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Internal Assesment Test – 2

Sub: Engineering Mathematics II				Code: 15MAT21
Date: 10/04/2017	Duration: 90 mins	Max Marks: 50	Sem: 2	Sections: ALL

Answer **Q1** and ANY SIX from **Q2** to **Q9**.

		Marks	OBE CO	OBE RBT
Q1.	Derive the one-dimensional wave equation in the form $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.	[8]	CO201.4	L3
Q2.	Obtain the PDE by eliminating the arbitrary function $\phi(x^2+y^2-z^2, x+y+z)=0$	[7]	CO201.2	L3
Q3.	Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$ *and* $z=0$ when y is an odd multiple of $\pi/2$.	[7]	CO201.2	L3
Q4.	Evaluate $\int_0^1 \int_x^1 e^{-y^2}\, dy\, dx$.	[7]	CO201.3	L3

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Q5.	Evaluate $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ by changing to polar coordinates.	[7]	CO201.4	L3
Q6.	Evaluate $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$.	[7]	CO201.3	L3
Q7.	Show that $\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}\, dx = 2^{p+q-1}\beta(p,q)$ using Beta and Gamma functions.	[7]	CO201.5	L1
Q8.	For $m>0, n>0$, show that $\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.	[7]	CO201.5	L1
Q9a.	Solve $\frac{\partial^2 z}{\partial x^2}+3\frac{\partial z}{\partial x}-4z=0$ subject to the conditions $z=1$ and $\frac{\partial z}{\partial x}=y$ when $x=0$.	[5]	CO201.2	L3
Q9b.	Show that $\beta(m,n)=\beta(n,m)$ for $m,n>0$.	[2]	CO201.5	L1

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Q9b.	Show that $\beta(m,n)=\beta(n,m)$ for $m,n>0$.	[2]	CO201.5	L1

IAT - II

Engineering Mathematics - II

1. Consider a flexible string tightly stretched between two fixed points at a distance l apart. Let ρ be the mass per unit length of the string

We assume the following

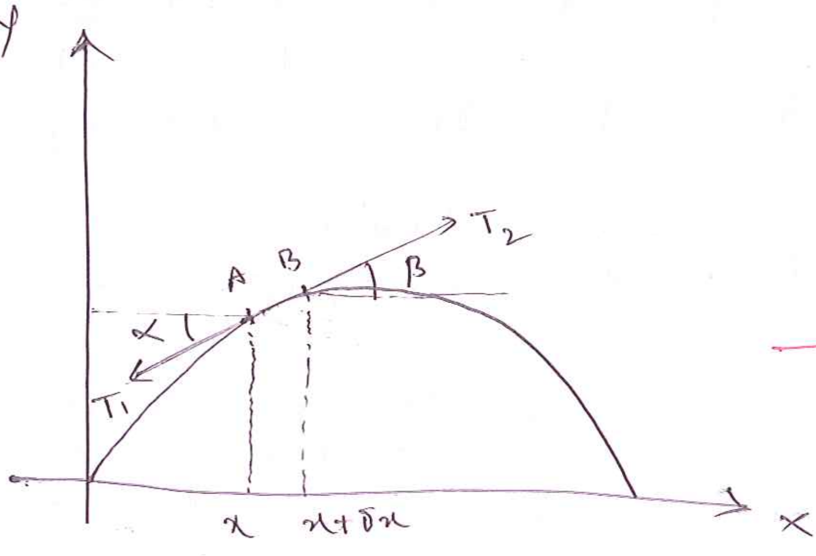
a) When the string vibrates, the motion takes place in the x-u-plane and we have transverse vibrations

b) The string is perfectly elastic

c) The tension at each point is tangential to the string

d) Other external forces on the string are zero. ——— 2

Let $u(x,t)$ be the displacement from x-axis at time t.



——— 1

Consider the forces acting on the small element δx

Let T_1 & T_2 be the tensions at pts A & B.

