CMR
INSTITUTE OF
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Internal Assessment Test I

Sub	ENGINEERING MATHEMATICS IV							Code	15MAT41
Date	27 / 3 / 2017	Duration	90 mins	Max Marks:	50	Sem	IV	Branch	ECE D, EEA,B(REGULAR)

First question(8 marks) is compulsory and anywer any SIX questions from the rest

Note: $z \in C, z = x + iy,\ x, y \in R$

	Marks	OBE: CO	OBE: RBT
1.Derive Rodrigue's formula for Legendre polynomials.	[08]	CO2	L3
2.Define conformal mapping and discuss the conformal transformation $w = z^2$.	[07]	CO3	L4
3.a) Show that $\lim\limits_{z \to 0} \frac{\bar{z}}{z}$ does not exist.	[07]	CO3	L3
b)Using Cauchy Riemann equations show that the polar curves $r^n = \alpha \sec(n\theta)$ and $r^n = \beta \operatorname{cosec}(n\theta)$ intersect orthogonally.			
4.Derive Cauchy Riemann equations in Cartesian form. [07]		CO3	L3
5.For a two-dimensional flow of an incompressible fluid, find the complex potential $w = f(z) = \phi(x, y) + i\psi(x, y)$, given $\phi(x, y) = (x^2 - y^2) + \frac{x}{x^2 + y^2}$	[07]	CO3	L3
6.If $f(z)$ is an analytic function, show that $\nabla^2 \lvert f(z) \rvert^2 = 4 \lvert f'(z) \rvert^2$	[07]	CO3	L3
7.Determine the poles of $f(z) = \frac{z^2}{(z-1)2(z+2)}$ and residue at each pole. Hence evaluate $\int_C f(z)dz$, C is the circle $\lvert z \rvert = 2.5$.	[07]	CO3	L4
8.If $F(a) = \int_C \frac{4z^2 + z + 5}{z - a} dz$, where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, find the values	[07]	CO3	L3
of $F(3.5), F(i), F'(-1) and\ F''(-i)$.	[07]	CO3	L3
9. Find the bilinear transformation that transforms the points $z = i, 1, -1$ onto the points $w = 1, 0, \infty$ respectively. Show that this transformation transforms the circle $\lvert z \rvert = 1$ into the real axis of the w plane and the interior of the circle onto the upper half of the w plane.	[07]	CO3	L3

Course Outcomes		PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1:	Use appropriate single-step and multi-step numerical methods to solve first and second order ordinary differential equations.	3	3	0	0	0	0	0	0	1	0	0	0
CO2:	Apply Frobenius method to find the power series solution of second order differential equations such as Bessel's and Legendre's differential equations.	3	3	0	0	0	0	0	0	1	0	0	0
CO3:	Apply the idea of analyticity and the calculus of residues to evaluate real and complex integrals and analyze conformal transformations.	2	3	0	0	0	0	0	0	1	0	0	0
CO4:	Describe random variables and probability distributions using rigorous statistical methods and translate real-world problems into probability models.	3	3	1	0	0	0	0	0	1	0	0	0
CO5:	Explain and successfully apply parametric testing techniques including single and multi-sample tests for mean and proportion.	3	3	0	0	0	0	0	0	1	0	0	0
CO6:	Estimate the nature and strength of relationship between two variables of interest using joint probability distribution and describe a discrete time Markov chain in terms of a transition matrix.	3	3	0	0	0	0	0	0	1	0	0	0

Cognitive level	KEYWORDS
L1	List, define, tell, describe, identify, show, label, collect, examine, tabulate, quote, name, who, when, where, etc.
L2	summarize, describe, interpret, contrast, predict, associate, distinguish, estimate, differentiate, discuss, extend
L3	Apply, demonstrate, calculate, complete, illustrate, show, solve, examine, modify, relate, change, classify, experiment, discover.
L4	Analyze, separate, order, explain, connect, classify, arrange, divide, compare, select, explain, infer.
L5	Assess, decide, rank, grade, test, measure, recommend, convince, select, judge, explain, discriminate, support, conclude, compare, summarize.

