CMR INSTITUTE OF TECHNOLOGY		
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INTERNAL ASSESSMENT TEST 1 – MARCH 2017

Sub:	ELECTROMAGNETIC FIELD THEORY							Code:	15EE45
Date:	30/03/2017	Duration:	90 mins	Max Marks:	50	Sem:	4	Branch:	EEE
Answer Any FIVE FULL Questions									

		Marks	OBE	
			CO	RBT
1(a)	State and explain Coulomb's law of force between two charges.	[03]	CO2	L1, L2
(b)	Find electric field intensity and electric flux density at the origin due to Q = 0.35 μC at (0, 4, 0).	[03]	CO2	L3
(c)	Prove that electric field intensity is negative potential gradient.	[04]	CO3, CO1	L2
2	Derive an expression for electric field strength due to finite and infinite line of linear charge density $\rho_L\ C/m$.	[10]	CO2	L2
3	Given that $\boldsymbol{D} = z\rho\ (\cos\phi)^2\ \boldsymbol{a_z}$ $\mathrm{C/m^2}$, calculate charge density at $(1,\frac{\pi}{4},3)$ and the total charge enclosed by the cylinder of radius 1 m with $-2m \le z \le 2m$. Verify using Divergence theorem.	[10]	CO2, CO1	L3
4(a)	Determine the work done in carrying a 2μC charge from (2, 1,-1) to (8, 2,-1) in the field $\boldsymbol{E} = y\ \boldsymbol{a_x}\ + x\ \boldsymbol{a_y}$ along the Hyperbola $x = \frac{8}{7-3y}$.	[04]	CO3	L3
(b)	Derive the expression for electric potential due to an electric dipole.	[06]	CO3	L2
5	Potential is given by $V = 2(x+1)^2\ (y+2)^2(z+3)^2$ volt in free space. Calculate Potential, Electric field intensity, Electric flux density and volume charge density at point (2,-1, 4).	[10]	CO3, CO1	L3
6	Obtain the expression for Gauss's Divergence theorem from Gauss's law.	[10]	CO2, CO1	L2
7(a)	Transform to cylindrical co-ordinates, the vector $\boldsymbol{F} = 10\ \boldsymbol{a_x} - 8\boldsymbol{a_y} + 6\boldsymbol{a_z}$ at point P(10,-8,6)	[05]	CO1	L3
(b)	Give the rectangular components of the vector $\boldsymbol{H} = 20\boldsymbol{a_\rho} - 10\boldsymbol{a_\phi} + 3\boldsymbol{a_z}$ at P(x=5, y=2, z=-1)	[05]	CO1	L3
8(a)	Find the energy stored in free space for the region 2 mm < r < 3 mm, 0 < θ < 90^0, 0< ϕ<90^0, given the potential function $V = \frac{200}{r}\ volts.$	[05]	CO3	L3
(b)	Determine electric flux density caused at P(6,8,-10) due to i) a point charge of 30 mC at origin. ii) a surface charge with $\rho_s = 57.2\ \mu\mathrm{C/m^2}$ on a plane Z = - 9 m.	[05]	CO2	L3

SCHEME OF EVALUATION

		Marks
1(a)	Statement	2
	Expression	1
(b)	Formula	1
	Approach	1
	Answer	1
(c)	Diagram	1
	Proof	3
2	Diagram	2
	Derivation	7
	Final Expression	1
3	Divergence Theorem	2
	Surface integral	4
	Volume Integral	4
4(a)	Formula	1
	Approach	2
	Answer	1
(b)	Diagram	1
	Derivation	4
	Final Expression	1
5	Potential	2
	Electric field intensity	3
	Electric flux density	2
	Volume charge density	3
6	Diagram	2
	Derivation	6
	Final Expression	2
7(a)	Formula	2
	Approach	2
	Answer	1
(b)	Formula	2
	Approach	2
	Answer	1
8(a)	Formula	2
	Approach	2
	Answer	1
(b)	Diagram	1
	Point Charge:	
	Formula and Approach	1
	Answer	1
	Surface Charge:	
	Formula and Approach	1
	Answer	1

1.a.

Coulomb's law:

The force between two very small objects separated in vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them.

$$\boxed{F = k\frac{(Q_1 Q_2)}{R^2}}$$

$(Q_1 Q_2)$ → positive or negative quantities of charge

k → Proportionality constant

R → separation

$$\boxed{K = \frac{1}{4\pi \varepsilon_0}}$$

ε_0 → Permittivity of free space.

$$\varepsilon_0 = \frac{10^{-9}}{36\pi} \text{ F/m.}$$

$$\varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$$

$$K = \frac{1}{\frac{4\pi \times 10^{-9}}{\underset{9}{36}\pi}}$$

$$K = 9 \times 10^{9}$$

$$\boxed{F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 R^2}}$$

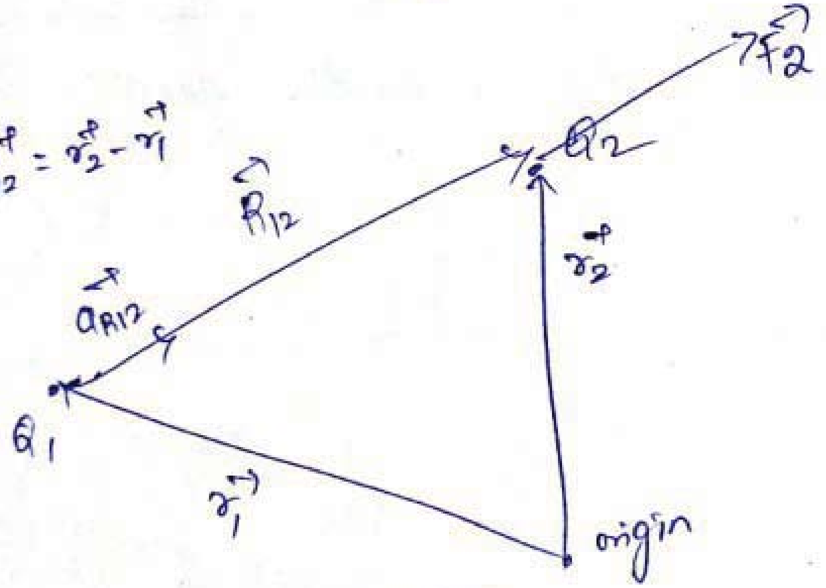
Vector form :

Force acts along line joining two charges.

Like charges → repulsive force

unlike charges → attractive force

$$\vec{R_{12}} = \vec{r_2} - \vec{r_1}$$



$$\vec{a_{R_{12}}} = \frac{\vec{R_{12}}}{|\vec{R_{12}}|}$$

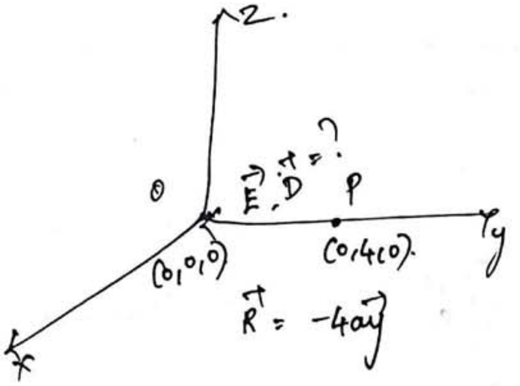
$$\boxed{\vec{F_2} = \frac{Q_1 Q_2}{4\pi\epsilon_0 (\vec{R_{12}})^2} \cdot \vec{a_{R_{12}}}}$$

$$\vec{F_2} = \frac{Q_1 Q_2 \vec{R_{12}}}{4\pi\epsilon_0 |\vec{R_{12}}|^3}$$

1.b. **Problem:**

Find the electric field intensity and flux density at the origin due to

$Q = 0.35 \mu C$ at $(0,4,0)$.



