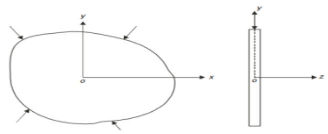
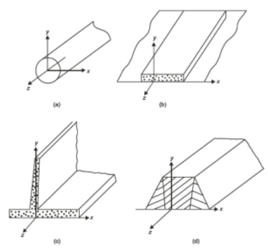
Q1.a) **Plane Stress problems**



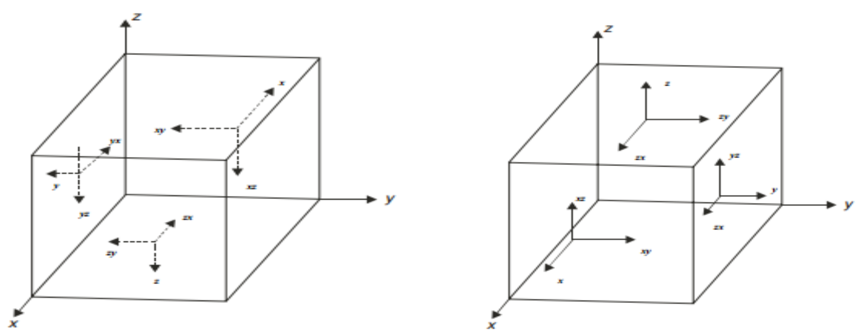
$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\frac{E}{1-\mu^2}\begin{bmatrix}1 & \mu & 0\\ \mu & 1 & 0\\ 0 & 0 & \frac{1-\mu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

Plane Strain problems



$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\frac{E}{(1+\mu)(1-2\mu)}\begin{pmatrix}1-\mu & \mu & 0\\ \mu & 1-\mu & 0\\ 0 & 0 & \frac{1-2\mu}{2}\end{pmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

Q1 b)



Face	*Stress on –ve Face*	*Stresses on +ve Face*
x	σ_x τ_{xy} τ_{xz}	$\sigma_x^+ = \sigma_x + \frac{\partial \sigma_x}{\partial x}dx$ $\tau_{xy}^+ = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}dx$ $\tau_{xz}^+ = \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x}dx$
y	σ_y τ_{yx} τ_{yz}	$\sigma_y^+ = \sigma_y + \frac{\partial \sigma_y}{\partial y}dy$ $\tau_{yx}^+ = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy$ $\tau_{yz}^+ = \tau_{yz} + \frac{\partial \tau_{yz}}{\partial y}dy$
z	σ_z τ_{zx} τ_{zy}	$\sigma_z^+ = \sigma_z + \frac{\partial \sigma_z}{\partial z}dz$ $\tau_{zx}^+ = \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}dz$ $\tau_{zy}^+ = \tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}dz$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial_z} + Y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial_z} + Z = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\tau_{yz} = \tau_{zy}$$

$$\tau_{xz} = \tau_{zx}$$

Q2 i)

Solⁿ:- Check if solution of $v = a(x^4 + l^2x^2 - 2lx^3)$ satisfy

i) @ $x = 0$ $\quad v = 0$. & $\frac{dv}{dx} = 0$.

ii) @ $x = L$ $\quad v = 0$. & $\frac{dv}{dx} = 0$.

This disp function can be used as it satisfy the boundary condition.

