1 List and define the rules for the construction of Root Locus.

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Table 6.1 Rules for construction of root loci of $1 + G(s)H(s) = 0$, where the open-loop transfer function $G(s)H(s)$ is known in pole-zero form with n = number of open-loop poles and m = number of open-loop zeros

S. No.	Rule
1.	The root locus is symmetrical about the real axis.
2.	Each branch of the root locus originates from an open-loop pole where $K = 0$, and terminates on an open-loop zero or on infinity where $K = \infty$. The number of branches of the root locus terminating on infinity is equal to $n - m$, i.e. the number of open-loop poles minus the number open-loop zeros.
3.	Segments of the real axis having an odd number of real axis open-loop poles plus zeros to their right are parts of the root locus.
4.	The $(n - m)$ root locus branches that tend to infinity do so along straight line asymptotes making angles with the real axis given by $$\theta_q = \frac{(2q+1)\pi}{n-m};\ q = 0, 1, 2, \dots, (n-m-1)$$
5.	The point of intersection of the asymptotes with the real axis called centroid is at $s = -\sigma$ where $$-\sigma = \frac{\text{sum of real parts of poles} - \text{sum of real parts of zeros}}{\text{number of poles} - \text{number of zeros}}$$
6.	The break points (breakaway and break-in points) of the root locus are determined from the roots of the equation $\frac{dK}{ds} = 0$. r branches of the root locus which meet at a point break away at an angle of $$\pm\frac{180^\circ}{r}$$
7.	The angle of departure from an open-loop pole is given by $$\theta_d = \pm(2q+1)\pi + \phi;\ q = 0, 1, 2, \dots$$ where ϕ is the net angle contribution at the pole, of all other open-loop poles and zeros. Similarly, the angle of arrival at an open-loop zero is given by $$\theta_a = \pm(2q+1)\pi - \phi;\ q = 0, 1, 2, \dots$$ where ϕ is the net angle contribution at the zero, of all other open-loop poles and zeros.
8.	The point of intersection of the root locus branches with the imaginary axis and the critical value of K can be determined by use of the Routh criterion.
9.	The open-loop gain K in pole-zero form at any point s_0 on the root locus is given by $$K = \frac{\text{product of phasor lengths from } s_0 \text{ to open-loop poles}}{\text{product of phasor lengths from } s_0 \text{ to open-loop zeros}}$$

6.4.1 Determination of Roots for a Specified Open-Loop Gain

Along a particular root locus b...

2 Draw the complete root locus for

$$G(s)H(s) = \frac{K}{s(s+2)(s+4)}$$

Determine the value of K for a damping ratio of ξ=0.5. With this value of K, find the closed loop transfer function.

Example 6.5 Draw the complete root locus for

$$G(s)H(s) = \frac{K}{s(s+2)(s+4)}$$

From the root locus plot, find the range of values of K for which the system will have damped oscillatory response. Also, determine the value of K for a damping ratio of $\xi = 0.5$. With this value of K, find the closed-loop transfer function.

Solution: For the given open-loop transfer function $G(s)H(s)$:

The open-loop poles are at $s = 0$, $s = -2$ and $s = -4$. Therefore, $n = 3$.

There are no finite open-loop zeros. Therefore, $m = 0$.

So the number of branches of root locus $n = 3$ and the number of asymptotes $n - m = 3 - 0 = 3$.

The complete root locus is drawn as shown in Figure 6.12, as per the rules given below.

1. All the open-loop poles and zeros are on the real axis only. So the root locus will be symmetrical about the real axis.
2. The three branches of the root locus start at the open-loop poles $s = 0$, $s = -2$ and $s = -4$, where $K = 0$ and terminate at the zeros at infinity, where $K = \infty$.
3. There are three asymptotes, and the angles of the asymptotes are given by

$$\theta_q = \frac{(2q+1)\pi}{n-m}, \; q = 0, 1, 2$$

i.e.

$$\theta_0 = \frac{\pi}{3}, \quad \theta_1 = \frac{3\pi}{3} = \pi, \quad \theta_2 = \frac{5\pi}{3}$$

4. The point of intersection of the asymptotes on the real axis (centroid) is given by

$$-\sigma = \frac{\text{sum of real parts of poles} - \text{sum of real parts of zeros}}{\text{number of poles} - \text{number of zeros}} = \frac{(0-2-4)-(0)}{3-0} = -2$$