$\vec{R} = -4\vec{a_y}$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{(\vec{R})}{|\vec{R}|^3}$$

$$= 0.35\times10^{-6} \times 9\times10^{9} \times \frac{(-4\vec{a_y})}{(4)^3}.$$

$$= 0.196875 \times 10^{3} (-\vec{a_y})$$

$$= -0.196875 \times 10^{3} \vec{a_y}$$

$$\boxed{\vec{E} = -196.875 \, \vec{a_y}} \; V/m$$

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$= -1743.13 \times 10^{-12} \; C/m^2 \, \vec{a_y}$$

$$= -1.743 \times 10^{-9} \; C/m^2 \, \vec{a_y}$$

$$\boxed{\vec{D} = -1.743} \; nC/m^2 \, \vec{a_y}.$$

Potential Gradient :

1.c.

① $\vec{E} \longrightarrow V = -\int \vec{E}\cdot d\vec{l}$.   ② $\int \rho_v \, dv \Rightarrow V = \int_v \frac{\rho_v \, dv}{4\pi\epsilon_0 r}$.

In practical problems,

neither $\vec{E}$, nor ρ_v is known.

Description of two equipotential surfaces

Eg: two ||el conductors of circular cross sections
at potentials 100 V & −100 V.

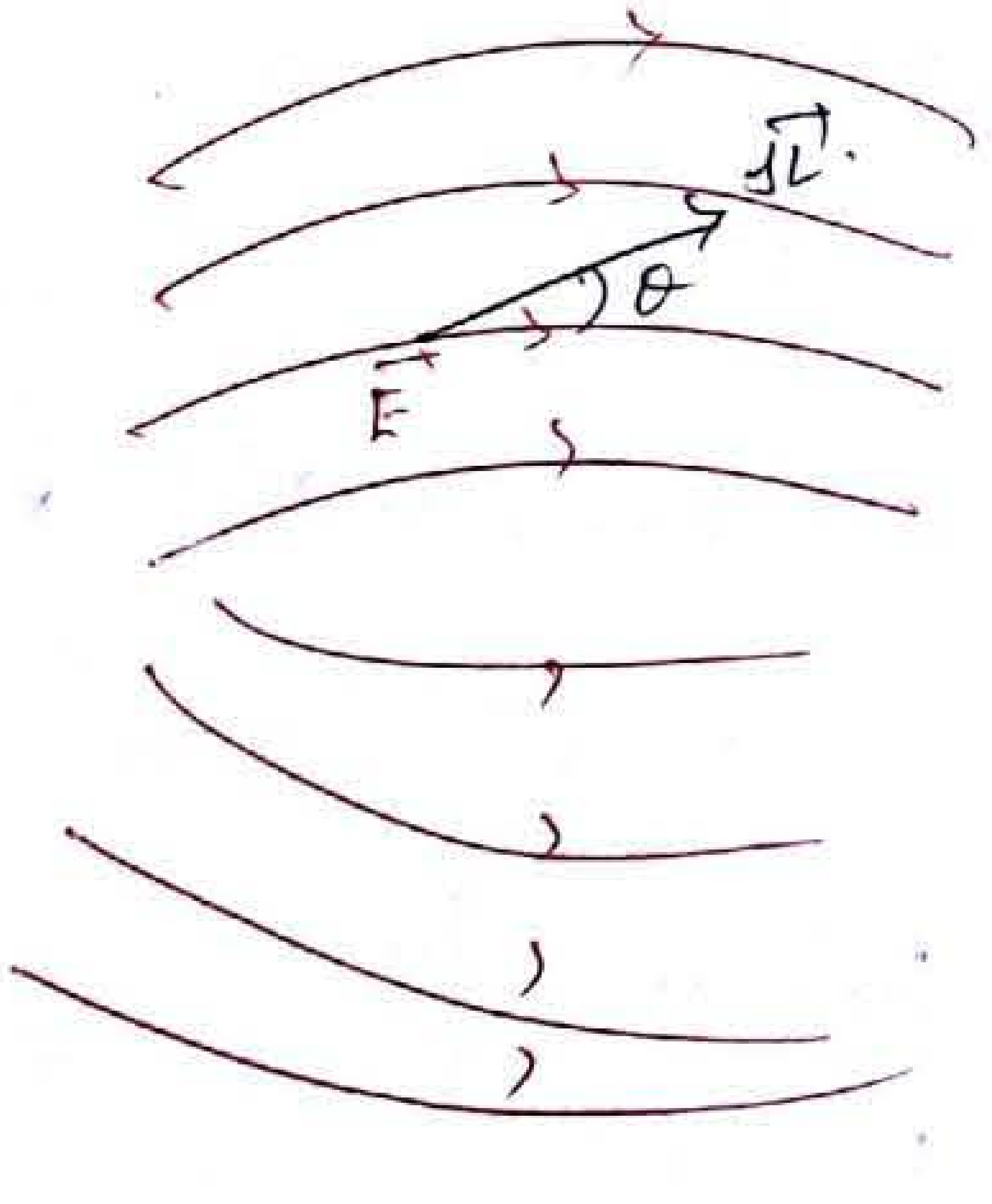
To find 1) capacitance between conductors.
2) charge or current distribution on conductors
↓ from which
3) Losses may be calculated.

Simple method to find $\vec{E}$ from potential:

$$V = -\int \vec{E}\cdot d\vec{L}$$

↓

applied to a very small length element, $\vec{\Delta L}$
along which $\vec{E}$ is constant.



$$\boxed{\Delta V = -\vec{E}\cdot\vec{\Delta L}}$$

$\vec{E}$ & V both change as we move from point to point

$$\vec{\Delta L} = \Delta L \, \vec{a_L}$$

$$\Delta V = -|\vec{E}| \cdot \Delta L \cdot \cos\theta$$

$$\boxed{\Delta V = -E \, \Delta L \cos\theta}$$

V may be interpreted as a fuction $V(x, y, z)$.

Consider derivative of $\frac{dV}{dL}$ = ?