$\sqrt{2} \times 2\sqrt{3}$

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx - P \cdot v_c.$$

$$v_c = a\left[\left(\frac{l}{2}\right)^4 + l^2\left(\frac{l}{2}\right)^2 - 2l\left(\frac{l}{2}\right)^3\right] = a\left[\frac{l^4}{16} + \frac{l^4}{4} - l^4\right] = \frac{11 a l^4}{16}$$

$$\Pi = \frac{EI}{2}\int_0^L \left[a^2\left(12x^2 + 2l^2 - 12lx\right)^2\right] dx - P\cdot\left(\frac{11 a l^4}{16}\right)$$

$$\frac{\partial \pi}{\partial a_1} = \frac{EI\pi^4}{4L^3}(2a_1) - \frac{2qL}{4\pi} - P$$

$$= \frac{EI\pi^4 a_1}{2L^3} - \frac{2qL}{4\pi} - P \quad \text{—— (5)}$$

$$\frac{\partial \pi}{\partial a_2} = \frac{EI\pi^4}{4L^3}\, 16\, 2a_2 - \frac{2qL}{3\pi} + P$$

$$= \frac{8EI\pi^4 a_2}{2} - \frac{2qL}{3\pi} + P \quad \text{—— (6)}$$

Solve for the Ritz parameter a_1 & a_2 using Eqn (5) & (6)

Mid span deflection @ $x = \frac{L}{2}$.

$$a_1 = \frac{2L^3}{EI\pi^4}\left(\frac{2qL}{\pi} + P\right)$$

$$a_2 = \frac{2L^3}{81EI\pi^4}\left(\frac{2qL}{3\pi} - P\right)$$

Mid span deflection Put $x = \frac{L}{2}$.

$$V_{max} = V_{\frac{L}{2}} = a_1 - a_2 = \frac{PL^3}{48.11EI} + \frac{qL^4}{76.92EI}$$

This value practically coincides with the exact value obtained in SOM.

$$v = \frac{PL^3}{48EI} + \frac{qL^4}{76.8EI}$$

ii)

Let E be the Young's modulus, I be the moment of Inertia of the beam.

The potential energy functional is given by

$$\Pi = SE + WP$$

Where, $$SE = \frac{EI}{2}\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx \text{ for the beam and } WP = -Py_{max}$$

$\therefore$ $$\Pi = \frac{EI}{2}\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx - Py_{max}$$

Given trial function is $$y = a\left(1-\cos\frac{\pi x}{2L}\right) \Rightarrow \left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{a\pi^2}{4L^2}\cos\frac{\pi x}{2L}$$

$\therefore$ $$\left(\frac{\partial^2 y}{\partial x^2}\right)^2 = \frac{\pi^4 a^2}{16L^4}\cos^2\frac{\pi x}{2L}$$

Also, $$\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx = \frac{\pi^4 a^2}{16L^4}\int_0^l \cos^2\frac{\pi x}{2L}dx = \frac{\pi^4 a^2}{32L^3} \text{ and}$$

Deflection will be maximum at $x = L \quad \therefore \quad y_{max} = a$

$\therefore$ $$\Pi = \frac{EI}{2}\times\frac{\pi^4 a^2}{32L^3} - Pa$$

Minimizing PE functional i.e., $\dfrac{\partial\pi}{\partial a} = 0$

$\therefore$ $$\frac{EI}{2}\times\frac{\pi^4}{32L^3}(2a) - P = 0 \quad \Rightarrow a = \frac{PL^3}{3.044EI}$$

$\therefore$ Deflection of cantilever beam at free end is $y_{max} = \dfrac{PL^3}{3.044EI}$

Q3

Let u be the axial displacement at any point x from the fixed end. When the bar subjected to uniaxial loading P at point 1, u_1 be the displacement at loading point 1.

i) Formulate the PE functional

PE functional is given by $\Pi = SE + WP$

Where, Strain energy stored in bar is given by

$$SE = \frac{AE}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \text{ and}$$

Work potential. $$WP = -Pu_1$$

$$\therefore \quad \Pi = \frac{AE}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx - Pu_1$$

$$\Pi = \frac{1}{2}\times 100\times 70\times 10^3 \int_L \left(\frac{\partial u}{\partial x}\right)^2 dx - 8\times 10^3 u_1$$

$$\Pi = 3.5\times 10^6 \int_L \left(\frac{\partial u}{\partial x}\right)^2 dx - 8\times 10^3 u_1 \quad \text{...... (1)}$$

ii) Assume a polynomial displacement function (2nd order $\because$ given)

$$u = a_0 + a_1 x + a_2 x^2 \quad \text{...... (2)}$$

Where, a_0, a_1 and a_2 are the generalized coordinates are to be determined from the boundary condition. i.e.,

