

7. Starting from the expressions of $J_{1/2}(x)$ and $J_{-1/2}(x)$ in the standard form prove the following results. [7] CO2 L3

(a) $J_{1/2}'(x)\, J_{-1/2}(x) - J_{-1/2}'(x)\, J_{1/2}(x) = \frac{2}{\pi x}$ (b) $\int_0^{\frac{\pi}{2}} \sqrt{x}\, J_{1/2}(2x) = \frac{1}{\sqrt{\pi}}$

8. Express $f(x)=x^4+3x^3-x^2+5x-2$ in terms of Legendre Polynomials by first deriving the expression for $P_4(x)$ and $P_3(x)$ from Rodrigue's formula. [7] CO2 L3

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IAT 1 - Solution - CSE-A, EEE-A, CIV-A, TCE, DIP-1, DIP-2

March 2018

1. Let $y' = z$, $y'' = z'$, $y_0 = 1$, $x_0 = 0$, $z_0 = 0$.

Given eq reduces to.

$$\frac{dz}{dx} = 1 + 2xy + x^2 z.$$

$f(x, y, z) = z$

$g(x, y, z) = 1 + 2xy + x^2 z$

2m

$k_1 = h f(x_0, y_0, z_0) = 0.1\, f(0, 1, 0) = 0$

$l_1 = h g(x_0, y_0 z_0) = 0.1 [1 + 0 + 0] = 0.1$

$k_2 = h f[x_0 + h/2, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}] = 0.1 [0.05]$

$= 0.005.$

$l_2 = h g[x_0 + h/2, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}] = 0.1 [1 + 2 \times 0.05 \times 1 + (0.05)^3]$

$= 0.11$

$k_3 = h f[x_0 + h/2, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}] = 0.1 \times 0.055 = 0.0055$

$l_3 = 0.1 [1 + 2 \times 0.05 \times 1.0025 + (0.05)^2 (0.055)]$

$= 0.11004.$

$l_3 = h g [x_0 + h/2, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}]$

$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3]$

$k_4 = 0.1\, f(0.1, 1.0055, 0.11004) = 0.1 \times 0.11004$

$= 0.011$

$l_4 = h g[x_0 + h, y_0 + k_3, z_0 + l_3]$

$= 0.1 [1 + 2(0.1)(1.0055) + (0.1)^2 (0.11004)]$

4m

$= 0.1202.$

$y(x_0 + h) = y(0.1) = y_0 + 1/6 (k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + 1/6 [0 + 2 \times 0.005 + 2(0.0055) + 0.011]$

$\therefore y(0.1) = \underline{1.0053}.$

2m

2. $f(x,y) = x - y^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$ 1m

I st stage: $f(x_0, y_0) = -1$, $x_1 = x_0 + h = 0.1$

To find: $y(x_1) = y_1 = ?$

$y_1^{(0)} = y_0 + h f(x_0, y_0) = 1 + (0.1)(-1) = 0.9$

By Modified Euler's formula.

$y_1^{(1)} = y_0 + h/2 \left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$

$= 1 + \frac{0.1}{2} \left[-1 + x_1 - (y_1^{(0)})^2 \right]$

$y_1^{(1)} = 1 + 0.05 \left[-0.9 - (0.9)^2 \right] = 0.9145$.

$y_1^{(2)} = y_0 + h/2 \left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$

$= 1 + \frac{0.1}{2} \left[-1 + x_1 - [y_1^{(1)}]^2 \right]$

$= 1 + 0.05 \left[-0.9 - (0.9145)^2 \right] = 0.9133$.

$\therefore y(0.1) = 0.9133$ 3m

2nd stage: $y(0.1) = 0.9133$

$x_0 = 0.1$, $y_0 = 0.9133$, $f(x,y) = x - y^2$

$f(x_0, y_0) = 0.1 - (0.9133)^2 = -0.7341$

$x_1 = x_0 + h = 0.1 + 0.1 = 0.2$

$y(x_1) = y(0.2) = ?$

$y_1^{(0)} = y_0 + h f(x_0, y_0) = 0.9133 + 0.1(-0.7341)$

$= 0.8399$

By Modified Euler's method.