V ← unique function of end point (x,y,z)

↓

V ← single valued function

$\vec{E}$ ← conservative field.

$$\boxed{\frac{dV}{dL} = -E\cos\theta}$$

In which dirn should $\vec{\Delta L}$ be placed to obtain maximum value of ΔV = ?

$$\boxed{\left.\frac{dV}{dL}\right|_{max} = E}$$

where $\cos\theta = -1$

$\theta = \pi$

Relationship b/w V & $\vec{E}$

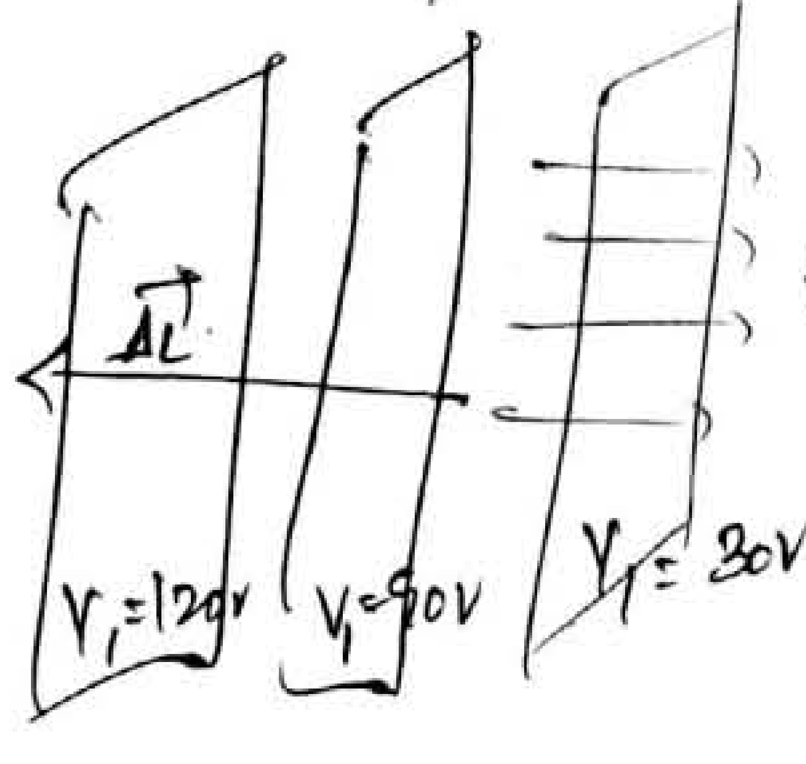
1) magnitude of $\vec{E}$ is given by maximum value of rate of change of potential with distance

2) This maximum value is attained when the direction of $\vec{\Delta L}$ is opposite to $\vec{E}$. Or

the direction of $\vec{E}$ is opposite to the direction in which the potential is increasing the most rapidly.

For equipotential surfaces,

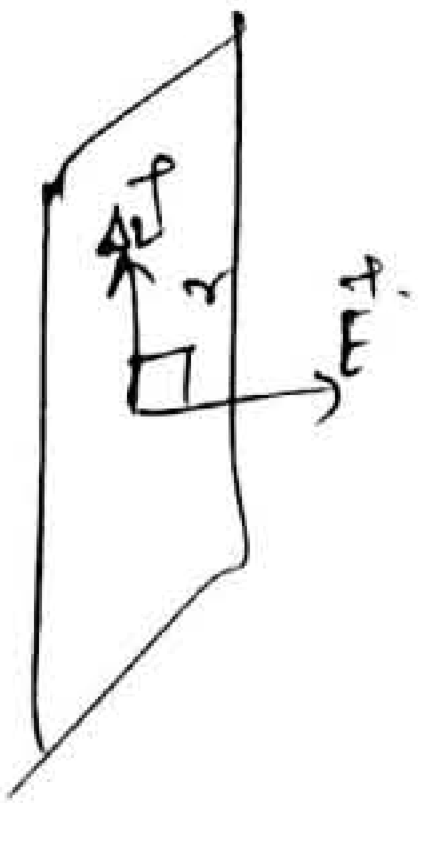
(i)



$\vec{E}$ ⊥r to the surface

$\vec{E}$ & $\vec{\Delta L}$ are oppositely directed

(ii)



Definition of equipotential surface

$\Delta V = -\vec{E}\cdot\vec{\Delta L} = 0$

$$\boxed{\vec{E} = -\left(\left.\frac{dV}{dL}\right|_{max}\right)\vec{n}}$$

direction of

1) $\vec{\delta L}$ is which maximum increase in potential

$\downarrow$

$\vec{a_L}$.

In terms of potential field (rather than $\vec{E}$).

2) $\vec{a_N}$ ← unit normal vector to the Equipotential surface & directed towards higher potentials

$$\boxed{\vec{E} = -\frac{dV}{dL}\Big|_{max} \vec{a_N}.}$$

$\downarrow$

max space rate of change of V.

$$\frac{dV}{dL}\Big|_{max} \vec{a_N} = \text{grad } V. \quad (\underline{\text{Gradient}})$$

$$\boxed{\vec{E} = -\text{grad } V.}$$

$V(x,y,z) = \& \; dV = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}.$

$dV = -\vec{E}\cdot\vec{dL} = -E_x dx - E_y dy - E_z dz.$

$E_x = -\frac{\partial V}{\partial x}$

$E_y = -\frac{\partial V}{\partial y}$

$E_z = -\frac{\partial V}{\partial z}.$

$$\boxed{-\nabla V = -\text{grad } V = \vec{E} = -\left(\frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}\right)}$$