(a) $u = 0$ at $x = 0$, $\Rightarrow$ $a_0 = 0$

$\therefore$ $$u = a_1x + a_2x^2$$

(b) $u = 0$ at $x = 40$, $\Rightarrow$ $$0 = 40\,a_1 + (40)^2 a_2$$

$\Rightarrow$ $$a_1 = -40a_2$$

$\therefore$ $$u = -40a_2x + a_2x^2 \quad \text{...... (3)}$$

$\Rightarrow$ $$\frac{\partial u}{\partial x} = -40a_2 + 2a_2x \quad \text{...... (4)}$$

(c) $u = u_1$ at $x = 20$

$\therefore$ Eq.(3) becomes $$u_1 = -40a_2 \times 20 + a_2\,(20)^2$$

i.e., $$u_1 = -400a_2 \quad \text{...... (5)}$$

iii) Substitute the displacement function into PE functional

Substituting Eq.(3); Eq.(4) and Eq.(5) into Eq.(1), we get

$$\Pi = 3.5 \times 10^6 \int_L \left(-40a_2 + 2a_2x\right)^2 dx - 8 \times 10^3 \times \left(-400a_2\right)$$

$$= 3.5 \times 10^6 \int_0^{40} \left(1600a_2^2 + 4a_2^2x^2 - 160a_2^2x\right)dx + 32 \times 10^5\,a_2$$

$$= 3.5\times10^6 \left[1600\,a_2^2\,x\Big|_0^{40} + 4a_2^2\frac{x^3}{3}\Big|_0^{40} - 160a_2^2\frac{x^2}{2}\Big|_0^{40}\right] + 32\times10^5\,a_2$$

$$\Pi = 7.465 \times 10^{10}\,a_2^2 + 32 \times 10^5\,a_2 \quad \text{...... (6)}$$

iv) Minimize PE functional

Condition for minimization is $$\frac{\partial \Pi}{\partial a_i} = 0$$

From Eq.(6), we have $$\frac{\partial \Pi}{\partial a_2} = 7.465 \times 10^{10} \times 2a_2 + 32 \times 10^5 = 0$$

$\Rightarrow$ $$a_2 = -2.143 \times 10^{-5}$$

v) Determination of displacement and stress

Substituting value of a_2 into Eq.(5), we get

$$u_1 = -400\,a_2 = -400 \times (-2.143 \times 10^{-5}) = 8.572 \times 10^{-3}$$

$\therefore$ Displacement at loading point

$$u_1 = 8.572 \times 10^{-3}\,\text{mm}$$

To find stress

From Hooke's Law $$\frac{\sigma}{\epsilon} = E$$

$$\Rightarrow \qquad \sigma = \in E = \left(\frac{\partial u}{\partial x}\right)E$$

$$= (-40a_2 + 2a_2x)E$$

At mid point $x = 20$; $\quad \sigma = \left(-40\left(-2.143\times10^{-5}\right)+2\left(-2.143\times10^{-5}\right)20\right)70\times10^{3}$

$\sigma = 0$ Ans.

At $x = 0$; $\quad \sigma = -40a_2 E = -40\,(-2.143\times10^{-5})\,70\times10^{3}$

$\sigma = +60$ N/mm^2

At $x = 40$; $\quad \sigma = \left(-40\left(-2.143\times10^{-5}\right)+2\left(-2.143\times10^{-5}\right)40\right)70\times10^{3}$

$\sigma = -60$ N/mm^2

Q3a

The Principle of Minimum Potential Energy (MPE)

Deformation & stress analysis of structural systems can be accomplished using the principle of Minimum PE which states that,

"For conservative structural systems, of all the kinematically admissible deformations, those corresponding to the equilibrium state extremize (i.e, minimize or maximize) the total potential energy. If the extremum is a minimum, the equilibrium state is stable."