5. The root locus exists on the real axis from $s = 0$ to $s = -2$ and to the left of $s = -4$.
6. The breakaway points are given by the solution of the equation $\frac{dK}{ds} = 0$.

$$|G(s)H(s)| = \left|\frac{K}{s(s+2)(s+4)}\right| = 1$$

∴

$$K = s(s+2)(s+4)$$

i.e.

$$\frac{d}{ds}[s(s+2)(s+4)] = 0$$

$$\frac{d}{ds}(s^3 + 6s^2 + 8s) = 3s^2 + 12s + 8 = 0$$

i.e. $s = \frac{-12 \pm \sqrt{144-96}}{6} = -2 \pm \sqrt{\frac{48}{36}} = -2 \pm 1.15$

Therefore, the break points are $s = -3.15$ and $s = -0.85$. Out of these two, $s = -0.85$ is the actual break point because the root locus exists there. $s = -3.15$ is not an actual break point because the root locus does not exist there, and so it can be ignored. The break angles at $s = -0.85$ are

$$\pm\frac{\pi}{r} = \pm\frac{180^\circ}{2} = \pm 90^\circ$$

7. There is no need to compute the angles of departure and arrival as there are no complex poles and zeros.
8. The point of intersection of the root locus with the imaginary axis, and the marginal value of K can be determined by applying the Routh criterion. The characteristic equation is

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+2)(s+4)}$$

i.e. $$s^3 + 6s^2 + 8s + K = 0$$

The Routh table is as follows:

s^3	1	8
s^2	6	K
s^1	$\frac{48-K}{6}$	0
s^0	K	

For stability, all the elements in the first column of the Routh array must be positive. Therefore,

$$K > 0$$

and $$48 - K > 0$$

i.e. $$K < 48$$

Therefore, the range of values of K for stability is

$$0 < K < 48$$

The marginal value of K for stability is $K_m = 48$. The frequency of sustained oscillations is given by the solution of the auxiliary equation.

$$6s^2 + K = 0$$

i.e. $$6s^2 + K_m = 0$$

i.e. $$6s^2 + 48 = 0$$

∴ $$s^2 = -8$$

Obtain the transfer function of the control system whose block diagram is shown below by block diagram reduction technique.

Solution: *(a) Block diagram reduction technique*

1. In the block diagram shown in Figure 3.14, blocks G_2 and ... be combined into a single block with a gain of $(G_2 - G$ ...

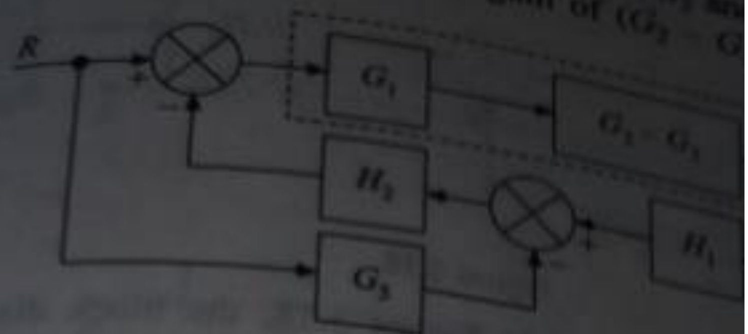


Figure 3.15

2. In Figure 3.15, blocks G_1 and $(G_2 - G_3)$ are in cascade. ... single block with a gain of $G_1(G_2 - G_3)$ as shown in F...