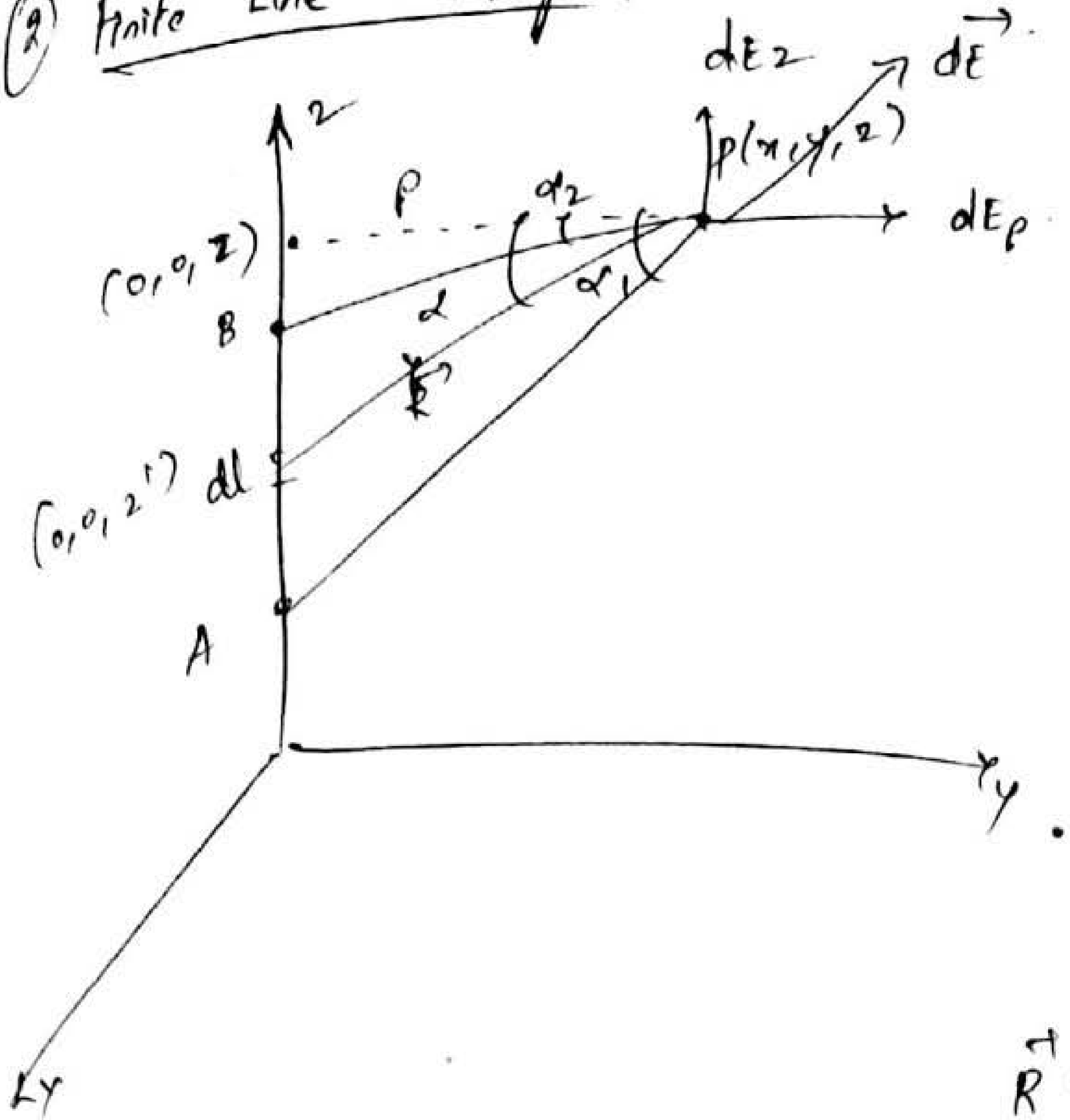
$$\boxed{\vec{E} = -\nabla V}$$

Rectangular $\Rightarrow$ $\boxed{\nabla V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}.}$

Cylindrical $\Rightarrow$ $\boxed{\nabla V = \frac{\partial V}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a_\phi} + \frac{\partial V}{\partial z}\vec{a_z}.}$

~~Cartesian~~ Spherical $\Rightarrow$ $\boxed{\nabla V = \frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi}}$

2. ② Finite Line charge:



Charge element

$$dQ = \rho_L \, dl$$

$$dQ = \rho_L \, dz' \qquad dl = dz'$$

Total charge on a the line, $Q = \int_{z_A}^{z_B} \rho_L \, dz'$.

$$\vec{R} = (z-z')\vec{a_z} + x\vec{a_x} + y\vec{a_y}$$

$$\vec{R} = (z-z')\vec{a_z} + \rho\vec{a_\rho}$$

$$\frac{\vec{a_R}}{|\vec{R}|^2} = \frac{\rho\vec{a_\rho} + (z-z')\vec{a_z}}{\left[\rho^2 + (z-z')^2\right]^{3/2}}$$

$$\frac{z-z'}{\rho} = \tan\alpha$$

$$\boxed{z-z' = \rho\tan\alpha}$$

$$d\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0} \cdot \frac{dz' \cdot \vec{a_R}}{|\vec{R}|^2}$$

$$|\vec{R}| = \sqrt{\rho^2 + (z-z')^2}$$

$$d\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0} \cdot \frac{dz' \cdot \vec{a}_R}{|\vec{R}|^2} \qquad |\vec{R}| = \sqrt{\rho^2+(z-z')^2}$$

$$d\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0} \cdot dz' \left(\frac{\rho\vec{a}_\rho + (z-z')\vec{a}_z}{[\rho^2+(z-z')^2]^{3/2}}\right) \qquad \frac{\rho}{|\vec{R}|} = \cos\alpha$$

$$|\vec{R}| = \rho\sec\alpha$$

$$\vec{E} = \int_{z=z_A}^{z_B} \frac{\rho_L}{4\pi\varepsilon_0} \frac{(\rho\vec{a}_\rho + (z-z')\vec{a}_z)}{[\rho^2+(z-z')^2]^{3/2}} \cdot$$

$$z' = OT - \rho\tan\alpha.$$

$$dz' = -\rho\sec^2\alpha\, d\alpha.$$

$$= +\int_{\alpha=\alpha_1}^{\alpha_2} \frac{\rho_L}{4\pi\varepsilon_0} \cdot \frac{(|\vec{R}|\cos\alpha\,\vec{a}_\rho + \rho\tan\alpha\,\vec{a}_z)\ \rho\,\cancel{\sec^2\alpha}}{\rho^3\sec^3\alpha}\,(-\rho\sec^2\alpha\, d\alpha)$$

z	z_A	z_B
α	α_1	α_2

$$= -\frac{\rho_L}{4\pi\varepsilon_0}\int_{\alpha=\alpha_1}^{\alpha_2} \frac{(\cancel{\rho}\sec\alpha\cos\alpha\,\vec{a}_\rho + \cancel{\rho}\tan\alpha\,\vec{a}_z)\,\cancel{\rho^2}\,\cancel{\sec^2}\alpha\, d\alpha}{\rho^{\cancel{2}}\sec^{\cancel{3}}\alpha.}$$

$$\vec{E} = -\frac{\rho_L}{4\pi\varepsilon_0\rho}\int_{\alpha=\alpha_1}^{\alpha_2} (\cos\alpha\,\vec{a}_\rho + \sin\alpha\,\vec{a}_z)\ d\alpha$$

[continued....]

[.... continuation]

$$\vec{E} = -\frac{\rho_L}{4\pi\varepsilon_0\rho}\left[(\sin\alpha_2 - \sin\alpha_1)\vec{a_\rho} - (\cos\alpha_2 - \cos\alpha_1)\vec{a_z}\right].$$

$$\boxed{\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\vec{a_\rho} + (\cos\alpha_2 - \cos\alpha_1)\vec{a_z}\right]}$$

Special case for an infinite line charge

B is at $(0,0,\infty)$ & A is at $(0,0,-\infty)$.

$\alpha_1 = \pi/2$., $\alpha_2 = -\pi/2$.

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0\rho} \cdot 2 \cdot \vec{a_\rho}$$

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a_\rho}}$$

Problem:

3. ⓘ Given that: $\vec{D} = z\rho\cos^2\phi\ \vec{a}_z$ C/m², Calculate charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1m with $-2 \leq z \leq 2$m. Verify using Divergence theorem.

