PB Verified

CMR INSTITUTE OF TECHNOLOGY

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CMR

Internal Assessment Test I

Sub:	**ENGINEERING MATHEMATICS IV**							Code:	15MAT41
Date:	27 / 3 / 2017	Duration:	90 mins	Max Marks:	50	Sem:	**IV**	Branch:	**ECE:A,B,C**

Answer Q1 OR Q2 and ANY SIX questions from Q3-Q9

	Marks	OBE CO	OBE RBT
1. Derive Rodrigue's formula for Legendre polynomials.	[08]	CO2	L3
OR			
2. Discuss the transformation $w = z^2$.	[08]	CO3	L4
3. Show that $u(r,\theta) = \left(r - \frac{1}{r}\right)\cos\theta$ is harmonic, find its harmonic conjugate and the corresponding analytic function.	[07]	CO3	L3
4. Given $u - v = (x-y)(x^2+4xy+y^2)$, find the analytic function $f(z) = u + iv$	[07]	CO3	L3
5. If $f(z)$ is an analytic function, show that $\left(\frac{\partial}{\partial x}\lvert f(z)\rvert\right)^2 + \left(\frac{\partial}{\partial y}\lvert f(z)\rvert\right)^2 = \lvert f'(z)\rvert^2$.	[07]	CO3	L3
6. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$ where C is the circle $\lvert z\rvert = 3$, using Cauchy's residue theorem.	[07]	CO3	L3
7. If $F(a) = \int_C \frac{4z^2+z+5}{z-a} dz$, where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, find the values of $F(3.5), F(i), F'(-1)$ and $F''(-i)$.	[07]	CO3	L3
8. Find the bilinear transformation that maps the points $z = \infty, i, 0$ into the points $w = -1, -i, 1$ respectively. What are the invariant points of this transformation?	[07]	CO3	L3
9. Express $x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre polynomials.	[07]	CO2	L2

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Course Outcomes		PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1:	Use appropriate single-step and multi-step numerical methods to solve first and second order ordinary differential equations.	3	3	0	0	0	0	0	0	1	0	0	0
CO2:	Apply Frobenius method to find the power series solution of second order differential equations such as Bessel's and Legendre's differential equations.	3	3	0	0	0	0	0	0	1	0	0	0
CO3:	Apply the idea of analyticity and the calculus of residues to evaluate real and complex integrals and analyze conformal transformations.	2	3	0	0	0	0	0	0	1	0	0	0
CO4:	Describe random variables and probability distributions using rigorous statistical methods and translate real-world problems into probability models.	3	3	1	0	0	0	0	0	1	0	0	0
CO5:	Explain and successfully apply parametric testing techniques including single and multi-sample tests for mean and proportion.	3	3	0	0	0	0	0	0	1	0	0	0
CO6:	Estimate the nature and strength of relationship between two variables of interest using joint probability distribution and describe a discrete time Markov chain in terms of a transition matrix.	3	3	0	0	0	0	0	0	1	0	0	0

Cognitive level	KEYWORDS
L1	List, define, tell, describe, identify, show, label, collect, examine, tabulate, quote, name, who, when, where, etc.
L2	summarize, describe, interpret, contrast, predict, associate, distinguish, estimate, differentiate, discuss, extend
L3	Apply, demonstrate, calculate, complete, illustrate, show, solve, examine, modify, relate, change, classify, experiment, discover.
L4	Analyze, separate, order, explain, connect, classify, arrange, divide, compare, select, explain, infer.
L5	Assess, decide, rank, grade, test, measure, recommend, convince, select, judge, explain, discriminate, support, conclude, compare, summarize.

PO1 - *Engineering knowledge*; PO2 - *Problem analysis*; PO3 - *Design/development of solutions*; PO4 - *Conduct investigations of complex problems*; PO5 - *Modern tool usage*; PO6 - *The Engineer and society*; PO7- *Environment and sustainability*; PO8 – *Ethics*; PO9 - *Individual and team work*; PO10 - *Communication*; PO11 - *Project management and finance*; PO12 - *Life-long learning*

IAT – I – Solution

1. $$P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n}(x^2-1)^n \qquad \text{—— (1)}$$

Proof: Let $v = (x^2-1)^n$

Diff. w.r.t x, $v_1 = n(x^2-1)^{n-1} \cdot 2x$

$$\Rightarrow (x^2-1)v_1 = 2nxv$$

Diff. again, $(x^2-1)v_2 + 2xv_1 = 2nxv_1 + 2nv$

$$\Rightarrow (x^2-1)v_2 + 2(1-n)xv_1 - 2nv = 0 \qquad \text{—— (2)}$$

Diff. n times using Leibnitz rule,

$$(x^2-1)v_{n+2} + 2nxv_{n+1} + \frac{n(n-1)}{2}\cdot 2v_n + 2(1-n)v_{n+1}x + 2(1-n)nv_n - 2nv_n = 0$$

ie $(x^2-1)v_{n+2} + (2nx + 2x - 2nx)v_{n+1} + (n^2 - n + 2n - 2n^2 - 2n)v_n = 0$

$$(x^2-1)v_{n+2} + 2xv_{n+1} - (n^2+n)v_n = 0$$

ie $(1-x^2)v_{n+2} - 2xv_{n+1} + n(n+1)v_n = 0$

$\Rightarrow v_n$ is a solution of the Legendre D.E —— (2)