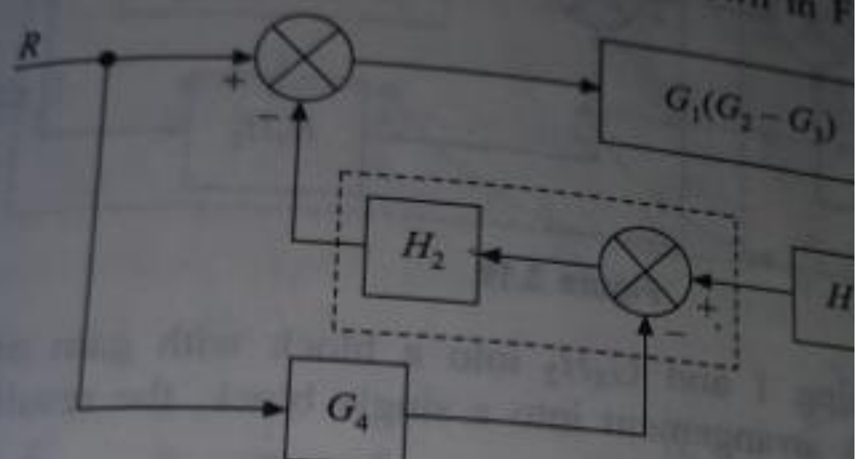
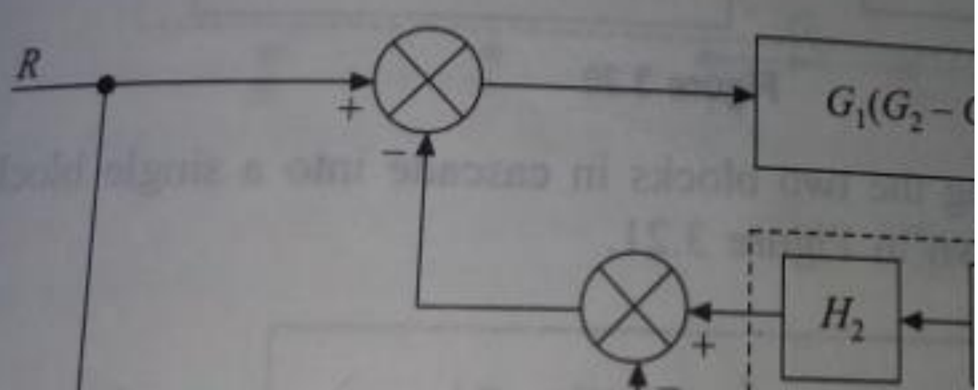


Figure 3.16

3. In Figure 3.16, move the summing point after the block ... is shown in Figure 3.17.



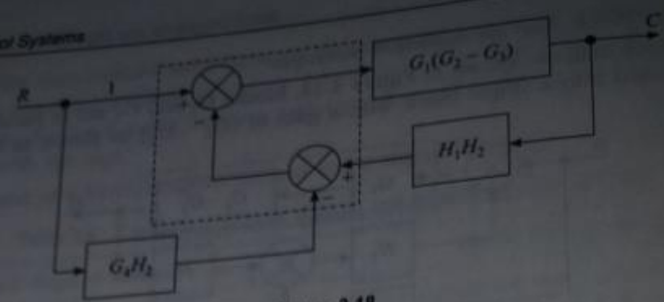


Figure 3.18

5. Interchanging the summing points in Figure 3.18, the block diagram is as shown in Figure 3.19.

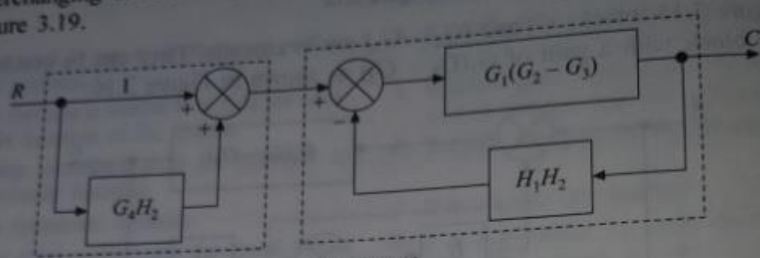


Figure 3.19

6. In Figure 3.19, combining 1 and G_4H_2 into a block with gain of $(1 + G_4H_2)$ and simplifying the feedback arrangement into a single block, the resultant block diagram is as shown in Figure 3.20.

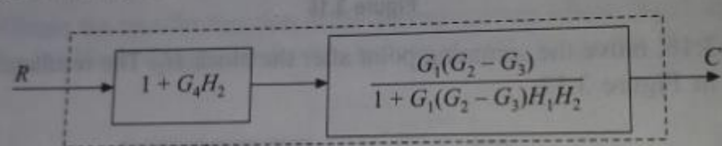


Figure 3.20

7. In Figure 3.20, combining the two blocks in cascade into a single block, the resultant block diagram is as shown in Figure 3.21.

