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CMRIT
CMR INSTITUTE OF TECHNOLOGY, BENGALURU.
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Improvement Test

Sub:	**Engineering Mathematics - IV**	Sub Code:	15MAT41	Branch:	**ISE/ECE/ ME**
Date:	21.05.2018	Duration:	90 mins.	Max Marks: 50	
Sem / Sec:	**IV/ ISE–A & B, EC–A &C ME–B & CV–B (regular)**				OBE

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1. JPD :-

X \ Y	−2	−1	4	5	Total
1	0·1	0·2	0	0·3	0·6
2	0·2	0·1	0·1	0	0·4
Total	0·3	0·3	0·1	0·3	1

Marginal dist. of X.

X	1	2	Total
f(x)	0·6	0·4	1

— (1)

Marginal dist of Y.

Y	−2	−1	4	5	Total
g(y)	0·3	0·3	0·1	0·3	1

— (1)

$$E(X) = \sum_x x\, f(x) = 1 \times 0\cdot6 + 2 \times 0\cdot4 = 1\cdot4$$

$$E(Y) = \sum_y y\, g(y) = -2 \times 0\cdot3 + -1 \times 0\cdot3 + 4 \times 0\cdot1 + 5 \times 0\cdot3$$

$$= 1$$

$$E(XY) = \sum_x \sum_y xy\, J_{xy} = 0\cdot9$$ — (1)

$$Cov(X, Y) = E(XY) - E(X)\,E(Y) = -0\cdot5$$ — (1)

$$E(X^2) = \sum_x x^2 f(x) = 0\cdot6 + 1\cdot6 = 2\cdot2$$

$$E(Y^2) = \sum_y y^2 g(y) = 1\cdot2 + 0\cdot3 + 1\cdot6 + 7\cdot5 = 10\cdot6$$

$$\sigma_x = \sqrt{E(X^2) - [E(X)]^2} = \sqrt{2\cdot2 - 1\cdot4^2} = \sqrt{0\cdot24} = 0\cdot49$$

$$\sigma_y = \sqrt{E(Y^2) - [E(Y)]^2} = \sqrt{10\cdot6 - 1} = 3\cdot1$$ — (1)

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-0.5}{0.49 \times 0.31} = \underline{\underline{-0.3292}}$$ — (1)

$$P(X+Y) > 0 = P(1,4) + P(1,5) + P(2,-1) + P(2,4) + P(2,5)$$

$$= 0 + 0.3 + 0.1 + 0.1 + 0 = 0.5$$ — (2)

2. $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ — ①

$(1-x^2) \neq 0$ at $x=0$

Let $y = \sum_{m=0}^{\infty} a_m x^m$ be a soln of ①.

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \quad \text{and} \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$ — (1)

Sub. in ①,

$$(1-x^2) \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + n(n+1) \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - 2 \sum_{m=1}^{\infty} m a_m x^m + n(n+1) \sum_{m=0}^{\infty} a_m x^m = 0$$ — (1)

Put $m-2=r$ in 1st sum and $m=r$ in others

$$\sum_{r=0}^{\infty} (r+2)(r+1) a_{r+2} x^r - \sum_{r=2}^{\infty} r(r-1) a_r x^r - 2 \sum_{r=1}^{\infty} r a_r x^r + n(n+1) \sum_{r=0}^{\infty} a_r x^r = 0$$

Equate coeff of x^0 on both sides, (put $r=0$)

$$a_2 = \frac{-n(n+1)}{2} a_0$$ — (1)

Equate coeff of x on b.s (put $r=1$)

$$a_3 = -\frac{(n-1)(n+2)}{6} a_1$$ — (1)

For $r \geq 2$,

$$a_{r+2} = \frac{-n(n+1)+r(r+1)}{(r+2)(r+1)} a_r$$

$$= -\frac{(n-r)(n+r+1)}{(r+1)(r+2)} a_r \quad \text{— (1)}$$

For $r=2$, $a_4 = \frac{-(n-2)(n+3)}{12} a_2 = \frac{n(n+1)(n-2)(n+3)}{4!} a_0$

" $r=3$, $a_5 = \frac{-(n-3)(n+4)}{20} a_3 = \frac{(n-1)(n+2)(n-3)(n+4)}{5!} a_1$ — (1)