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{—— (i)}$$

v is a polynomial of degree $2n \Rightarrow v_n$ is a polynomial of degree n —— (ii)

—— (1)

(i) and (ii) $\Longrightarrow P_n(x) = k\,v_n$

$$= k\frac{d^n}{dx^n}(x^2-1)^n \quad \text{—— (I)}$$

$$= k\frac{d^n}{dx^n}\left[(x+1)^n(x-1)^n\right]$$

$$= k\left[(x+1)^n n! + \text{terms containing pwrs of } (x-1)\right]$$

$P_n(1) = k \cdot 2^n n!$

ie $1 = k \cdot 2^n n! \Longrightarrow k = \dfrac{1}{2^n n!}$

Sub in (I), $$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \qquad \text{—— (2)}$$

2. $w = z^2$

Conformality: $f(z) = z^2$ is analytic

$f'(z) = 2z$. $f'(z) = 0$ at $z = 0$ (critical pt.)

$\therefore$ the mapping is conformal except at $z = 0$. —— (1)

Using polar forms

$z = re^{i\theta}$ and $w = Re^{i\varphi}$, we have

$$Re^{i\varphi} = r^2 e^{2i\theta}$$

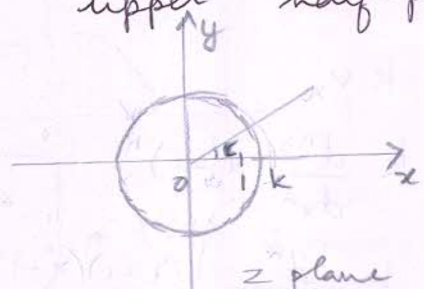
$\Rightarrow$ $R = r^2$ and $\varphi = 2\theta$

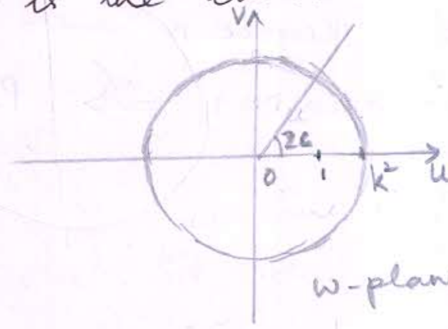
Let $r = k$, a const. Then $R = k^2$.

$\therefore$ the image of a circle of radius k is a circle of radius k^2.

Let $\theta = c$, a const. Then $\varphi = 2c$.

$\Rightarrow$ the image of a radial line that makes an angle c with the horizontal axis is a ray that makes an angle $2c$ with the u axis. As θ varies from 0 to π, φ varies from 0 to 2π. $\therefore$ The image of the upper half-plane is the entire w-plane.





—— (3)

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Using cartesian forms

$z = x + iy$, $w = u + iv$

$u + iv = (x+iy)^2 = x^2 - y^2 + i\cdot 2xy$

$u = x^2 - y^2$, $v = 2xy$.

Image of vertical lines $x = c$

$u = c^2 - y^2, \quad v = 2cy$

$y = \frac{v}{2c}$

$u = c^2 - \frac{v^2}{4c^2}$

$v^2 = -4c^2(u - c^2)$ which is a parabola with vertex $(c^2, 0)$, opening left.