Since there is no motion in the horizontal direction, we have

$T_1 \cos\alpha = T_2 \cos\beta = T$ where α & β are the angles shown

(1)

The resulting force acting upwards is given by $T_2 \sin\beta - T_1 \sin\alpha$ (1)

By Newton's second law of motion

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho\,\delta x)\frac{\partial^2 u}{\partial t^2} \quad (1)$$

$$\frac{T_2}{T}\sin\beta - \frac{T_1}{T}\sin\alpha = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 u}{\partial t^2} \quad (\div T)$$

$$\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha} = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 u}{\partial t^2} \quad (\text{from } (1))$$

$$\tan\beta - \tan\alpha = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 u}{\partial t^2} \quad (1)$$

~~Dividing by δx & taking lt $\delta x \to 0$.~~

$$\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 u}{\partial t^2}$$

[$\because$ $\tan\beta$ & $\tan\alpha$ represent slopes at $B(x+\delta x)$ & $A(x)$ respectively]

Dividing by δx & taking lt $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x} = \cancel{\lim_{\delta x \to 0}}\ \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \quad (1)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

Let $T/\rho = c^2$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

2. $\phi(x^2+y^2-z^2,\ x+y+z) = 0$

Let $u = x^2+y^2-z^2 \qquad v = x+y+z$ —— (1)

$$\frac{\partial u}{\partial x} = 2x - 2z\frac{\partial z}{\partial x} = 2(x - zp)$$

$$\frac{\partial u}{\partial y} = 2y - 2z\frac{\partial z}{\partial y} = 2(y - zq)$$

$$\frac{\partial v}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1 + p$$

$$\frac{\partial v}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1 + q$$ —— (2)

we have $\phi(u, v) = 0$

Diff wrt x & y using chain rule

$$\frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \quad \text{or} \quad \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x} \quad (2)$$

$$\frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y} \quad (3)$$

—— (2)

(2) ÷ (3) $\Rightarrow$

$$\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} \quad \text{———— (1)}$$

$$\Rightarrow \frac{1+p}{1+q} = \frac{\cancel{2}(x-zp)}{\cancel{2}(y-zq)} \quad \text{———— (1)}$$

$$y - zq + py - pqz = x - zp + qx - pqz$$

$$p(y+z) - q(x+z) = x - y$$

(3). $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$; $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$

& $z = 0$ if y is an odd multiple of $\pi/2$

The given pde can be written as

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \sin y$$

Integrating wrt x treating y as constant

$$\frac{\partial z}{\partial y} = -\sin y \cos x + f(y) \rightarrow (1) \quad \text{———— (2)}$$

Integrating wrt y treating x as constant

$$z = \cos x \cos y + F(y) + g(x) \rightarrow (2)$$

where $F(y) = \int f(y)\, dy$

Put $x=0$ in (1) ———— (2)

$$-2\sin y = -\sin y + f(y) \Rightarrow f(y) = -\sin y \quad \text{———— (1)}$$

$$\therefore F(y) = \int f(y)\, dy = \int -\sin y\, dy = \cos y. \quad \text{———— (1)}$$

$\therefore$ (2) becomes

$$Z = \cos x \cos y + \cos y + g(x) \rightarrow (3)$$

Given that $Z = 0$ if $y = (2n+1)\pi/2$

(3) becomes

$$0 = \cos x \cos (2n+1)\pi/2 + \cos (2n+1)\pi/2 + g(x)$$

$$\cos ((2n+1)\pi/2) = 0$$

$$\therefore \quad g(x) = 0$$ ——— (1)

Thus the solution to the pde is

$$Z = \cos x \cos y + \cos y$$

or

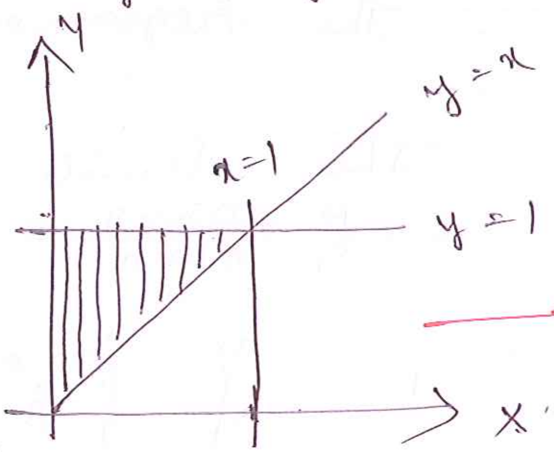
$$Z = \cos y (1 + \cos x)$$

(4) $\displaystyle\int_0^1 \int_x^1 e^{-y^2}\, dy\, dx$

Clearly x varies from $x=0$ to $x=1$

& y varies from $y=x$ to $y=1$

The region of integration is the shaded region



——— (1)