1. a) Rodrigue's Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$ (1M)

Let $u = (x^2-1)^n$ (1)

$$\frac{du}{dx} = n(x^2-1)^{n-1}(2x) = 2nx\frac{(x^2-1)^n}{(x^2-1)}$$

$$(x^2-1)\frac{du}{dx} = 2nx(x^2-1)^n$$

$$(x^2-1)\frac{du}{dx} = 2nxu$$

$$(x^2-1)u_1 = 2nxu$$

$$(x^2-1)u_2 + 2xu_1 = 2n(xu_1 + 1.u)$$

$$(x^2-1)u_2 + 2xu_1 - 2nxu_1 - 2nu = 0$$

$$(x^2-1)u_2 + (2-2n)xu_1 - 2nu = 0$$

$$(1-x^2)u_2 - (2n-2)xu_1 + 2nu = 0$$ (2) (3M)

By Leibniz

$$D^n((1-x^2)u_2) - (2n-2)D^n(xu_1) + 2nD^n(u) = 0$$

$$\left\{(1-x^2)u_{n+2} + nc_1(-2x)u_{n+1} + nc_2(-2)u_n\right\} - (2n-2)\left\{xu_{n+1} + nc_1 1.u_n\right\}$$

$$+ 2nu_n = 0$$

$$\left\{(1-x^2)u_{n+2} - 2nxu_{n+1} - \frac{n(n-1)}{\cancel{2}}\cancel{2}u_n\right\}$$

$$-(2n-2)\left\{xu_{n+1}+nu_n\right\}+2nu_n=0$$

$$(1-x^2)u_{n+2} - xu_{n+1}(2n+2n-2)$$
$$+(2n-n^2+n+2n^2-2n)u_n$$
$$=0$$

$$(1-x^2)u_{n+2} - 2xu_{n+1}+(n^2+n)u_n=0$$

$$(1-x^2)\frac{d^2u_n}{dx^2} - 2x\frac{du_n}{dx}+n(n+1)u_n=0$$

$\Rightarrow$ $u_n=(x^2-1)^n$ is the soln.

$\Rightarrow$ cu_n is also the solution

$$P_n(x) = cu_n = c\frac{d^n}{dx^n}(x^2-1)^n$$

at $x=1$

We K.T $P_n(x)=1$ at $x=1$

$$1 = c\frac{d^n}{dx^n}(x^2-1)^n$$

$$1 = c.\frac{d^n}{dx^n}\left\{(x-1)^n(x+1)^n\right\}$$

$1 = c\{(x+1)^n n! +$ terms containing $x-1$ and its powers$\}$

$$1 = c\, 2^n n!$$

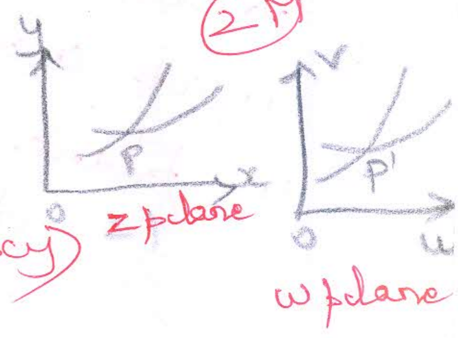
$$c = \frac{1}{2^n n!}$$

(3M)

$$P_n(x) = c\, u_n$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$ (1M)

2. Suppose two curves C and C' in the z plane intersect at the point P and the corresponding curves C_1' and C_1'' intersect at P' in the w plane. If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P' in magnitude and sense then the transformation is said to be conformal. (2M)



$$w = z^2$$

$$u + iv = (x+iy)^2 = (x^2 - y^2) + i(2xy)$$

$$u = x^2 - y^2, \quad v = 2xy$$

If u is a constant say a then $x^2 - y^2 = a$ is a rectangular hyperbola. If v is a constant $xy = \frac{b}{2}$ also rep a rectangular hyperbola.