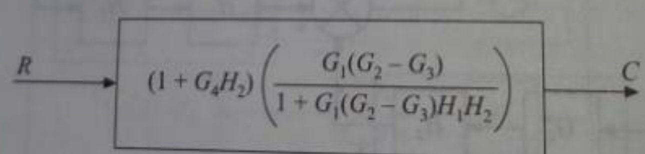


Figure 3.21

From Figure 3.21, the overall transfer function is

$$\frac{C}{R} = \frac{G_1G_2 - G_1G_3 + G_1G_2G_4H_2 - G_1G_3G_4H_2}{1 + G_1G_2H_1H_2 - G_1G_3H_1H_2}$$

4 Find the transfer function C(s)/R(s) for the system shown below using signal flow graph method.

...root locus for

$$G(s)H(s) = \frac{K}{s(s+2)(s+4)}$$

Determine the value of K for a damping ratio of ζ=0.5. With ... the closed loop transfer function. [10] 3 L3, L4

3 Obtain the transfer ... below by ...

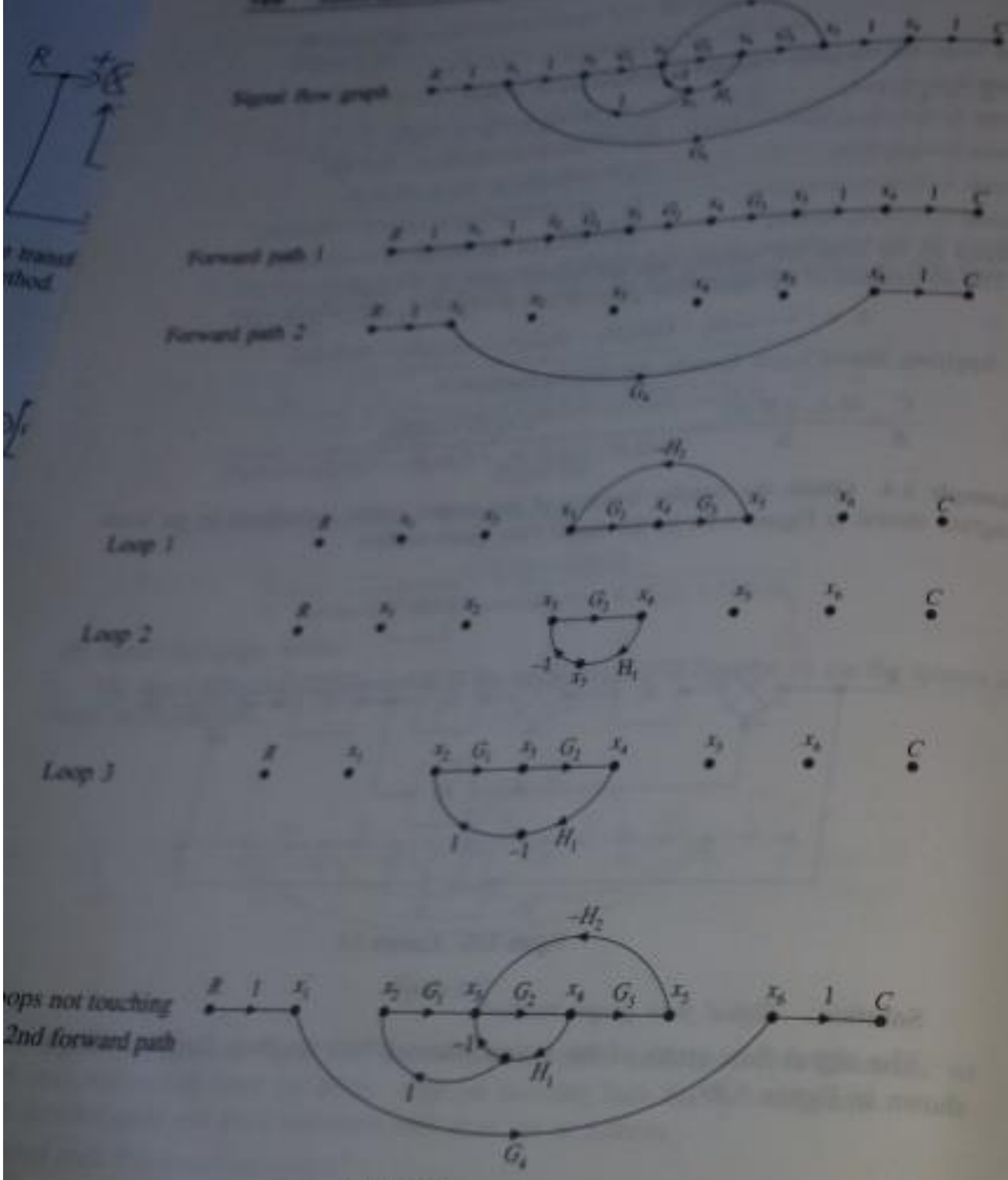


Figure 3.39

5 Given

$$G(s)H(s) = \frac{\mathrm{K}}{s^2(\mathrm{s}+2)}$$

Show that the system is unstable for all values of K, by sketching Root Locus Plot.