Changing the order of integration

* y varies from $y=0$ to $y=1$

x varies from $x=0$ to $x=y$

$$\therefore \int_0^1 \int_x^1 e^{-y^2}\, dy\, dx = \int_0^1 \int_0^y e^{-y^2}\, dx\, dy$$ ——— (2)

$$= \int_0^1 e^{-y^2} \cdot x \Big|_0^y \, dxy$$ ——— (1)

$$= \int_0^1 e^{-y^2} \cdot y \, dy$$ ——— (1)

$y^2 = t$

$2y\,dy = dt \Rightarrow y\,dy = \frac{dt}{2}$

$y = 0,\ t = 0$

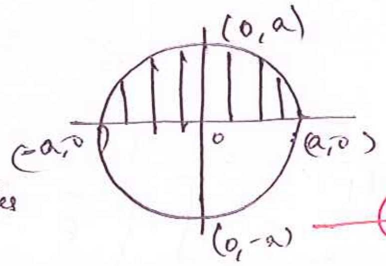
$y = 1,\ t = 1$

$$= \frac{1}{2}\int_0^1 e^{-t} \, dt$$

$$= \frac{1}{2} \frac{e^{-t}}{-1}\Big]_0^1 = \frac{1}{2}\left[\frac{-1}{e} + 1\right]$$

$$= \frac{e-1}{2e}$$ ——— (2)

(5) Let $I = \int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2} \, dy\, dx$



Changing to polar coordinates ——— (1)

$x = a\cos\theta \qquad y = a\sin\theta$

$\Rightarrow x^2 + y^2 = a^2 \qquad dy\,dx = r\,dr\,d\theta$ ——— (2) + (1)

The region of integration is the shaded region

The limits of r & θ are

$\theta : 0 \to \pi$

$r : 0 \to a$

$$\therefore I = \int_{\theta=0}^{\pi} \int_{r=0}^{a} r \cdot r\,dr\,d\theta$$ ——— (1)

$$= \int_{\theta=0}^{\pi} \frac{r^3}{3}\Big|_0^a d\theta = \frac{a^3}{3}\int_{\theta=0}^{\pi} d\theta = \frac{\pi a^3}{3}.$$ ——— (2)

Q6. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$

Sol^n:-

put, $k = \sqrt{1-x^2-y^2}$

$$\Rightarrow \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{k} \frac{dz\,dy\,dx}{\sqrt{k^2-z^2}} \quad \text{— (1)}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left[\frac{z}{k}\right]\right]_0^k dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}1 - \sin^{-1}0\right] dy\,dx \quad \text{— (2)}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\,dy\,dx$$

$$= \frac{\pi}{2} \int_0^1 \left[y\right]_0^{\sqrt{1-x^2}} dx \quad \text{— (2)}$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\,dx$$

put $x = \sin\theta$

$dx = \cos\theta\,d\theta$

as $x \to 0,\ \theta \to 0$

$x \to 1,\ \theta \to \pi/2$

$$= \frac{\pi}{2} \int_0^{\pi/2} \cos\theta \cos\theta\,d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{\pi}{2}\,\frac{1}{2}\,\frac{\pi}{2}$$

$$= \frac{\pi^2}{8} \quad \text{— (2)}$$

(4) S.T $\displaystyle\int_{-1}^{1} (1+x)^{p-1} (1-x)^{q-1}\, dx = 2^{p+q-1} \beta(p,q)$ using Beta and gamma functions.