Explanation

A constrained structural system, i.e. a structure that is fixed at some portions, will deform when forces are applied on it.

Deformation of a structural system refers to the incremental change to the new deformed state from the original undeformed state.

The deformation is the principal unknown in structural analyses as the strains depend upon the deformation, and the stresses are in turn dependent on the strains. Therefore, our sole objective is to determine the deformation.

The deformed state a structure attains upon the application of forces is the equilibrium state of a structural system.

The Potential energy (PE) of a structural system is defined as the sum of the strain energy (SE) and the work potential (WP).

$$P.E = SE + WP$$

The strain energy is the elastic energy stored in deformed structure. It is computed by integrating the strain energy density (i.e strain energy per unit volume) over the entire volume of the structure.

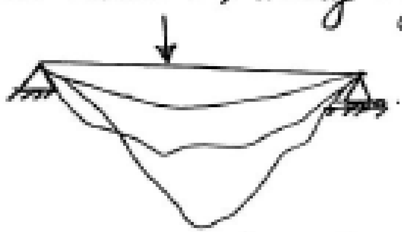
$$SE = \int_V (\text{strain energy density})\, dV$$

$$= \int_V (\text{Stress} \times \text{Strain})\, dV$$

The work potential (WP), is the negative of the work done by the external forces acting on the structure

Work done by the external forces is simply the forces multiplied by the displacements at the points of application of forces.

For a structure, many deformations are possible.

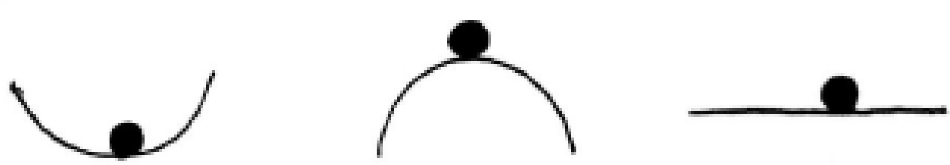


But for a given force it will only attain a unique deformation to achieve equilibrium

The principle of minimum potential energy implies ~~is~~ that this unique deformation corresponds to the extremum value of the potential Energy.

I.e, inorder to determine the equilibrium deformation, we have to extremize the P.E. The extremum can be a minimum or a maximum.

For a minimum P.E, the equilibrium state is said to be stable.



Stable Unstable Neutrally stable

Q3b

Steps in Finite Element analysis

1) Discretize and select the element types.

2) Select displacement function

3) Define stress / strain and strain / displacement relationship

(Determination of equilibrium equation for the finite element)

4) Derive the element stiffness matrix and equation.

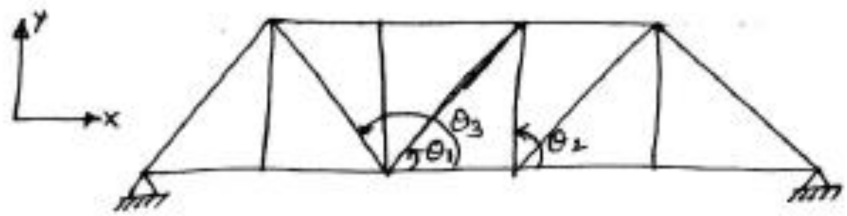
5) Assemble element equation to obtain the global or total equation and introduce boundary condition.

6) Solve for unknown DOF (generalised displacement)

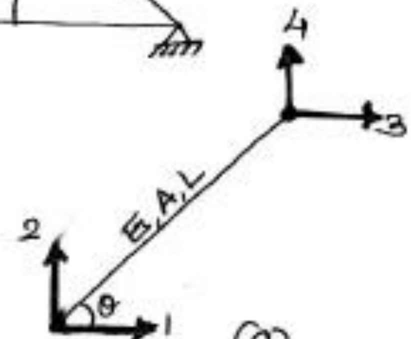
7) Solve F.E strain and stress.