$$y_1^{(1)} = 0.9133 + \frac{0.1}{2}\left[-0.7341 + 0.2 - (0.8399)^2\right]$$

$$= 0.9133 + 0.05\left[-0.5341 - (0.8399)^2\right] = 0.8513$$

$$y_1^{(2)} = 0.9133 + 0.05\left[-0.5341 - (0.8513)^2\right] = 0.8504$$

$$\therefore y(0.2) = \underline{0.8504}. \qquad 3m$$

3. $y' = x^2 + y^2$, $y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$

$y'(x_0) = 0 + 1 = 1$

$y'' = 2x + 2yy'$, $y''(x_0) = 2(0) + 2y(0)y'(0)$

$= 2(1)(1) = 2.$

$y''' = 2 + 2yy'' + 2(y')^2$

$= 2 + 2\times1\times2 + 2\times1 \Rightarrow y'''(x_0) = 8$

$y^{IV} = 2yy''' + 2y''\cdot y' + 4y'y''$

$= 2\times1\times8 + 2\times1\times2 + 4\times1\times2 = 16 + 4 + 8$

$= 28 \qquad 4m$

$\therefore$ Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{IV}(x_0) + \cdots$$

$$= 1 + x\cdot1 + \frac{x^2}{2}\cdot2 + \frac{x^3}{6}\cdot8 + \frac{x^4}{24}\cdot28.$$

$$= 1 + x + x^2 + \frac{4x^3}{3} + \frac{7}{6}x^4 \qquad 2m$$

$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)^3}{3} + \frac{7}{6}(0.1)^4 = 1.1113$

$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{4(0.2)^3}{3} + \frac{7}{6}(0.2)^4 = 1.2507$

$y(0.3) = \underline{\underline{1.426}}$.

1m

4. $y' = 2e^x - y$

x	:	0	0.1	0.2	0.3
y	:	2	2.010	2.040	2.090
$y' = 2e^x - y$	:	0	0.2003	0.4028	0.6097

2m

To find: $y(0.4)$

$y_4^{(P)} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$

$\therefore y_4^{(P)} = 2.09 + \frac{0.1}{24}[55(0.6097) - 59(0.4028) + 37(0.2003) - 9(0)]$

$= 2.1616$

$y_4' = 2e^{0.4} - 2.1616 = 0.822$

$y_4^{(C)} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$ 3m

$y_4^{(C)} = 2.09 + \frac{0.1}{24}[9 \times 0.822 + 19(0.6097) - 5(0.4028) + 0.2003]$

$= 2.1615$

$y_4' = 2e^{0.4} - 2.1615 = 0.8221$

$$y_4^{(c)} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$$

$$= 2.1615$$

$\therefore y(0.4) = \underline{\underline{2.1615}}.$ 2m

5. Let $y' = z \Rightarrow y'' = \frac{dz}{dx} = z'$

$\therefore$ The given eq becomes $z' = 2yz$ 1m

x :	0	0.2	0.4	0.6
y :	0	0.2027	0.4228	0.6841
$y' = z$:	1	1.041	1.179	1.468
$z' = 2yz$:	0	0.422	0.997	2.009.

2m

Predictor formula.

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2z_1 - z_2 + 2z_3]$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}[2z_1' - z_2' + 2z_3']$$

$$y_4^{(P)} = 0 + \frac{4 \times 0.2}{3}[2 \times 1.041 - 1.179 + 2 \times 1.468]$$

$$= 1.0237.$$

$$z_4^{(P)} = 1 + \frac{0.2 \times 4}{3}[2 \times 0.422 - 0.997 + 2 \times 2.009]$$

$$= 2.0307.$$

2m

$$z_4' = 2y_4^{(P)} z_4^{(P)} = 2 \times 1.0237 \times 2.0307 = 4.1577$$

$$y_4^{(c)} = y_2 + h/3 [z_2 + 4z_3 + z_4]$$

$$z_4^{(c)} = z_2 + h/3 [z_2' + 4z_3' + z_4']$$

$$y_4^{(c)} = 0.4228 + \frac{0.2}{3} [1.179 + 4 \times 1.468 + 2.0307]$$

$$= 1.0282.$$

$$z_4^{(c)} = 1.179 + \frac{0.2}{3} [0.997 + 4 \times 2.009 + 4.1577]$$

$$= 2.0584.$$

$$z_4' = 2 y_4^{(c)} z_4^{(c)} = 4.1577.$$

2m

$$\therefore y_4^{(c)} = \underline{1.0301}, \quad \therefore y(0.8) = 1.0301.$$

7. a)

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad , \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

1m

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin x}{\sqrt{x}} \quad , \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{\cos x}{\sqrt{x}}$$

$$J_{1/2}'(x) = \sqrt{\frac{2}{\pi}} \left[\frac{1}{\sqrt{x}} \cos x - \frac{\sin x}{(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{x}} \left[\cos x - \frac{\sin x}{2x} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{x}} \left[\frac{2x \cos x - \sin x}{2x} \right]$$

