CMR INSTITUTE OF TECHNOLOGY

USN

Internal Assessment Test II

Sub:	ENGINEERING MATHEMATICS IV					Code:	15MAT41
Date:	8 / 05 / 2017	Duration:	90 mins	Max Marks:	50	Sem: IV	Branch: ECE:A,B,C

Question 1 is compulsory. Answer any SIX questions from the rest.

Question	Marks	OBE CO	OBE RBT
1. Derive mean and variance of a binomial distribution.	[08]	CO4	L3
2. With usual notations, show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.	[07]	CO2	L3
3. Show that $\int_0^1 x\, J_n(\alpha x) J_n(\beta x) dx = 0$, where α and β are roots of $J_n(x) = 0$ and $\alpha \neq \beta$.	[07]	CO2	L3
4. A random variable X has the density function $p(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$. Determine k and evaluate i) $P(x \geq 0)$ ii) $P(0 < x < 1)$	[07]	CO4	L3
5. Given that 2% of fuses manufactured by a firm are defective, find by using Poisson distribution, the probability that a box containing 200 fuses has i) at least one defective fuse ii) 3 or more defective fuses.	[07]	CO4	L3

CMR INSTITUTE OF TECHNOLOGY

USN

Internal Assessment Test II

Sub:	ENGINEERING MATHEMATICS IV					Code:	15MAT51
Date:	8 / 05 / 2017	Duration:	90 mins	Max Marks:	50	Sem: IV	Branch: ECE:A,B,C

Question 1 is compulsory. Answer any SIX questions from the rest.

ii) 3 or more defective fuses.

Question	Marks	OBE CO	OBE RBT
1. Derive mean and variance of a binomial distribution.	[08]	CO4	L3
2. With usual notations, show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.	[07]	CO2	L3
3. Show that $\int_0^1 x\, J_n(\alpha x) J_n(\beta x) dx = 0$, where α and β are roots of $J_n(x) = 0$ and $\alpha \neq \beta$.	[07]	CO2	L3
4. A random variable X has the density function $p(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$. Determine k and evaluate i) $P(x \geq 0)$ ii) $P(0 < x < 1)$	[07]	CO4	L3
5. Given that 2% of fuses manufactured by a firm are defective, find by using Poisson distribution, the probability that a box containing 200 fuses has i) at least one defective fuse ii) 3 or more defective fuses.	[07]	CO4	L3

6. The average daily turnout in a medical store is ₹10,000 and the net profit is 8%. If the turnout has an exponential distribution, find the probability that the net profit will exceed ₹ 3000 each on two consecutive days. [07] CO4 L3

7. The marks of 1000 students in an examination form a normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be (i) less than 65 (ii) more than 75 (iii) between 65 and 75, given that $\varphi(1) = 0.3413$. [07] CO4 L3

8. The joint probability distribution of two random variables X and Y is as follows: [07] CO6 L3

X \ Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Find (a) marginal distributions of X and Y (b) $E(X)$ and $E(Y)$ (c) $E(XY)$ (d) $Cov\,(X,Y)$

9. Show that the Markov chain whose transition probability matrix is $\begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is irreducible and find the stationary probability vector. [07] CO6 L3

6. The average daily turnout in a medical store is ₹10,000 and the net profit is 8%. If the turnout has an exponential distribution, find the probability that the net profit will exceed ₹ 3000 each on two consecutive days. [07] CO4 L3

7. The marks of 1000 students in an examination form a normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be (i) less than 65 (ii) more than 75 (iii) between 65 and 75, given that $\varphi(1) = 0.3413$. [07] CO4 L3

8. The joint probability distribution of two random variables X and Y is as follows: [07] CO6 L3

X \ Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Find (a) marginal distributions of X and Y (b) $E(X)$ and $E(Y)$ (c) $E(XY)$ (d) $Cov\,(X,Y)$

(e) (X,Y) . Are X and Y independent random variables?