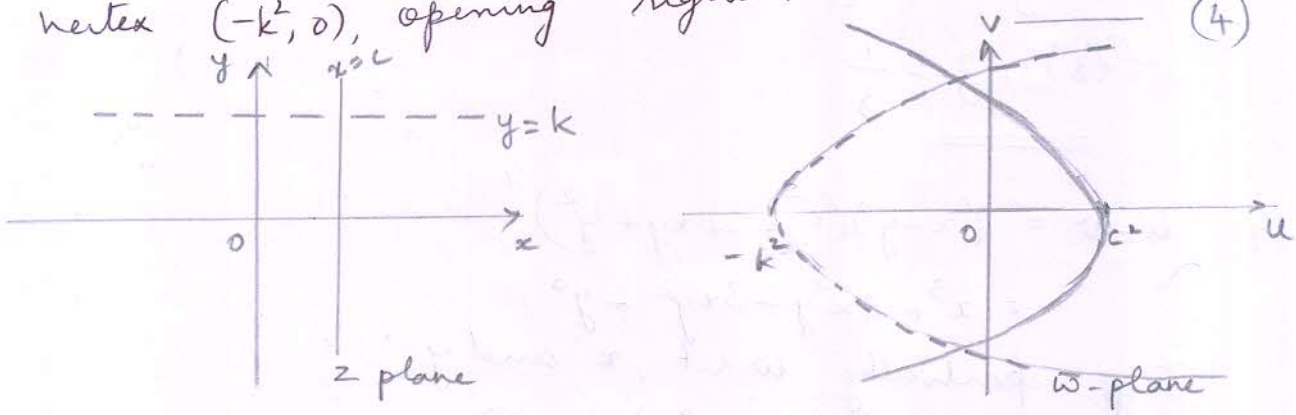
Image of horizontal lines $y = k$

$u = x^2 - k^2, \quad v = 2kx.$

$u = \frac{v^2}{4k^2} - k^2$

$v^2 = 4k^2(u + k^2)$ which is a parabola with vertex $(-k^2, 0)$, opening right.

(4)



3. $u(r,\theta) = \left(r - \frac{1}{r}\right)\cos\theta.$

$u_r = \left(1 + \frac{1}{r^2}\right)\cos\theta, \quad u_\theta = -\left(r - \frac{1}{r}\right)\sin\theta$

$u_{rr} = \frac{-2}{r^3}\cos\theta \qquad u_{\theta\theta} = -\left(r - \frac{1}{r}\right)\cos\theta$ ——— (2)

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -\frac{2}{r^3}\cos\theta + \left(\frac{1}{r} + \frac{1}{r^3}\right)\cos\theta - \frac{1}{r^2}\left(r - \frac{1}{r}\right)\cos\theta$

$= \left(-\frac{2}{r^3} + \frac{1}{r} + \frac{1}{r^3} - \frac{1}{r} + \frac{1}{r^3}\right)\cos\theta = \underline{\underline{0}}$

$\therefore$ u is harmonic ——— (4)

We have $u_r = \frac{1}{r} v_\theta$ and $v_r = \frac{-1}{r} u_\theta$.

$\therefore$ $v_\theta = \left(r + \frac{1}{r}\right) \cos\theta$ and $v_r = \left(1 - \frac{1}{r^2}\right) \sin\theta$

Integrating w.r.t θ and r resp.,

$$V = \left(r + \frac{1}{r}\right) \sin\theta + g(r)$$

and $V = \left(r + \frac{1}{r}\right) \sin\theta + g(\theta)$

$$\therefore \quad V = \left(r + \frac{1}{r}\right) \sin\theta \qquad \text{—— (2)}$$

$$f(z) = u + iv = \left(r - \frac{1}{r}\right) \cos\theta + i \left(r + \frac{1}{r}\right) \sin\theta.$$

Putting $r = z$ and $\theta = 0$,

$$f(z) = z - \frac{1}{z} \qquad \text{—— (2)}$$

4. $u - v = (x-y)(x^2 + 4xy + y^2)$

$\quad = x^3 + 3x^2y - 3xy^2 - y^3.$

Diff partially w.r.t x and y,

$u_x - v_x = 3x^2 + 6xy - 3y^2$ —— ①

and $u_y - v_y = 3x^2 - 6xy - 3y^2$ —— ② —— (2)

Using C-R eqns in ②,

$-v_x - u_x = 3x^2 - 6xy - 3y^2$ —— ③ —— (1)

① + ③ $\rightarrow$ $-2v_x = 6(x^2 - y^2)$

$\quad v_x = 3(y^2 - x^2)$

Sub in ①, $u_x = 6xy$ —— (2)

Thus $f'(z) = u_x + iv_x = 6xy + i \cdot 3(y^2 - x^2)$

Replacing x with z and y with 0,

$f'(z) = i(-3z^2) \Rightarrow f(z) = -iz^3 + c$ —— (2)

5. To s.t $\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = |f'(z)|^2$

$|f(z)| = \sqrt{u^2+v^2}$ ~~where~~ $f(z) = u + iv$

$$\frac{\partial}{\partial x}|f(z)| = \frac{1}{2\sqrt{u^2+v^2}} \cdot (2uu_x + 2vv_x) = \frac{uu_x + vv_x}{\sqrt{u^2+v^2}}$$

$$\frac{\partial}{\partial y}|f(z)| = \frac{1}{2\sqrt{u^2+v^2}} \cdot (2uu_y + 2vv_y) = \frac{uu_y + vv_y}{\sqrt{u^2+v^2}} \quad \text{— (2)}$$

$$\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = \frac{u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y}{u^2+v^2}$$

$$= \frac{u^2(u_x^2+u_y^2) + v^2(v_x^2+v_y^2) + 2uv(u_xv_x + u_yv_y)}{u^2+v^2}$$

$$= \frac{u^2(u_x^2+(v_x)^2) + v^2(v_x^2+u_x^2) + 2uv(u_xv_x - v_xu_x)}{u^2+v^2}$$