Solution:

$$\rho_v = \nabla\cdot\vec{D} = \frac{\partial D_z}{\partial z} \quad \boxed{\rho_v = \rho\cos^2\phi} \ \text{C/m}^3$$

$$\boxed{\rho_v\Big|_{(1,\pi/4,3)} = 0.5} \ \text{C/m}^3$$

$$Q_{enc} = \int_v \rho_v\, dv = \int_{z=-2}^{2}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{1} \rho\cos^2\phi\, \rho d\rho\, d\phi\, dz$$

$$= \left[\frac{\rho^3}{3}\right]_0^1 \times \left[\frac{\phi + \frac{\sin 2\phi}{2}}{2}\right]_0^{2\pi} \times [z]_{-2}^{2}$$

$$= \frac{1}{3} \times \frac{2\pi}{2} \times 4$$

$$\boxed{Q_{enc} = \frac{4\pi}{3}} \ \text{C}$$

Divergence theorem:

$$\oint_s \vec{D}\cdot\vec{ds} = \int_v \rho_v\, dv.$$

$$\oint_s \vec{D}\cdot\vec{ds} = \int_{top} \vec{D}\cdot\vec{ds}_z + \int_{bottom} (-\vec{D})\cdot\vec{ds}_{-z} + \cancelto{0}{\int_{Lateral} \vec{D}\cdot\vec{ds}_\rho}.$$

$$= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{1} z\rho\cos^2\phi\ \rho d\rho\, d\phi\Big|_{z=2} - \int_{\phi=0}^{2\pi}\int_{\rho=0}^{1} z\rho\cos^2\phi\ \rho d\rho\, d\phi\Big|_{z=-2}$$

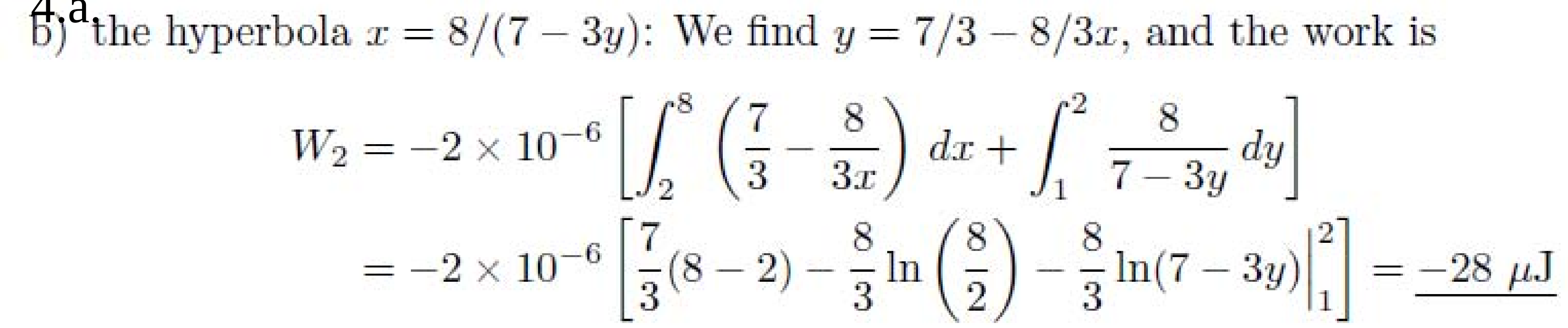
$$= 2\left[\frac{\rho^3}{3}\right]_0^1 \left[\frac{\phi}{2} + \frac{\sin 2\phi}{4}\right]_0^{2\pi} + 2\left[\frac{\rho^3}{3}\right]_0^1 \left[\frac{\phi}{2} + \frac{\sin 2\phi}{4}\right]_0^{2\pi}$$

$$= \frac{2\pi}{3} + \frac{2\pi}{3}$$

$$\boxed{\oint_s \vec{D}\cdot\vec{ds} = \frac{4\pi}{3} \ \text{C}}$$

Hence Verified.

4.a.



b) the hyperbola $x = 8/(7-3y)$: We find $y = 7/3 - 8/3x$, and the work is

$$W_2 = -2 \times 10^{-6} \left[\int_2^8 \left(\frac{7}{3} - \frac{8}{3x} \right) dx + \int_1^2 \frac{8}{7-3y}\, dy \right]$$

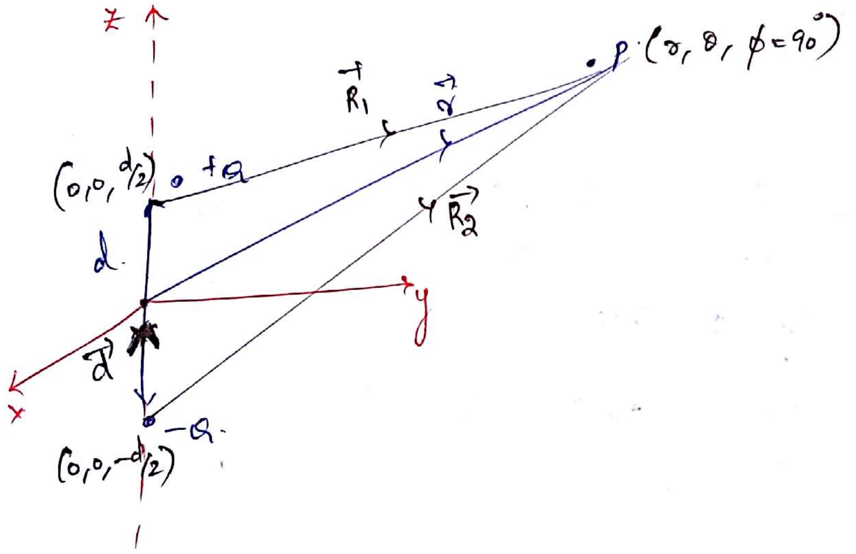
$$= -2 \times 10^{-6} \left[\frac{7}{3}(8-2) - \frac{8}{3} \ln \left(\frac{8}{2} \right) - \frac{8}{3} \ln(7-3y) \Big|_1^2 \right] = \underline{-28\ \mu\mathrm{J}}$$

4.b.

The electric dipole:

↓ study behaviour of dielectric materials in electric fields.

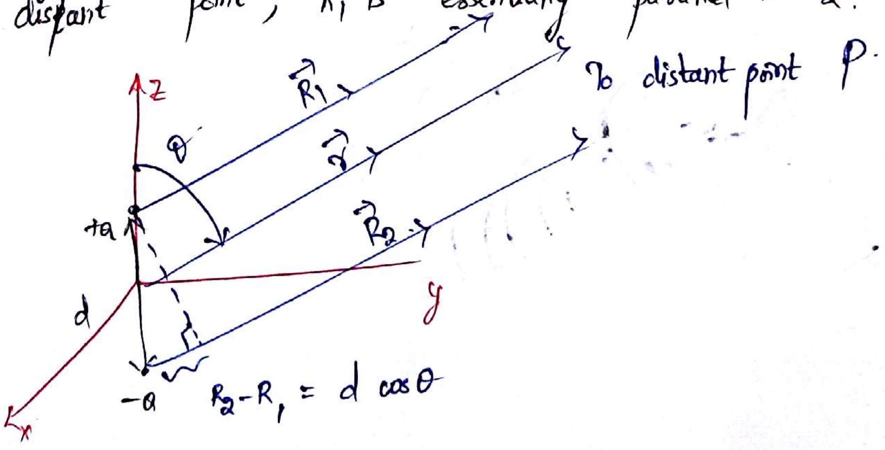
Electric dipole → two point charges of equal magnitude & opposite polarity, separated by a distance that is small compared to the distance at which we evaluate electric & potential fields.



Potential at P, $V = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$.

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{R_2 - R_1}{R_1 R_2}\right)$$

for a distant point, R_1 is essentially parallel to R_2.



$R_2 - R_1 = d\cos\theta$

For a distant point, $R_1 \simeq R_2 = r \Rightarrow R_1 R_2 = r^2$

$R_2 - R_1 = d\cos\theta$

If we more very far from the dipole, $R_2 - R_1 \simeq 0$

$\Downarrow$

potential field approaches zero.