We have, $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \cdots) + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots)$$

$$= a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 - \cdots\right]$$

$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!} - \cdots\right]$$

$$= a_0 y_1(x) + a_1 y_2(x) \quad \text{— (2)}$$

This is the required series soln of Legendre's d.e

3. $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$
— (1)

$$\Rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x) \Big\} \text{— (2)}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) \quad \text{— (2)}$$

$$\therefore x^3 + 2x^2 - x + 1 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{3}P_0(x) - P_1(x) + P_0(x) \text{— (}$$

$$= \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) + \frac{5}{3}P_0(x)$$

— (1)

4. $$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \quad \text{——} \ (1)$$

$$J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+\frac{3}{2})} \left(\frac{x}{2}\right)^{\frac{1}{2}+2r} \quad \text{——} \ (1)$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma(\frac{3}{2})} - \frac{1}{1!\,\Gamma(\frac{5}{2})} \left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(\frac{7}{2})} \left(\frac{x}{2}\right)^4 \cdots \right] \quad \text{——} (1)$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})} \frac{x^2}{4} + \frac{1}{\cancel{2}\cdot\frac{5}{\cancel{2}}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})} \frac{x^4}{16} - - - - \right] \text{—} (1)$$

2.5

(using $\Gamma(n+1) = n\Gamma(n)$)

$$= \sqrt{\frac{x}{2}} \cdot \frac{2}{\sqrt{\pi}} \left[1 - \frac{x^2}{6} + \frac{1}{120} x^4 \cdots \right] \quad \text{——} \ (2)$$

$$= \sqrt{\frac{x}{2}} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{1}{x} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x \quad \text{——} \ (1)$$

5 The state space S is $\{A, B, C\}$.

The t.p.m is
$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & \frac{2}{3} & 0 & \frac{1}{3} \\ C & \frac{2}{3} & \frac{1}{3} & 0 \end{array} \quad \text{——} \ (2)$$

To find a vector $V = (x, y, z)$, such that

$x + y + z = 1$ —① and $VP = V$ —— (1)

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} = (x, \ y \ z)$$

$\frac{2}{3}y + \frac{2}{3}z = x$, $x + \frac{1}{3}z = y$, $\frac{1}{3}y = z$.

$3x - 2y - 2z = 0$ $3x - 3y + z = 0$ —② $y = 3z$ —③

Solving ①, ② and ③.

$x = \frac{8}{20}, \ y = \frac{9}{20}, \ z = \frac{3}{20}$ ——— (3)

i.e. In the long run, he sells in City A with probability $\frac{8}{20}$, City B with probability $\frac{9}{20}$ and City C with probability $\frac{3}{20}$. ——— (1)

6. Prob. of the equipment supplied to the factory conformal to the specifications $p = 0.95$

$\Rightarrow q = 0.05$ ——— (1)

$H_0 : p_0 = 0.95$ i.e. claim is correct

$H_1 : p < 0.95$ (one tailed test) ——— (1)

Expected no. $= np = 200 \times 95\% = 190$

Actual no $= 182$ ——— (1)

$\sigma = \sqrt{npq} = 3.082$

Consider $Z = \frac{x - np}{\sqrt{npq}} = 2.6$ ——— (2)

$\therefore Z = 2.6 > 1.645 = Z_{0.05}$ $> 2.33 = Z_{0.01}$ } one tailed test ——— (1)

$\therefore$ The null hypothesis is rejected at 5% as well as 1% level of significance. ——— (1)

7. H_0 : Mean height $= 66$

H_1 : Mean height $\neq 66$ ——— (1)

$n = 10, \quad \mu = 66$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{(63+63+\dots+71)}{10} = 67.8 \longleftarrow (1)$$

$$s^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \frac{1}{9}\left[(63-67.8)^2 + \dots + (71-67.8)^2\right] = 9.067$$

$\Rightarrow S = 3.011$ —— (2)

Now $t = \frac{\bar{x}-\mu}{s}\sqrt{n} = 1.89 < 2.262$ —— (2)

∴ Hypothesis is accepted at 5% level of significance.