Centroid will be

$$-\sigma = \frac{(-2)-(-a)}{3-1} = \frac{a-2}{2}$$

There are two asymptotes, and the asymptotic angles are

$$\theta_q = \frac{(2q+1)\pi}{n-m}, \; q = 0, 1$$

$$\theta_0 = \frac{\pi}{2} \text{ and } \theta_1 = \frac{3\pi}{2}$$

∴ The root locus will now be as shown in Figure 6.14(a). Since the root locus lies in the left-half of the s-plane for all positive values of K, we can conclude that the addition of a zero at $s = -a$, $(0 \le a < 2)$ stabilizes the system.

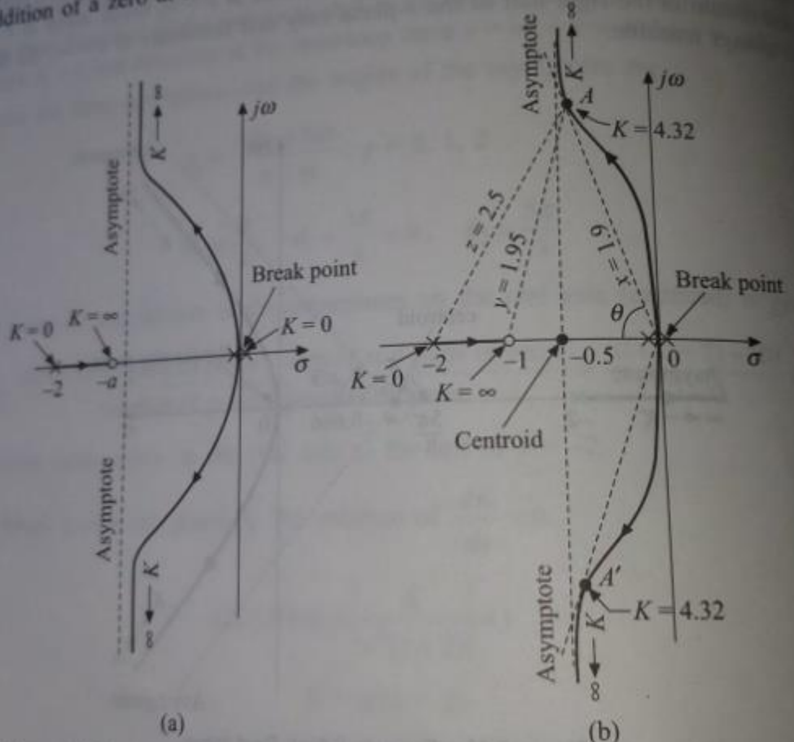


Figure 6.14 (a) Root locus (Example 6.6(b)) and, (b) root locus (Example 6.6(c)).

(c) If $a = 1$, the open-loop transfer function is

$$G(s)H(s) = \frac{K(s+1)}{s^2(s+2)}$$

6 Find the transfer function of the system whose signal flow graph is shown below.

$\ldots + G_2G_3H_2G_4H_3)$

All loops are touching the first and third forward paths and L_3 is not touching the second forward path

$\therefore \quad \Delta_1 = 1,\ \Delta_2 = 1 - L_3 = 1 - (-G_4H_4) = 1 + G_4H_4,\ \Delta_3 = 1$

Applying Mason's gain formula, the transfer function is

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}$$

i.e.

$$\frac{C}{R} = \frac{G_1G_2G_3G_4G_5G_6 + G_1G_2G_7G_6(1 + G_4H_4) + G_1G_2G_3G_4G_8}{\Delta}$$

Example 3.18 Find the transfer function of the system whose signal flow graph is shown in Figure 3.79.

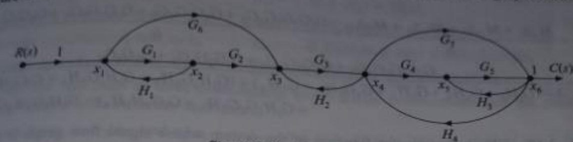


Figure 3.79 Example 3.18.