Distant potential field of dipole:

$$\boxed{V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}}$$

Spherical coordinates:

$$\vec{E} = -\vec{\nabla}V = -\left[\frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi\right]$$

$$= -\left[\frac{Qd\cos\theta}{4\pi\varepsilon_0}\cdot(-2)\cdot\frac{1}{r^3}\vec{a}_r + \frac{1}{r}\cdot\frac{Qd}{4\pi\varepsilon_0 r^2}\cdot(-\sin\theta)\vec{a}_\theta + 0\right]$$

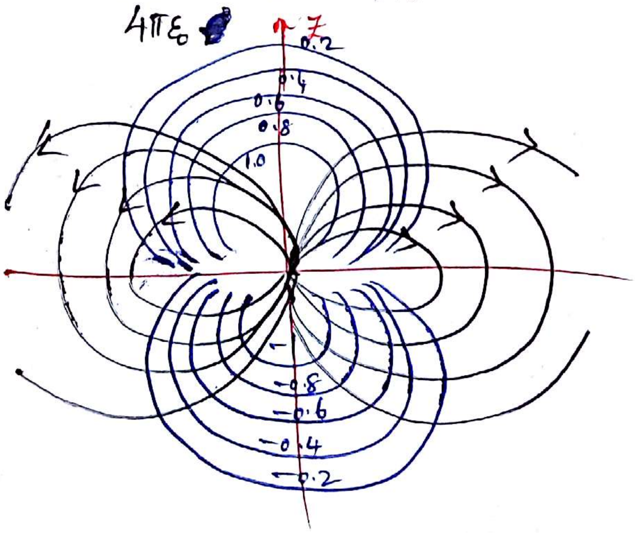
Distant electric field of dipole

$$\boxed{\vec{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}\left[2\cancel{d}\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta\right]}$$

Plot of potential field:

We choose a dipole, such that

$$\frac{Qd}{4\pi\varepsilon_0} = 1 \text{ and } \cos\theta = Vr^2$$



ii) 5.

$V = 2(x+1)^2 (y+2)^2 (z+3)^2 \qquad P(2,-1,4)$

$V \text{ at } P = 2(3)^2 (1)^2 (7)^2 = \underline{882 V}$

$E = -\nabla V = -\left[\frac{\partial V}{\partial x}\hat{a_x} + \frac{\partial V}{\partial y}\hat{a_y} + \frac{\partial V}{\partial z}\hat{a_z}\right]$

$= -\left[4(x+1)(y+2)^2(z+3)^2\hat{a_x} + 4(x+1)^2(y+2)(z+3)^2\hat{a_y} + 4(x+1)^2(y+2)^2(z+3)\hat{a_z}\right]$

$=$ E at $P(2,-1,4)$

$= -\left[4(3)(7)^2 a_x + 4(7)^2\hat{a_y} + 4(3)^2(7)\hat{a_z}\right]$

$$\boxed{\vec{E} = -588\hat{a_x} - 1764\hat{a_y} - 252\hat{a_z}}$$

$D = \epsilon_0\vec{E} = -\epsilon_0(588\hat{a_x} + 1764\hat{a_y} + 252\hat{a_z}]$

$= (-5.2061\hat{a}_x - 15.618\hat{a}_y - 2.231\hat{a}_z)$ nC/

$\rho_v = \nabla\cdot D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$

$= -\epsilon_0\left[4(y+2)^2(z+3)^2 + 4(x+1)^2(z+3)^2 + 4(x+1)^2(y+2)^2\right]$

$= -\epsilon_0\left[4(1)^2(7)^2 + 4(3)^2(7)^2 + 4(3)^2(1)^2\right] = 17.6726$

6.

Application of Gauss' law:

Differential Volume element.

No Symmetry.

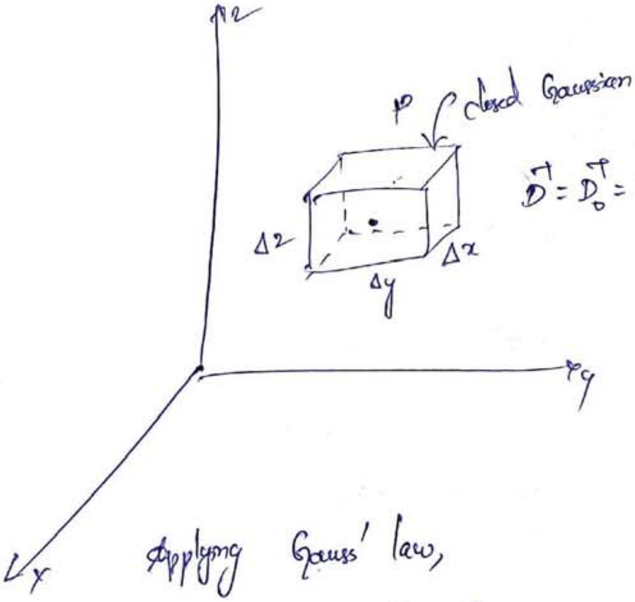
Choose a small gaussian surface → almost $\vec{D}$ is constant over that surface.

Result becomes correct only when volume $\Delta v \rightarrow 0$ (shrinks).

We will not obtain $\vec{D}$, obtain the valuable information about the way $\vec{D}$ varies in the region.

⇓

one of Maxwell's four equations (base to all electromagnetic theory).



$$\vec{D} = \vec{D}_0 = D_{x0}\vec{a}_x + D_{y0}\vec{a}_y + D_{z0}\vec{a}_z.$$

Applying Gauss' law,

$$\oint_S \vec{D}\cdot d\vec{s} = Q$$

$$\int_{front} + \int_{back} + \int_{left} + \int_{Right} + \int_{top} + \int_{bottom} \vec{D}\cdot d\vec{s} = Q.$$

Surface element is very small → $\vec{D}$ is constant over the surface.

$$\int_{front} \vec{D}\cdot d\vec{s} = \vec{D}_{front}\cdot \Delta\vec{S}_{front}$$

$$= \vec{D}_{front}\cdot \Delta y \Delta z\, \vec{a}_x$$

$$= D_{x,front}\cdot \Delta y \Delta z$$

front face ⇒ distance of $\frac{\Delta x}{2}$ from P.

$$\boxed{D_{x,front} = D_{x0} + \frac{\Delta x}{2} \cdot \frac{\partial D_x}{\partial x}}$$

value of D_x at P.

Rate of change of D_x with x

∵ D_x varies with y & z.