2m

$$J_{-1/2}'(x) = \sqrt{\frac{2}{\pi}} \left[-\frac{\sin x}{\sqrt{x}} + \cos x \cdot \frac{(-1)}{(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} \right] = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} \left[-\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{2x} \right]$$

$J_{1/2}'(x) J_{-1/2}(x) - J_{-1/2}'(x) J_{1/2}(x)$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{+\cos x \cdot 2x - \sin x}{2x} \right] \sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x \cdot \left[\frac{-2x \sin x - \cos x}{2x} \right] \sqrt{\frac{2}{\pi x}}$$

$$= \frac{2}{\pi x} \left(\frac{\cos x \, 2x - \sin x}{2x} \right) \cos x - \frac{2}{\pi x} \left[\frac{-2x \sin x - \cos x}{2x} \right] \sin x$$

$$= \frac{2}{\pi x} \left[\frac{\cos^2 x \cdot 2x - \sin x \cancel{\cos x} + 2x \sin^2 x + \cancel{\cos x} \sin x}{2x} \right]$$

$$= \frac{2}{\pi x} \left[2\cancel{x} \right] \left[\frac{\cos^2 x + \sin^2 x}{2\cancel{x}} \right]$$

$$= \underline{\underline{\frac{2}{\pi x}}}.$$

2m

b) $\int_0^{\pi/2} \sqrt{x} \, J_{1/2}(2x) \, dx = \frac{1}{\sqrt{\pi}}$

$$J_{1/2}(2x) = \sqrt{\frac{\cancel{2}}{\pi \cdot \cancel{2} x}} \sin(2x)$$

$$= \sqrt{\frac{1}{\pi x}} \sin 2x.$$

1m

$$\int_0^{\pi/2} \sqrt{x} \, J_{1/2}(2x) dx = \int_0^{\pi/2} \cancel{\sqrt{x}} \frac{1}{\sqrt{\pi} \cancel{\sqrt{x}}} \sin 2x \, dx$$

$$= \frac{1}{\sqrt{\pi}} \left[\frac{-\cos 2x}{2} \right]_0^{\pi/2} = \frac{-1}{2\sqrt{\pi}} [\cos \pi - 1]$$

$$= + \frac{2}{2\sqrt{\pi}} = \underline{\frac{1}{\sqrt{\pi}}}$$

1m

8.

$$P_3(x) = \frac{1}{2^3 \cdot 3!} \frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48} \frac{d^3}{dx^3}[x^6 - 3x^4 + 3x^2 - 1]$$

$$= \frac{1}{48}[120x^3 - 72x] = \frac{24}{48}[5x^3 - 3x]$$

$$P_3(x) = \frac{1}{2}[5x^3 - 3x]$$ 1m

$$P_4(x) = \frac{1}{2^4 \cdot 4!} \frac{d^4}{dx^4}(x^2-1)^4.$$

$$= \frac{1}{16 \times 24} \frac{d^4}{dx^4}(x^8 - 4x^6 + 6x^4 - 4x^2 + 1)$$

$$(x-a)^n = x^n - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 - {}^nC_3 x^{n-3} a^3 + \cdots$$
$$\cdots (-1)^n a^n.$$

$$\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!} x^{m-n}, \quad m > n.$$

$$P_4(x) = \frac{1}{16 \times 24}\left[\frac{8!}{4!} x^4 - 4 \times \frac{6!}{2!} x^2 + 6 \times \frac{4!}{0!} x^0\right]$$

$$= \frac{48}{16 \times 24}[35x^4 - 30x^2 + 3]$$

$$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$$ 2m

$$x^4 + 3x^3 - x^2 + 5x - 2 = f(x)$$

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$, $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

$\Rightarrow x^2 = \frac{1}{3}(2P_2(x)+P_0(x))$, $x^3 = 1/5\,(2P_3(x)+3P_1(x))$

$x^4 = 1/35\,[8P_4(x)+30x^2-3]$, $x^4 = \frac{1}{35}(8P_4(x)+20P_2(x)+7P_0(x))$

$\therefore f(x) = \frac{1}{35}[8P_4(x)+20P_2(x)+7P_0(x)] + 3/5\,[2P_3(x)+3P_1(x)] - \frac{1}{3}(2P_2(x)+P_0(x)) + 5P_1(x) - 2P_0(x)$ 2m

$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) + \left(\frac{20}{35}-\frac{2}{3}\right)P_2(x) + \left(\frac{9}{5}+5\right)P_1(x) + \left(7/35 - 1/3 - 2\right)P_0(x)$

$= \frac{8}{35}P_4(x) + 6/5\,P_3(x) - \frac{2}{21}P_2(x) + 34/5\,P_1(x) - \frac{224}{105}P_0(x).$ 2m