8) Interpretation of results.

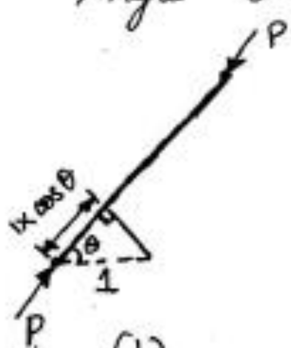
Q4



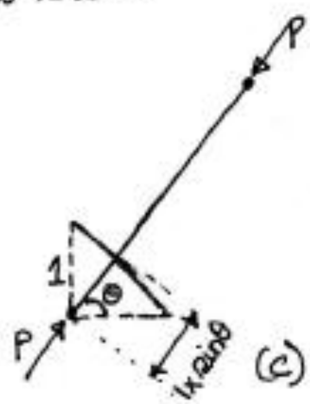
Young's Modulus E
Cross sectional area A
Length L
Angle θ to x axis



(a)



(b)



(c)

we know that;

$$P = \frac{EA}{L}\delta$$

I) Unit displacement of end '1' in x-direction. → ref (b)

Displacement along the axis of the member is = $1 \times \cos\theta$

$$P = \frac{EA}{L}\cos\theta$$

from the definition of elements of stiffness matrix,

$$k_{11} = P\cos\theta = \frac{EA}{L}\cos^2\theta.$$

$$k_{21} = P\sin\theta = \frac{EA}{L}\cos\theta\sin\theta$$

$k_{31} = -P\cos\theta = -\frac{EA}{L}\cos^2\theta$

$k_{41} = -P\sin\theta = -\frac{EA}{L}\cos\theta\sin\theta.$

$k_{31} = -P\cos\theta = -\frac{EA}{L}\cos^2\theta$

$k_{41} = -P\sin\theta = \frac{-EA}{L}\cos\theta\sin\theta$

Unit displacement in coordinate direction 2 → ref ⓪

Displacement along the axis of the member 'δ' $= 1 \times \sin\theta$

Forces developed at each end are as shown.

$$\therefore P = \frac{EA}{L}\sin\theta.$$

$k_{12} = P\cos\theta = \frac{EA}{L}\sin\theta\cos\theta$

$k_{22} = P\sin\theta = \frac{EA}{L}\sin^2\theta$

$k_{32} = -P\cos\theta = \frac{-EA}{L}\sin\theta\cos\theta$

$k_{42} = -P\sin\theta = -\frac{EA}{L}\sin^2\theta$

Unit displacement in coordinate direction 3:-

Displacement along the axis of the member

$$'\delta' = 1 \times \cos\theta$$

$$\therefore P = \frac{EA}{L}\cos\theta$$

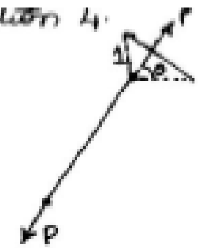
$k_{13} = -P\cos\theta = -\frac{EA}{L}\cos^2\theta.$

$k_{23} = -P\sin\theta = -\frac{EA}{L}\cos\theta\sin\theta$

$k_{33} = P\cos\theta = \frac{EA}{L}\cos^2\theta.$

$k_{43} = P\sin\theta = \frac{EA}{L}\cos\theta\sin\theta.$

(vi) Due to unit displacement in coordinate direction 4.