—— (1)

8.

x	0	1	2	3	4	5
f	173	168	37	18	3	1

The prob. func. for Poisson dist is given by

$$P(x) = \frac{e^{-m} . m^x}{x!}$$

where m = mean $= \frac{\Sigma fx}{\Sigma f} = 0.7825$ —— (1)

The expected frequency func. is given by $f(x) = 400.P(x)$

Put $x = 0, 1, 2, 3, 4, 5$

$f(x=0) = 183$, $f(x=4) = 3$

$f(x=1) = 143$, $f(x=5) = 0$

$f(x=2) = 56$

$f(x=3) = 15$

—— (3)

Oi	173	168	37	18	3	1
ei	183	143	56	15	3	0

Now we'll club the last two frequencies as $e_6 = 0$

∴ New table is

Oi	173	168	37	18	3
ei	183	143	56	15	3+0 = 3

—— (1)

$$\chi^2 = \sum \frac{(oi-ei)^2}{ei} = 12.297 > 9.49$$ —— (1)

∴ The hypothesis that the fitness is good is rejected.

—— (1)

9. Consider a diff. eqn. of the form

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0 \quad \text{—— (1)}$$

we know $J_n(\lambda x)$ is a solution of (1)

Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ are the solutions of the following D.E.

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \quad \text{—— (2)}$$

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \quad \text{—— (3)}$$

—— (2)

$$(2) \times \frac{v}{x} - (3) \times \frac{u}{x}$$

$$x[vu'' - uv''] + [u'v - uv'] + [\alpha^2 - \beta^2]xuv = 0$$

$$\frac{d}{dx}\{x(u'v - uv')\} = (\beta^2 - \alpha^2)xuv$$ —— (2)

Integrating on both sides b/w 0 & 1

$$x(u'v - uv')\Big|_0^1 = \int_0^1 (\beta^2 - \alpha^2)xuv\,dx$$

—— (1)

we have $u = J_n(\alpha x)$, $v = J_n(\beta x) \Rightarrow u' = \alpha J_n'(\alpha x)$ & $v' = \beta J_n'(\beta x$

$$J_n(\beta x)\,\alpha J_n'(\alpha x) - J_n(\alpha x)\,\beta J_n'(\beta x)\Big|_0^1 = (\beta^2-\alpha^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$$

$$\therefore \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{(\beta^2-\alpha^2)}\left[\alpha J_n(\beta) J_n'(\alpha) - J_n(\alpha).J_n'(\beta)\right] \quad \text{—(1)}$$

—(5)

$\because$ α and β are distinct roots of $J_n(x) = 0$

$\therefore$ $J_n(\alpha) = 0 = J_n(\beta)$

$\therefore$ Eqⁿ (5) reduces to

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$$ —— (1)

10.

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \Rightarrow P^2 = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

$\because$ in P^2 all the entries are +ve. $\therefore$ t.p.m. P is regular

Hence the Markov chain is irreducible. —— (3)

For fixed prob. vector we've $vP = v$ where $v = (a, b, c)$

& $(a+b+c = 1)$ —— (1)

$$[a \;\; b \;\; c]\begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [a \;\; b \;\; c]$$

$$\Rightarrow \quad \frac{b+c}{2} = a \;; \quad \frac{2}{3}a + \frac{c}{2} = b \;; \quad \frac{a}{3} + \frac{b}{2} = c$$

Solving $a = \frac{1}{3}$, $b = \frac{10}{27}$, $c = \frac{8}{27}$ $\quad \therefore v = \left(\frac{1}{3}, \frac{10}{27}, \frac{8}{27}\right)$ —— (3)