$(f(z)$ is analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x)$

$$= \frac{\cancel{(u^2+v^2)}(u_x^2+v_x^2)}{\cancel{u^2+v^2}} \quad \text{— (5)}$$

$$= u_x^2 + v_x^2 = \underline{\underline{|f'(z)|^2}}$$

$\left(f'(z) = u_x + iv_x \Rightarrow |f'(z)|^2 = u_x^2 + v_x^2\right)$

6. $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}\,dz \qquad C: |z| = 3$

$z = 1$ is a pole of order 2 and $z = 2$ is a simple pole.

Both lie within C. — (1)

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\frac{\cancel{(z-1)^2}[\sin \pi z^2 + \cos \pi z^2]}{\cancel{[z-1]^2}[z-2]}$$

$$= \lim_{z\to 1}\frac{(z-2)[\cos \pi z^2 - \sin \pi z^2](2\pi z) - [\sin \pi z^2 + \cos \pi z^2]}{(z-2)^2}$$

$$= \frac{(-1)(-1)\cdot 2\pi + 1}{(-1)^2} = 2\pi + 1 \quad \text{— (3)}$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2}\frac{\cancel{(z-2)}[\sin \pi z^2 + \cos \pi z^2]}{(z-1)^2\cancel{(z-2)}} = 1 \quad \text{— (2)}$$

$$\int f(z)\,dz = 2\pi i[2\pi + 1 + 1] = \underline{4\pi i(\pi + 1)} \quad \text{— (1)}$$

7. $F(a) = \int_C \frac{4z^2+z+5}{z-a}\,dz \qquad C: \frac{x^2}{4}+\frac{y^2}{9}=1$

$F(3.5) = 0$ since $z = 3.5$ lies outside C. by Cauchy's theorem — (1)

If a lies within C,

$F(a) = \int_C \frac{4z^2+z+5}{z-a}\,dz = 2\pi i\, f(a)$ where $f(z) = 4z^2+z+5$

by Cauchy's integral formula — (1)

ie $F(a) = 2\pi i\,(4a^2+a+5)$

$\therefore F'(a) = 2\pi i\,(8a+1)$ — (2)

& $F''(a) = 2\pi i \cdot 8 = 16\pi i$

$i, -1$ and $-i$ lie within C.

$\therefore F(i) = 2\pi i\,(4i^2+i+5) = 2\pi i\,(-4+i+5) = 2\pi i\,(1+i)$

$= 2\pi\,(-1+i)$ — (1)

$F'(-1) = 2\pi i\,(-8+1) = -14\pi i$ — (1)

$F''(-i) = \underline{16\pi i}$ — (1)

8. $z = \infty, i, 0$ to $w = -1, -i, 1$

$w = \frac{az+b}{cz+d}$

$-1 = \frac{\cancel{z}(a+b/z)}{\cancel{z}(c+d/z)} = \frac{a}{c} \Rightarrow a+c=0$ — (1)

$-i = \frac{ia+b}{ic+d} \Rightarrow ia+b-c+id=0$ — (2)

$1 = \frac{b}{d} \Rightarrow b=d.$ — (3) — (3)

(1) $\rightarrow a+0b+c=0$

(2) $\rightarrow ia+(1+i)b-c=0$

$\frac{a}{\begin{vmatrix} 0 & 1 \\ 1+i & -1 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & 1 \\ i & -1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 0 \\ i & 1+i \end{vmatrix}}$

$\frac{a}{-(1+i)} = \frac{-b}{-1-i} = \frac{c}{1+i}$

$a = -(1+i) \quad b = (1+i) \quad c = 1+i \quad d = (1+i)$

$$w = \frac{-(1+i)z + (1+i)}{(1+i)z + (1+i)} = \frac{\cancel{(1+i)}\,(-z+1)}{\cancel{(1+i)}\,(z+1)}$$

$$w = \frac{1-z}{1+z} \quad \text{—— (3)}$$

Invariant points :-

$$z = \frac{1-z}{1+z}$$

$$z + z^2 = 1 - z$$

$$z^2 + 2z - 1 = 0$$

$$z = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= -1 \pm \sqrt{2} \quad \text{—— (1)}$$

9. $x^4 + 3x^3 - x^2 + 5x - 2$

$1 = P_0(x)$, $x = P_1(x)$, $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$

$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$, $x^4 = \frac{8}{35}P_4(x) + \frac{20}{35}P_2(x) + \frac{7}{35}P_0(x)$ —— (4)

$$x^4 + 3x^3 - x^2 + 5x - 2 = \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1(x) - \frac{224}{105}P_0(x)$$

—— (3)