A pair of lines $u=a$, $v=b$ parallel to the axes in the w plane map into a pair of orthogonal rectangular hyperbolas in the z plane.

Again if x is constant say c then $y=\frac{v}{2c}$, $y^2=c^2-u$. Eliminating y from these eqns, $v^2=4c^2(c^2-u)$ rep a parabola. If y is a constant say d, elimination of x from (1) gives $v^2=4d^2(d^2+u)$, a parabola.

A pair of lines $x=c$, $y=d$ parallel to the axes in the z plane map into orthogonal parabolas in the w plane.

$$\frac{dw}{dz}=2z=0 \quad \text{for } z=0$$

$z=0$ is the critical point

Take $z=re^{i\theta}$ $w=Re^{i\phi}$

$w=z^2$ becomes

$$Re^{i\phi}=(re^{i\theta})^2=r^2e^{i2\theta}$$

$$R=r^2 \quad \phi=2\theta$$

This shows that upper half of the z plane $0 < \theta < \pi$ transforms into the entire ω plane $0 \leq \phi < 2\pi$. The same is true for the lower half. (5M)

3.a) Suppose $z \to 0$ along the x axis (for which $y = 0$)

$$z = \bar{z} = x$$

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{x}{x} = 1 \quad -(1)$$ (2M)

Suppose $z \to 0$ along the y axis for which $x = 0$

$$z = iy \qquad \bar{z} = -iy$$

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0} \frac{-iy}{iy} = -1 \quad -(2)$$ (2M)

From (1) & (2) the reqd limit does not exist.

b) $r^n = a \sec(n\theta)$ $\qquad r^n = b \operatorname{cosec}(n\theta)$

$r^n \cos(n\theta) = a$ $\qquad r^n \sin(n\theta) = b$

(2M)

Let $u(r, \theta) = r^n \cos(n\theta) = a$

$v(r, \theta) = r^n \sin(n\theta) = b$

$u(r,\theta) + iv(r,\theta) = \alpha + i\beta$

$= r^n \cos n\theta + i r^n \sin n\theta$

$= r^n(\cos n\theta + i \sin n\theta)$

$= r^n e^{in\theta} = (re^{i\theta})^n = z^n$

1M

Thus $f(z) = u + iv$ an analytic fn. gives the curves $u = \alpha$ and $v = \beta$ intersect orthogonally.

4. The necessary and sufficient conditions for the derivative of the function $w = u(x,y) + iv(x,y) = f(z)$ to exist for all values of z in a region R are

a) u_x, u_y, v_x, v_y are continuous fns of x and y in R

b) $u_x = v_y$ $u_y = -v_x$ (CR eqns)

2M

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -iu_y + v_y \quad (2) \qquad (1M)$$

(1M)

$(1) + (2) \quad u_x = v_y \quad u_y = -v_x$

5. $$\phi(x, y) = x^2 - y^2 + \frac{x}{x^2+y^2}$$

$$\phi_x = 2x + \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2}$$

$$2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\phi_y = -2y + \frac{-2yx}{(x^2+y^2)^2} \qquad (2M)$$

$f(z) = \phi + i\psi$, the complex potential

$\phi_x = \psi_y, \quad \phi_y = -\psi_x$

$$f'(z) = \phi_x + i\psi_x$$

$$= \phi_x - i\phi_y$$

$$f'(z) = \left\{2x + \frac{y^2 - x^2}{(x^2+y^2)^2}\right\} + i\left\{2y + \frac{2xy}{(x^2+y^2)^2}\right\}$$

(7)

a) If $f(z)$ possesses a unique derivative at $P(z)$ then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)\} - \{u(x,y) + iv(x,y)\}}{\delta x + i\delta y}$$

When δz is wholly real, $\delta y = 0$

$$\delta z = \delta x$$

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$f'(z) = u_x + iv_x \quad (1)$$

(3M)