Solⁿ:- put

$$\left.\begin{aligned} x &= \cos 2\theta \\ dx &= -2\sin 2\theta\, d\theta \\ \text{as } x \to -1, &\quad \theta \to +\frac{\pi}{2} \\ x \to 1, &\quad \theta \to 0 \end{aligned}\right\} \text{ — } 2$$

LHS

$$\therefore \; = \int_{+\frac{\pi}{2}}^{0} (1+\cos 2\theta)^{p-1} (1-\cos 2\theta)^{q-1} (-2\sin 2\theta\, d\theta)$$

$$= -\int_{0}^{+\frac{\pi}{2}} (2\cos^2\theta)^{p-1} (2\sin^2\theta)^{q-1} (-4\sin\theta\cos\theta)\, d\theta \quad \text{— } 2$$

$$= \int_{0}^{\pi/2} 2^{p+q} \cos^{2p-1}\theta \, \sin^{2q-1}\theta \, d\theta \quad \text{— } 1$$

$$= 2^{p+q} \int_{0}^{\pi/2} \cos^{2p-1}\theta \, \sin^{2q-1}\theta \, d\theta \quad \text{— } 1$$

$$= 2^{p+q} \frac{1}{2} \beta\left[\frac{2q-1+1}{2}, \frac{2p-1+1}{2}\right] = 2^{p+q-1} \beta(q,p)$$

$$= 2^{p+q-1} \beta(p,q) = \text{RHS} \quad \text{— } 1$$

8. For $m>0$, $n>0$, S.T $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Solⁿ:-

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{—— } 1$$

$$\Gamma(m) = 2\int_0^{\infty} e^{-x^2} x^{2m-1} dx \quad \text{—— } 1$$

$$\Gamma(n) = 2\int_0^{\infty} e^{-y^2} y^{2n-1} dy \quad \text{—— } 1$$

$$\Gamma(m)\cdot\Gamma(n) = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\,dy$$

We've,

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$, $x^2+y^2 = r^2$ —— 1

and $r \to 0$ to ∞, $\theta \to 0$ to ∞

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r^{2m-1} \sin^{2m-1}\theta \, r^{2n-1} \cos^{2m-1}\theta \; r\,dr\,d\theta$$

$$= 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr \times 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{—— } 2$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\,\beta(m,n)$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{—— } 1$$

9

(a) Solve $\frac{\partial^2 Z}{\partial x^2} + 3\frac{\partial Z}{\partial x} - 4Z = 0$ subject to the condition

$Z = 1$ and $\frac{\partial Z}{\partial x} = y$ when $x = 0$

Soln:- Given P.D.E is homogeneous, So, we can solve this in the form of ODE.

$$\therefore \frac{d^2 Z}{dx^2} + 3\frac{dZ}{dx} - 4Z = 0$$

$$[D^2 + 3D - 4]Z = 0$$

$$f(D)Z = 0 \quad \text{—— (1)}$$

$$Z = Z_c$$

A.Eqn is,

$$f(m) = 0$$

$$m^2 + 3m - 4 = 0$$

$$m = -4, 1$$

$$\therefore Z_c = C_1 e^{-4x} + C_2 e^{x}$$

$$\therefore Z = C_1 e^{-4x} + C_2 e^{x} \quad \text{—— (2)}$$

Condn (1)

$Z = 1$ when $x = 0$,

$\Rightarrow 1 = C_1 + C_2 \rightarrow (1)$

Condn (2)

$\frac{\partial Z}{\partial x} = y$ when $x = 0$

$\frac{\partial Z}{\partial x} = -4e^{-4x} C_1 + C_2 e^{x}$

$y = -4C_1 + C_2 \rightarrow (2)$

$$\begin{aligned} -4C_1 + \cancel{C_2} &= y \\ -\; C_1 + \cancel{C_2} &= 1 \\ \hline -5C_1 &= y - 1 \end{aligned}$$

$C_1 = \frac{1-y}{5}$ —— (1)

$C_1 + C_2 = 1$

$C_2 = 1 - C_1$

$= 1 - \frac{1-y}{5}$

$C_2 = \frac{4+y}{5}$ —— (1)

$$Z = \left(\frac{1-y}{5}\right)e^{-4x} + \left(\frac{4+y}{5}\right)e^{x}$$

$$\Rightarrow 5Z = (1-y)\,e^{-4x} + (4+y)\,e^{x}$$

(b) S.T $\beta(m,n) = \beta(n,m)$ for $m, n > 0$

proof :- $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $1-x = t$

$-dx = dt$ (1)

as $x \to 0,\ t \to 1$

as $x \to 1,\ t \to 0$

$$\therefore \beta(m,n) = \int_1^0 (1-t)^{m-1}\, t^{n-1}(-dt)$$

$$= \int_0^1 t^{n-1}(1-t)^{m-1}\,dt$$

$$= \beta(n,m)$$ (1)