Expansion of Taylor's series.

$$\boxed{\int_{front} = \left(D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z}$$

Integral over the back surface

$$\int_{back} = \vec{D}_{back} \cdot \Delta \vec{S}_{back}$$

$$= \vec{D}_{back} \cdot \left(-\Delta y\,\Delta z\,\vec{a}_x\right)$$

$$\int_{back} = -D_{x,back}\,\Delta y\,\Delta z$$

$$\int_{back} = -\left[D_{x0} + \left(-\frac{\Delta x}{2}\right)\frac{\partial D_x}{\partial x}\right]\Delta y\,\Delta z$$

$$\boxed{\int_{back} = \left(-D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z}$$

$$\boxed{\int_{front} + \int_{back} = \frac{\partial D_x}{\partial x}\,\Delta x\,\Delta y\,\Delta z}$$

Exactly in the same way

$$\int_{right} + \int_{left} = \frac{\partial D_y}{\partial y}\,\Delta x\,\Delta y\,\Delta z$$

&

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z}\,\Delta x\,\Delta y\,\Delta z$$

$$\therefore \oint_S \vec{D}\cdot d\vec{s} = \frac{\partial D_x}{\partial x}\Delta x\Delta y\Delta z + \frac{\partial D_y}{\partial y}\Delta x\Delta y\Delta z + \frac{\partial D_z}{\partial z}\Delta x\Delta y\Delta z$$

$$\boxed{\oint_S \vec{D}\cdot d\vec{s} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x\Delta y\Delta z = Q}$$

↓
It is an approximation which is better when ΔV becomes smaller.

$\therefore \Delta v \to 0.$ when $\Delta v = \Delta x\Delta y\Delta z$

$$\boxed{\text{Charge enclosed in volume } \Delta v = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\cdot\Delta v}$$

Problem:
Find an approximate value for the total charge enclosed in an incremental volume of $10^{-9}\,m^3$ located at origin, if $\vec{D} = e^{-x}\sin y\,\vec{a_x} - e^{-x}\cos y\,\vec{a_y} + 2z\vec{a_z}$ C/m²

$$\left.\begin{aligned}\frac{\partial D_x}{\partial x} &= -e^{-x}\sin y.\\ \frac{\partial D_y}{\partial y} &= e^{-x}\sin y.\end{aligned}\right\} \Rightarrow 0.$$

↑ At origin

$\frac{\partial D_z}{\partial z} = 2.$

$Q = (2)\times\Delta v.$

$\therefore \boxed{Q = 2\ nC}$.

We now obtain the exact relationship by $\Delta v \to 0.$

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \frac{\oint \vec{D}\cdot d\vec{s}}{\Delta v} = \frac{Q}{\Delta v}.$$

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v\to 0}\frac{\oint_S \vec{D}\cdot d\vec{s}}{\Delta v} = \left(\lim_{\Delta v\to 0}\frac{Q}{\Delta v}\right) \to \text{volume charge density}$$

$$\boxed{\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta v\to 0}\frac{\oint_S \vec{D}\cdot d\vec{s}}{\Delta v}} \to ①$$

&

$$\boxed{\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v,} \to ②$$

Divergence of $\vec{D}$: $\text{div}\,\vec{D} = \lim_{\Delta v \to 0} \dfrac{\oint_S \vec{D} \cdot d\vec{s}}{\Delta v}$

Divergence of the vector flux density $\vec{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

$$\boxed{\text{div}\,\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}}$$ Rectangular

$$\boxed{\text{div}\,\vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}}$$ cylindrical

$$\boxed{\text{div}\,\vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}.}$$ spherical

Problem:

Find $\text{Div}\,\vec{D}$ at origin $\vec{D} = e^{-x}\sin y\,\vec{a_x} - e^{-x}\cos y\,\vec{a_y} + 2z\vec{a_z}$.

$\text{div}\,\vec{D} = 2 = \rho_v$. C/m^3.

$\downarrow$

volume charge density.

(X.) MAXWELL's FIRST EQUATION OF ELECTROSTATICS:

1) $\operatorname{div} \vec{D} = \lim_{\Delta v \to 0} \dfrac{\oint_S \vec{D}\cdot d\vec{s}}{\Delta v}$ (Definition of divergence)

2) $\operatorname{div} \vec{D} = \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z}$ (Result of applying The definition to a differential volume element (In Rectangular co-ordinates))

3) $\operatorname{div} \vec{D} = \rho_v$

<u>Gauss's law</u> flux leaving any closed surface

$$\boxed{\oint_S \vec{D}\cdot d\vec{s} = Q} \leftarrow \text{charge enclosed}$$

Gauss' law per unit volume,

$$\frac{\oint_S \vec{D}\cdot d\vec{s}}{\Delta v} = \frac{Q}{\Delta V}.$$

As the volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{\oint \vec{D}\cdot d\vec{s}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} \leftarrow \text{volume charge density}$$

Divergence →

$$\boxed{\text{div } \vec{D} = \rho_v.}$$

(X.)

⇓ First of Maxwell's four equations

Point form of Gauss's law (or) Maxwell's first equation

Statement:

The electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there.

Gauss's law → $$\boxed{\oint_S \vec{D}\cdot d\vec{s} = Q = \int_v \rho_v \cdot dv.}$$

(XI) Integral form of Maxwell's first equation

Specific Illustration,

$\vec{D}$ in the region about point charge

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{a_r}.$$

(Spherical system)

$$\text{div } \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} D_\phi$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2 \cdot Q}{4\pi r^2}\right)$$

$$\boxed{\text{div.} \vec{D} = 0} \quad \text{if } r \neq 0.$$

$\rho_v = 0$ everywhere except at origin.

$\rho_v = \infty$ (at origin)

Vector operator ∇ & Divergence theorem:

∇ (Del) operator:

$$\nabla = \frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z}.$$

$$\vec{D} = D_x\vec{a_x} + D_y\vec{a_y} + D_z\vec{a_z}.$$

Dot operation

$$\boxed{\text{div }\vec{D} = \nabla\cdot\vec{D} = \frac{\partial}{\partial x}D_x + \frac{\partial}{\partial y}D_y + \frac{\partial}{\partial z}D_z}.$$

∇ operator on a scalar:

$$\boxed{\nabla u = \frac{\partial}{\partial x}u\,\vec{a_x} + \frac{\partial}{\partial y}u\,\vec{a_y} + \frac{\partial}{\partial z}u\,\vec{a_z}} \leftarrow \text{vector}.$$

Gradient of a scalar.

Divergence theorem: (For Electric flux density).

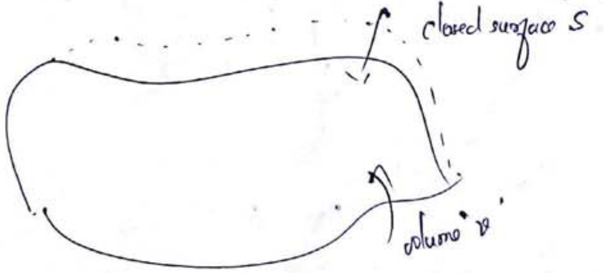
Gauss' law:

$$\oint_S \vec{D}\cdot\vec{ds} = Q = \int_V \rho_v\,dv.$$

$$\boxed{\oint_S \vec{D}\cdot\vec{ds} = \int_V \nabla\cdot\vec{D}\,dV.} \leftarrow \text{Divergence theorem.}$$

Statement:

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.



Divergence of flux density throughout a volume
=
net flux crossing the enclosing surface.

7.a.