Displacement along the axis of member
'δ' = $1 \times \sin\theta$.

$$\therefore P = \frac{EA}{L} \sin\theta$$

$$k_{14} = -P\cos\theta = -\frac{EA}{L}\sin\theta\cos\theta$$

$$k_{24} = -P\sin\theta = -\frac{EA}{L}\sin^2\theta$$

$$k_{34} = P\cos\theta = \frac{EA}{L}\sin\theta\cos\theta$$

$$k_{44} = P\sin\theta = \frac{EA}{L}\sin^2\theta.$$

$\therefore$ The stiffness Matrix is

$$K = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$= \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Where l & m are the direction cosines of the member

i.e $l = \cos\theta$ and

$m = \cos(90-\theta) = \sin\theta$

Q5 a)

Convergence Requirement-

→ Solution must converge to the exact solution

3 conditions to be satisfied by displacement function for convergence.

1) displacement function must be continuous within the element.

→ Achieved by choosing polynomials

2) displacement function must be capable of representing rigid body displacements of the element.

→ When nodes are given such displacements corresponding to a rigid body motion,
the element should not exp. any strain & hence leads to zero nodal forces.

→ Achieved by the constant terms in the polynomial.

3) displacement function must be capable of representing constant strain states within the element.

→ As the element approach infinitesimal size, the strains in each element also approach constant values

→ Achieved by linear terms present in the polynomial for 1D, 2D & 3D elasticity problem.

→ In beam, plate & shell element → this condition will be referred as constant curvature.

Compatibility Requirement

→ Displacement must be compatible between adjacent element.
→ when element deform there must not be any discontinuity bet. elements i.e elements must not overlap or seperate

Geometric Invariance

→ Imp. consideration is choosing proper terms in the polynomial expansion as the element should have no prefered directi

→ i.e displacement shapes will not change with a chang in local coordinate System. This property is known as geometric isotropy or geometric invariance.

→ Achieved, if the polynomial Includes all the terms i.e the polynomial is a complete one.

Q5b)

1D

2D

1 Constant term 1

1 Linear term x $p=1$

1 Quadratic term x^2 $p=2$

1 ~~Quadratic~~ Cubic term x^3. $p=3$.

2D

Constant term 1

linear terms x y. $p=1$ (3)

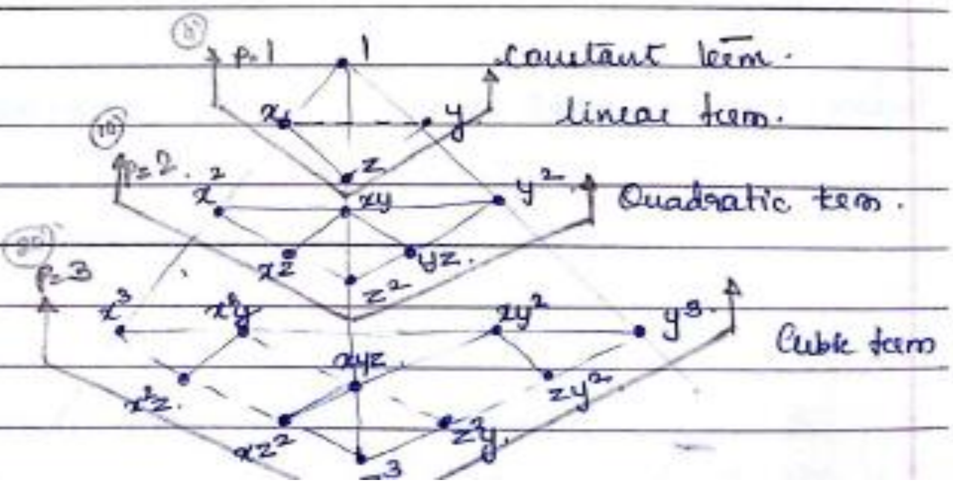
Quadratic terms x^2 xy y^2 $p=2$ (6)

Cubic terms x^3 x^2y xy^2 y^3 $p=3$ (10)

Quadratic terms x^4 x^3y x^2y^2 xy^3 y^4 $p=4$ (15)