6. Orthogonal property of Bessel Functions:

Statement: If α & β are 2 distinct roots of $J_n(x)=0$ then $\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$

$$= \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{1}{2}[J_n'(\alpha)]^2 & \text{if } \alpha = \beta \\ = \frac{1}{2}[J_{n+1}(\alpha)]^2 & \text{if } \alpha = \beta \end{cases}$$ 1m

Proof: W.K.T $J_n(\lambda x)$ is a solution of the equation.

$$x^2y''+xy'+(\lambda^2x^2-n^2)y=0$$

If $u=J_n(\alpha x)$, $v=J_n(\beta x)$ the associated diff eqs are

$$x^2u''+xu'+(\alpha^2x^2-n^2)u=0 \quad -(1)$$

$$x^2v''+xv'+(\beta^2x^2-n^2)v=0 \quad -(2)$$

×ly (1) by v/x & (2) by u/x. we get,

$$xvu''+vu'+\alpha^2uvx-\frac{n^2uv}{x}=0 \quad -(3)$$

$$xuv''+uv'+\beta^2uvx-\frac{n^2uv}{x}=0 \quad -(4)$$

(3) - (4) gives

$$x(vu''-uv'')+(vu'-uv')+(\alpha^2-\beta^2)uvx=0$$

$$\frac{d}{dx}[x(vu'-uv')]=(\beta^2-\alpha^2)xuv$$

Integrate b.s w.r.to 'x' b/w 0 to 1.

$$\left[x(u'v-uv')\right]_{x=0}^{1}=(\beta^2-\alpha^2)\int_{x=0}^{1}xuv\,dx$$

ie $(vu'-uv')_{x=1}-0=(\beta^2-\alpha^2)\int_0^1 xuv\,dx.$ -(5)

$u=J_n(\alpha x)$, $u'=\alpha J_n'(\alpha x)$, $v'=\beta J_n'(\beta x)$.

(5) becomes

$$\left[J_n(\beta x)\alpha J_n'(\alpha x)-J_n(\alpha x)\beta J_n'(\beta x)\right]_{x=1}=\beta^2-\alpha^2\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx$$

Hence $\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx=\frac{1}{\beta^2-\alpha^2}\left[\alpha J_n(\beta)J_n'(\alpha)-\beta J_n(\alpha)J_n'(\beta)\right]$ -(6)

∴ α & β are distinct roots of $J_n(x)=0$ we have

$J_n(\alpha)=0$ & $J_n(\beta)=0$ with the result the R.H.S of ⑥ becomes zero

$\therefore \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$ provided $\beta^2-\alpha^2 \neq 0$

$\Rightarrow \alpha \neq \beta$.

3m

<u>When $\alpha=\beta$:</u>

The R.H.S of ⑥ becomes an indeterminate form of the type $\left(\frac{0}{0}\right)$ when $\alpha=\beta$.

We shall evaluate by taking limits on b.s as $\beta\to\alpha$ Keeping α fixed, by applying L' Hospital's rule.

i.e $\lim_{\beta\to\alpha} \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx =$

$$\lim_{\beta\to\alpha}\left\{\frac{1}{\beta^2-\alpha^2}\left(\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\right)\right\}$$

$\because \alpha$ is fixed we must have $J_n(\alpha)=0$ as α is a root of $J_n(x)=0$

$$\therefore \lim_{\beta\to\alpha}\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \lim_{\beta\to\alpha}\frac{1}{\beta^2-\alpha^2}\left(\alpha J_n(\beta) J_n'(\alpha)\right)$$

$$= \lim_{\beta\to\alpha}\frac{1}{2\beta}\left(\alpha J_n'(\beta) J_n'(\alpha)\right)$$ by L. Hospital's rule

we now have

$$\int_0^1 x\left[J_n(\alpha x)\right]^2 dx = \frac{1}{2\alpha}\cdot \alpha J_n'(\alpha) J_n'(\alpha)$$

$$= \frac{1}{2}\left[J_n'(\alpha)\right]^2.$$

$$\therefore \int_0^1 x J_n^2(\alpha x)\,dx = \frac{1}{2}\left[J_n'(\alpha)\right]^2 \quad \text{--- (7)}$$

2m

from Recurrence relation

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$$J_n'(\alpha) = \frac{n}{\alpha} J_n(\alpha) - J_{n+1}(\alpha)$$

$$\because J_n(\alpha) = 0, \; J_n'(\alpha) = -J_{n+1}(\alpha)$$

$\therefore$ (7) becomes

$$\int_0^1 x J_n^2(\alpha x)\,dx = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2$$

1m