When δz is wholly imaginary, $\delta x = 0$

$$\delta z = i\delta y$$

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

Replace x by z, y by 0

$$f'(z) = 2z - \frac{z^2}{(z^2)^2} + i(0)$$

Intg $$\int f'(z)\,dz = \int \left(2z - \frac{1}{z^2}\right) dz$$

$$f(z) = 2\int z\,dz - \int z^{-2}\,dz$$

$$f(z) = z^2 + z^{-1} + c$$

$$f(z) = (x+iy)^2 + \frac{1}{x+iy} + c$$

$$x^2 - y^2 + i2xy + \frac{1}{x+iy}\,\frac{(x-iy)}{(x-iy)} + c$$

$$= x^2 - y^2 + i(2xy) + \frac{x-iy}{x^2+y^2} + c$$

(4M)

$$\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + i\left(2xy - \frac{y}{x^2+y^2}\right) + c$$

(1M)

Stream fn $\psi = 2xy - \frac{y}{x^2+y^2}$

5. $f(z) = u + iv$

$|f(z)| = \sqrt{u^2 + v^2}$

$|f(z)|^2 = u^2 + v^2$ (1M)

Consider $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2)$

$= \frac{\partial^2}{\partial x^2}(u^2 + v^2) + \frac{\partial^2}{\partial y^2}(u^2 + v^2)$

$= \frac{\partial^2}{\partial x^2} u^2 + \frac{\partial^2}{\partial x^2} v^2 + \frac{\partial^2}{\partial y^2} u^2 + \frac{\partial^2}{\partial y^2} v^2$

Consider $\frac{\partial^2 u^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} u^2\right)$

$= \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial u}(u^2)\frac{\partial u}{\partial x}\right\}$

$\cancel{= \frac{\partial}{\partial x}\left\{2u\frac{\partial u}{\partial x} + u^2 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right\}}$

$\cancel{= 2\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)}$

$= \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right)$

$= 2\left\{\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)\right\}$

(11)

$$= 2\left\{\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right\}$$

$$= 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right\}$$

|||ly $\frac{\partial^2 v^2}{\partial x^2} = 2\{(v_x)^2 + v v_{xx}\}$

$$\frac{\partial^2}{\partial y^2}u^2 = 2\{u_y^2 + u u_{yy}\}$$

$$\frac{\partial^2}{\partial y^2}v^2 = 2\{v_y^2 + v v_{yy}\}$$ (3M)

$$\nabla^2(u^2+v^2) = 2\{(u_x)^2 + u u_{xx} + (v_x)^2 + v v_{xx} + (u_y)^2 + u u_{yy} + v_y^2 + v v_{yy}\}$$

$$= 2\{(u_x)^2 + (v_x)^2 + (u_y)^2 + (v_y)^2 + v(v_{xx}+v_{yy}) + u(u_{xx}+u_{yy})\}$$

$f(z) = u + iv$ is an analytic fn.

u and v are harmonic

$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0$

$u_x = v_y \quad u_y = -v_x$

$$= 2\{(u_x)^2 + (v_x)^2 + (-v_x)^2 + (u_x)^2\}$$

$$= 2\{2u_x^2 + 2v_x^2\}$$

$$= 4(u_x^2 + v_x^2) = 4|f'(z)|^2$$

where $f'(z) = u_x + i v_x$

$|f'(z)| = \sqrt{u_x^2 + v_x^2}$

$|f'(z)|^2 = u_x^2 + v_x^2$ (3M)

7. $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$, analytic $\forall z$, except $z = 1, -2$

To determine the poles, equate Dr to 0

$(z-1)^2(z+2) = 0$

$z = 1, 1, -2$

$z = 1$ is a double pole $z = -2$ is a simple pole (1M)

Residue $f(a) = \lim\limits_{z \to a} (z-a) f(a)$

$\operatorname{Res} f(-2) = \lim\limits_{z \to -2} (z+2) \dfrac{z^2}{(z-1)^2(z+2)}$