Pascal's triangle. → Short note

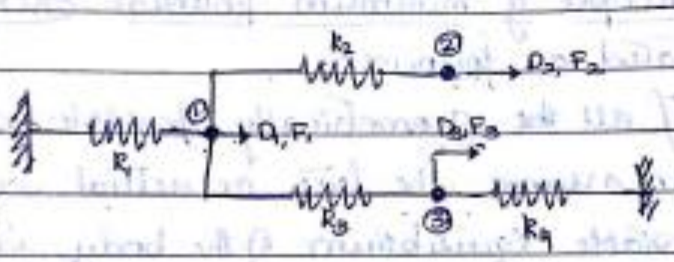
3D



Q6

Obtain the equilibrium equations for the spring system shown in fig. By using the principle of minimum potential energy.
(OR Det the stiffness matrix of the spring system shown).

~~There are~~.



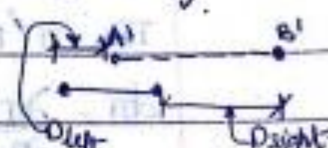
There are 3DOF, namely D_1, D_2 & D_3.

Step①:- Compute the total potential energy.

$$\Pi = \Pi_1 + \Pi_2 + \Pi_3 + \Pi_{4\,spring}$$
$$= U + \Omega$$

$$\Pi = (U_1 + U_2 + U_3 + U_4) + (\Omega_1 + \Omega_2 + \Omega_3)$$

$$\Pi = \left(\frac{1}{2}K_1\delta_1^2 + \frac{1}{2}K_2\delta_2^2 + \frac{1}{2}K_3\delta_3^2 + \frac{1}{2}K_4\delta_4^2\right) - \left(F_1D_1 + F_2D_2 + F_3D_3\right)$$



In the present prob, $\delta_1 = D_1$

$$\delta_2 = D_2 - D_1$$
$$\delta_3 = D_3 - D_1$$
$$\delta_4 = 0 - D_3 = -D_3.$$

$\therefore \delta = D_{right} - D_{left}$

Substituting these we have,

$$\Pi = \frac{1}{2}K_1D_1^2 + \frac{1}{2}K_2(D_2-D_1)^2 + \frac{1}{2}K_3(D_3-D_1)^2 + \frac{1}{2}K_4(-D_3)^2 - F_1D_1 - F_2D_2 - F_3D_3 \quad \text{—ⓐ}$$

<u>Step 2</u> Apply the principle of minimum potential energy

$$\frac{\partial \Pi}{\partial D_1} = 0, \Rightarrow K_1D_1 + \frac{1}{2}K_2(D_2^2 - 2D_1D_2 + D_1^2) + \frac{1}{2}K_3(D_3^2 - 2D_1D_3 + D_1^2) - F_1 = 0$$

$$= K_1D_1 + \frac{1}{2}K_2(-2D_2 + 2D_1) + \frac{1}{2}K_3(-2D_3 + 2D_1) - F_1 = 0$$

$$= K_1D_1 - K_2D_2 + K_2D_1 - K_3D_3 + D_1K_3 - F_1 = 0$$

$$= (K_1 + K_2 + K_3)D_1 - K_2D_2 - K_3D_3 = F_1 \quad \text{—ⓑ}$$

$$\frac{\partial \Pi}{\partial D_2} = \frac{1}{2}K_2(D_2^2 - 2D_1D_2 + D_1^2) + \frac{1}{2}K_3(D_3^2 - 2D_3D_1 + D_1^2) - F_2$$

$$= D_2K_2 - K_2D_1 - F_2$$

$$= K_2(D_2 - D_1) = F_2 \quad \text{—ⓒ}$$

$$\frac{\partial \Pi}{\partial D_3} = K_3D_3 - K_3D_1 + K_4D_3 - F_3$$

$$= -K_3D_1 + (K_3 + K_4)D_3 = F_3 \quad \text{—ⓓ}.$$

Express eqn ⓑ, ⓒ & ⓓ in matrix form:

$$\begin{bmatrix} K_1+K_2+K_3 & -K_2 & -K_3 \\ -K_2 & K_2 & 0 \\ -K_3 & 0 & K_3+K_4 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