Solution: The given signal flow graph shown in Figure 3.79 has four forward paths and five loops. All the loops are touching all the forward paths. There are five pairs of two nontouching loops and one combination of three nontouching loops.

The forward paths and the gains associated with them are as follows:

Forward path $R\text{-}x_1\text{-}x_2\text{-}x_3\text{-}x_4\text{-}x_5\text{-}x_6\text{-}C$ $\quad M_1 = (1)(G_1)(G_2)(G_3)(G_4)(G_5)(1) = G_1G_2G_3G_4G_5$

Forward path $R\text{-}x_1\text{-}x_3\text{-}x_4\text{-}x_5\text{-}x_6\text{-}C$ $\quad M_2 = (1)(G_6)(G_3)(G_4)(G_5)(1) = G_6G_3G_4G_5$

Forward path $R\text{-}x_1\text{-}x_2\text{-}x_3\text{-}x_4\text{-}x_6\text{-}C$ $\quad M_3 = (1)(G_1)(G_2)(G_3)(G_7)(1) = G_1G_2G_3G_7$

Forward path $R\text{-}x_1\text{-}x_3\text{-}x_4\text{-}x_6\text{-}C$ $\quad M_4 = (1)(G_6)(G_3)(G_7)(1) = G_6G_3G_7$

The loops and the gains associated with them are as follows:

Loop $x_1\text{-}x_2\text{-}x_1$ $\quad L_1 = (G_1)(H_1) = G_1H_1$

Loop $x_3\text{-}x_4\text{-}x_3$ $\quad L_2 = (G_3)(H_2) = G_3H_2$

Loop x_3-x_5-x_3 $\quad L_3 = (G_5)(H_3) = G_5H_3$

Loop x_4-x_5-x_6-x_4 $\quad L_4 = (G_4)(G_5)(H_4) = G_4G_5H_4$

Loop x_4-x_6-x_4 $\quad L_5 = (G_7)(H_4) = G_7H_4$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops x_1-x_2-x_1 and x_3-x_4-x_3 $\quad L_{12} = (G_1H_1)(G_3H_2) = G_1H_1G_3H_2$

Loops x_1-x_2-x_1 and x_5-x_6-x_5 $\quad L_{13} = (G_1H_1)(G_5H_3) = G_1H_1G_5H_3$

Loops x_3-x_4-x_3 and x_5-x_6-x_5 $\quad L_{23} = (G_3H_2)(G_5H_3) = G_3G_5H_2H_3$

Loops x_1-x_2-x_1 and x_4-x_5-x_6-x_4 $\quad L_{14} = (G_1H_1)(G_4G_5H_4) = G_1H_1G_4G_5H_4$

Loops x_1-x_2-x_1 and x_4-x_6-x_4 $\quad L_{15} = (G_1H_1)(G_7H_4) = G_1G_7H_1H_4$

Combinations of three nontouching loops and the gains associated with them are as follows:

Loops L_1, L_2 and L_3

$$L_{123} = (G_1H_1)(G_3H_2)(G_5H_3) = G_1H_1G_3H_2G_5H_3$$

Since all the loops are touching all the forward paths, $\Delta_1 = 1$, $\Delta_2 = 1$, $\Delta_3 = 1$, and $\Delta_4 = 1$.
The determinant of the signal flow graph is given by

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_{12} + L_{13} + L_{14} + L_{15} + L_{23}) - (L_{123})$$

Therefore, the closed-loop transfer function $\dfrac{C}{R}$ is

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3 + M_4\Delta_4}{\Delta} = \frac{G_1G_2G_3G_4G_5 + G_6G_3G_4G_5 + G_1G_2G_3G_7 + G_6G_3G_7}{\Delta}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_6G_3G_4G_5 + G_1G_2G_3G_7 + G_6G_3G_7}{1 - (G_1H_1 + G_3H_2 + G_5H_3 + G_4G_5H_4 + G_7H_4) + (G_1H_1G_3H_2 + G_1H_1G_5H_3 + G_3G_5H_2H_3 + G_1H_1G_4G_5H_4 + G_1G_7H_1H_4) - G_1H_1G_3H_2G_5H_3}$$

Example 3.19 Obtain the transfer function of the system whose signal flow graph is shown in Figure 3.80. Also determine $\dfrac{x_2}{x_1}$, $\dfrac{x_4}{x_1}$, $\dfrac{x_7}{x_2}$, $\dfrac{x_4}{x_2}$ and $\dfrac{x_7}{x_4}$.