$= \lim\limits_{z \to -2} \dfrac{z^2}{(z-1)^2}$

$= \dfrac{(-2)^2}{(-2-1)^2} = \dfrac{4}{9}$

(2M)

$$\text{Res } f(a) = \lim_{z \to a} (n-1)! \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

$n = 2$

$$\text{Res } f(1) = \lim_{z \to 1} \frac{d}{dz} (z-1)^2 \frac{z^2}{(z-1)^2 (z+2)}$$

$$= \lim_{z \to 1} \frac{d}{dz} \frac{z^2}{z+2}$$

$$= \frac{2z(z+2) - (z^2)}{(z+2)^2} \quad \text{at } z = 1$$

$$= \frac{2(3) - 1}{3^2} = \frac{5}{9}$$ (2M)

By Residue theorem

$$\oint_C f(z)\,dz = 2\pi i \left(f(-2) + f(1) \right)$$

$$= 2\pi i \left(\frac{4}{9} + \frac{5}{9} \right) = 2\pi i$$ (2M)

8. $$F(a) = \int_C \frac{4z^2 + z + 5}{z - a}\,dz$$

$C \quad \frac{x^2}{4} + \frac{y^2}{9} = 1$

$F(3.5)$, $F(i)$ $F'(-1)$ $F''(-i)$

$a = 3.5$ lies outside C.

$f(z)$ is analytic on C

By Cauchy's theorem

$$\oint_C f(z)\,dz = 0$$

(1M)

$$F(3.5) = 0$$

b) $a = i \quad (0, 1)$ lies inside C

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\oint_C \frac{4z^2+z+5}{z-i}\,dz = 2\pi i f(i)$$

$$= 2\pi i \{4i^2+i+5\}$$

$$= 2\pi i \{-4+i+5\}$$

$$\boxed{F(i) = 2\pi i(1+i)}$$

$a = -1, -i$ lie inside C

$$F(a) = 2\pi i f(z)$$

$$= 2\pi i (4z^2+z+5)$$

$$F(a) = 2\pi i (4a^2+a+5)$$

(4M)

$$F'(a) = 2\pi i (8a+1)$$

$$F''(a) = 16\pi i$$

$$F'(-1) = 2\pi i(-7) = -14\pi i$$

$$F''(-i) = 16\pi i$$

(1M)

10. $z_1 = i \quad z_2 = 1, \quad z_3 = -1$

$w_1 = 1 \quad w_2 = 0 \quad w_3 = \infty$

$\frac{1}{w_3} = 0$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-1)(0-1)}{\left(\frac{w}{w_3}-1\right)(0-1)} = \frac{(z-i)(1+1)}{(z+1)(1-i)}$$

$$\frac{(w-1)}{(-1)} = \frac{2(z-i)}{(z+i)(1-i)}$$

$$(w-1) = -\frac{2(z-i)}{(z+i)(1-i)}$$

$$w = 1 - \frac{2(z-i)}{(z+i)(1-i)}$$

$$w = \frac{(z+i)(1-i)-2(z-i)}{(z+i)(1-i)}$$

$$w = \frac{(1+i)(1-z)}{(1-i)(1+z)}$$

$$w = \frac{(1+i)^2(1-z)}{(1^2-i^2)(1+z)}$$

$$\omega = \frac{2i(1-z)}{2(1+z)}$$

$$\omega = i\left(\frac{1-z}{1+z}\right)$$

This is the reqd BLT

$$\omega(1+z) = i(1-z)$$

$$z(\omega+i) = i-\omega$$

$$z = \frac{i-\omega}{i+\omega}$$ (4M)

For $|z|=1$

$$|i+\omega| = |i-\omega|$$

$$\Rightarrow |i+u+iv| = |i-(u+iv)|$$

$$|u+i(1+v)| = |-u+i(1-v)|$$

$$u^2+(v+1)^2 = u^2+(1-v)^2$$ (2M)

$$v=-v$$

$$2v=0 \quad v=0$$

$|z|=1$ correspond to the line $v=0$

This means that the given transformation transforms $|z|=1$ to the real axis in the ω plane. (1M)