System stiffness matrix / Global stiffness matrix

$$[K]\{D\} = \{F\}.$$

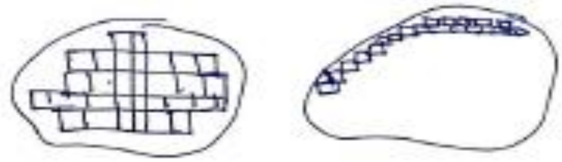
These are the equilibrium eqn. for the given spring system

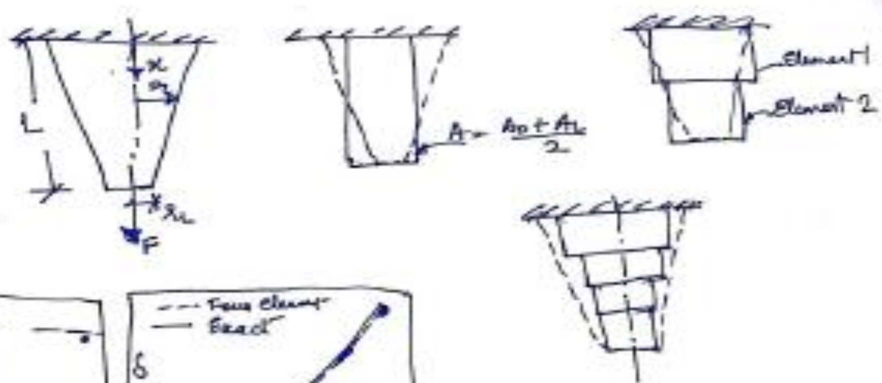
Q7

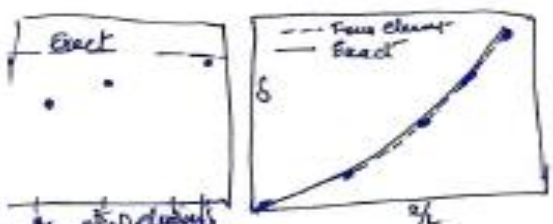
FEM vs Exact Sol^n

Representation of ~~phy~~ physical domain → in FEM → is Meshing → FE mesh

Commonly used element → straight boundaries → difficult in curved boundaries.





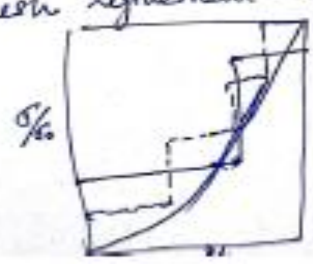


Strains & stress → referred to as "derived variables"

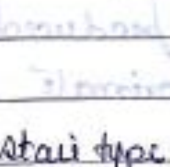
Primary field variable → continuous from element to element

Derived variables → not necessarily continuous.

Densed mesh refinement → derived variable becomes closer & close to continuity



Finite Difference method & Finite Element method

No.	FDM	FEM
1)	Makes Pointwise approximatn to the governing eqn. Ensures continuity only at node pts. The continuity along the sides of the gridlines are not ensured.	Makes piece-wise approximatn. Ensures continuity at the nodal pts as well as along the sides of the element.
2)	Does not give the values at any pt except at node point.	Can give values at any pt. values @ nodes are obt. directly value @ other pt are obt. using nodal values or shape or interpolatn functn
3)	Makes stair type approximatn to sloping & curved boundaries. 	When curved elements are used, FEM models the curved boundaries more exactly.
4)	Needs larger no. of nodes to get good results	Needs fewer nodes
5)	can handle fairly complicated Prob.	can handle All complicated Prob.