5) a) Tranform $\vec{F} = 10\hat{a}_x - 8\hat{a}_y + 6\hat{a}_z$ to cylindrical co-ordi at $P(10, -8, -6)$

$$F_\rho = \vec{F}\cdot\hat{a}_\rho$$

$$= 10\hat{a}_x\cdot\hat{a}_\rho - 8\hat{a}_y\cdot\hat{a}_\rho + 6\hat{a}_z\cdot\hat{a}_\rho$$

$$= 10\cos\phi - 8\sin\phi + 0$$

$$\phi = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{-8}{10}\right) = -38.66^\circ$$

$$F_\rho = 10\cos(-38.60) - 8\sin(-38.66)$$

$$= 12.81$$

$$F_\phi = 10\hat{a}_x\cdot\hat{a}_\phi - 8\hat{a}_y\cdot\hat{a}_\phi + 6\hat{a}_z\cdot\hat{a}_\phi$$

$$= -10\sin\phi - 8\cos\phi$$

$$= -10\sin(-38.66) - 8\cos(-38.66)$$

$$= 4.28\times10^{-5} \approx 0$$

$$F_z = \vec{F}\cdot\hat{a}_z = 6$$

$$\vec{F} = 12.81\hat{a}_\rho + 4.28\times10^{-5}\hat{a}_\phi + 6\hat{a}_z$$

$$\vec{F} \simeq 12.81\hat{a}_\rho + 6\hat{a}_z$$

7.b. b) Find rectangular components of :

$$H = 20\,\hat{a}_\rho - 10\,\hat{a}_\phi + 3\,\hat{a}_z \text{ at } P(5,2,-1)$$

$$\phi = \tan^{-1}(y/x)$$

$$= \tan^{-1}(2/5)$$

$$= 21.8$$

$$H_x = H \cdot \hat{a}_x$$

$$= 20\,\hat{a}_\rho \cdot \hat{a}_x - 10\,\hat{a}_\phi \cdot \hat{a}_x + 3\,\hat{a}_z \cdot \hat{a}_x$$

$$= 20\cos\phi + 10\sin\phi + 0$$

$$= 20\cos(21.8) + 10\sin(21.8)$$

$$= 22.28$$

$$H_y = H\hat{a}_y = 20\,\hat{a}_\rho \cdot \hat{a}_y - 10\,\hat{a}_\phi \cdot \hat{a}_y + 3\,\hat{a}_z \cdot \hat{a}_y$$

$$= 20\sin\phi - 10\cos\phi + 0$$

$$= -1.857$$

$$H_z = 3$$

$$H = 22.28\,\hat{a}_x - 1.857\,\hat{a}_y + 3\,\hat{a}_z$$

8. a. $\vec{E} = +\frac{200}{r^2}\vec{a}_r$

$$\vec{D} = \varepsilon_0\vec{E} = \varepsilon_0 \times \frac{200}{r^2}\vec{a}_r$$

$$W_E = \frac{1}{2}\int_v \varepsilon_0 |\vec{E}|^2 dv$$

$$= \frac{1}{2}\int_v \varepsilon_0 \cdot \left[\frac{200}{r^2}\right]^2 \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{1}{2}\cdot\varepsilon_0\cdot(200)^2 \iiint \frac{1}{r^2} \, dr \, \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{2}\cdot\varepsilon_0 \times (200)^2 \times \left[-\frac{1}{r}\right]_{2mm}^{3m} \left[-\cos\theta\right]_0^{\pi/2} \left[\phi\right]_0^{\pi/2}$$

$$= \varepsilon_0\pi \cdot \frac{40000}{4} \times \frac{1}{6\times10^{-3}}$$

$$= \frac{10^{-9}}{36} \times \frac{10000}{6\times10^{-3}}$$

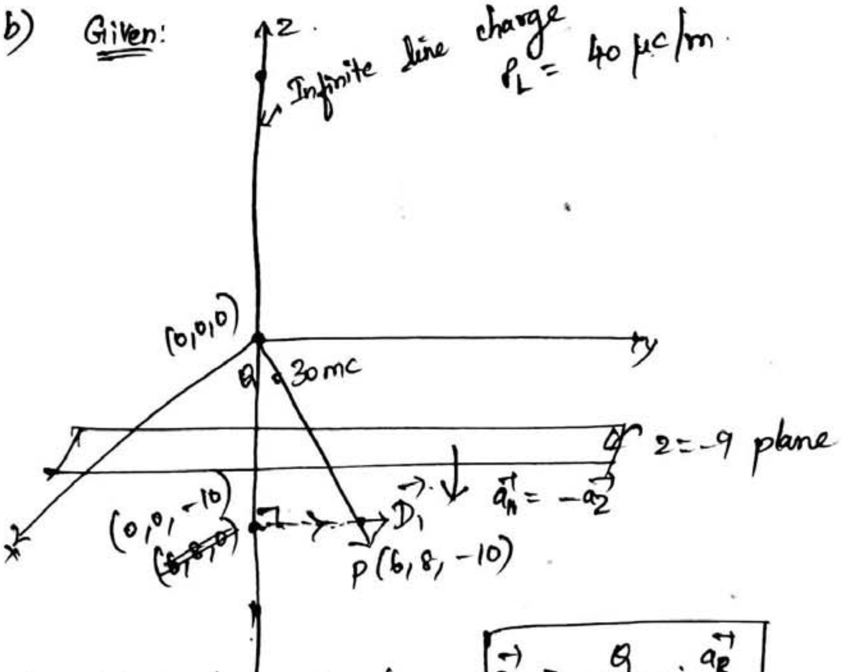
$$= 46.296\times10^{4}\times10^{-9}\times10^{+3}\times10^{4}$$

$$= 46.296\times10^{-6}\ J$$

$$\boxed{W_E = 46.296\,\mu J}$$

8.b.

1) b) Given:



i) $\vec{D}$ due to point charge, $\boxed{\vec{D_1} = \frac{Q}{4\pi|\vec{R}|^2} \cdot \vec{a_R}}$

$\vec{R} = 6\vec{a_x} + 8\vec{a_y} - 10\vec{a_z}$

$|\vec{R}| = \sqrt{36+64+100} = 10\sqrt{2}$

$$= \frac{30\times10^{-3}}{4\pi} \times \frac{[6\vec{a_x} + 8\vec{a_y} - 10\vec{a_z}]}{(10\sqrt{2})^3}$$

$$\boxed{\vec{D} = 5.06\,\vec{a_x} + 6.75\,\vec{a_y} - 8.44\,\vec{a_z}}\ \mu C/m^2$$

(ii) $\vec{D}$ due to infinite line charge, $\boxed{\vec{D_2} = \frac{\rho_L}{2\pi\rho}\,\vec{a_\rho}}$

$\vec{a_R} = 6\vec{a_x} + 8\vec{a_y}$

$|\vec{a_R}| = \sqrt{6^2+8^2} = 10$

$$= \frac{40\times10^{-6}}{2\pi\times10} \times \frac{(6\vec{a_x} + 8\vec{a_y})}{10}$$

$$\boxed{\vec{D} = 0.381\,\vec{a_x} + 0.509\,\vec{a_y}}\ \mu C/m^2$$

(iii) $\vec{D}$ due to surface charge, $\vec{D_3} = \frac{\rho_s}{2}\,\vec{a_N}$

$$= \frac{57.2\times10^{-6}}{2}\,(-\vec{a_z})$$

$$\boxed{\vec{D_3} = -28.6\,\vec{a_z}}\ \mu C/m^2$$