9. Show that the Markov chain whose transition probability matrix is $\begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is irreducible and find the stationary probability vector. [07] CO6 L3

1. Mean and variance of binomial distribution

$P(x) = nC_x p^x q^{n-x}$, where p is the probability of success and q, the prob. of failure

1

$$\text{Mean } \mu = \sum_{x=0}^{n} x\,P(x) = \sum_{x=0}^{n} x \cdot nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{(n-1)!\, np\, p^{x-1} q^{(n-1)-(x-1)}}{(x-1)!\,(n-1-\overline{x-1})!}$$

$$= np \sum_{x=1}^{n} (n-1)C_{(x-1)} p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\,(q+p)^{n-1} = \underline{np}$$

2

$$\text{Variance } V = \sum_{x=0}^{n} x^2 P(x) - \mu^2$$

$$= \sum_{x=0}^{n} \left[x(x-1) + x\right] P(x) - n^2p^2$$

$$= \sum_{x=0}^{n} x(x-1)P(x) + \sum_{x=0}^{n} x\,P(x) - n^2p^2$$

$$= \sum_{x=0}^{n} x(x-1)P(x) + np - n^2p^2$$

$$= \sum_{x=0}^{n} x(x-1)\, nC_x p^x q^{n-x} + np - n^2p^2$$

$$= \sum_{x=0}^{n} \frac{\cancel{x(x-1)}\, n!}{\cancel{x(x-1)}(x-2)!\,(n-x)!} p^x q^{n-x} + np - n^2p^2$$

$$= p^2 \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(x-2)!\,\{(n-2)-(x-2)\}!} p^{x-2} q^{n-x} + np - n^2p^2$$

$$= n(n-1)p^2 \sum_{2} (n-2)C_{(x-2)} p^{x-2} q^{n-x} + np - n^2p^2$$

$$= n(n-1)p^2 (q+p)^{n-2} + np - n^2p^2$$

4 marks

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np(1-p) = \underline{npq}$$

2. To s.t $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \quad \text{— 1 mark}$$

$$\therefore J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{\frac{1}{2}+2r} \frac{1}{\Gamma\left(r+\frac{3}{2}\right) r!}$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma\left(\frac{5}{2}\right)1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma\left(\frac{7}{2}\right)2!} + \cdots\right] \quad \text{— (1)} \quad \text{— 2 marks}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}, \quad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma\left(\tfrac{7}{2}\right) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \frac{15\sqrt{\pi}}{8}. \quad \text{— 1 mark}$$

Sub. in (1), $J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4} \cdot \frac{4}{3\sqrt{\pi}} + \frac{x^4}{16} \cdot \frac{8}{15\sqrt{\pi}\cdot 2} - \cdots\right]$

$$= \sqrt{\frac{x}{2\pi}} \left[2 - \frac{x^2}{3} + \frac{x^4}{60} - \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}} \cdot \frac{2}{x} \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x \quad \text{— 3 marks}$$

3. To s.t $\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0$, given $J_n(\alpha) = J_n(\beta) = 0$

$J_n(\lambda x)$ is a solution of the eqn:

$$x^2 y'' + x y' + (\lambda^2 x^2 - n^2) y = 0 \quad \text{— 1 mark}$$

Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$. then,

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0 \quad \text{— (1)}$$

$$\& \quad x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0 \quad \text{— (2)}$$

Multiplying ① by $\frac{v}{x}$ and ② by $\frac{u}{x}$, we get

$$xvu'' + vu' + \alpha^2 uvx - n^2 \frac{uv}{x} = 0$$

$$xuv'' + uv' + \beta^2 uvx - n^2 \frac{uv}{x} = 0$$ — 1 mark

Subtracting,

$$x(vu'' - uv'') + (vu' - uv') + (\alpha^2 - \beta^2) uvx = 0$$

i.e. $$\frac{d}{dx}\left[x(vu' - uv')\right] = (\beta^2 - \alpha^2) uvx$$ — 1 mark

Integrating both sides w.r.t x b/w 0 & 1,

$$x(vu' - uv')\Big]_0^1 = (\beta^2 - \alpha^2)\int_0^1 uvx\,dx \quad \text{—— ③}$$ — 1 mark

$u = J_n(\alpha x) \Rightarrow u' = \alpha J_n'(\alpha x)$

$v = J_n(\beta x) \Rightarrow v' = \beta J_n'(\beta x)$

$\therefore$ ③ $\Rightarrow$ $$x\left[\alpha J_n(\beta x) J_n'(\alpha x) - \beta J_n(\alpha x) J_n'(\beta x)\right]\Big)_0^1$$

$$= (\beta^2 - \alpha^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$$

When $x = 1$, $J_n(\beta x) = J_n(\beta) = 0$ and $J_n(\alpha x) = J_n(\alpha) = 0$.

$\therefore$ L.S $= 0$.

$\therefore$ $$(\beta^2 - \alpha^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$$

— 3 marks

$$\Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0 \quad (\because \alpha \neq \beta)$$

4. $p(x) = \dfrac{k}{1+x^2}, \quad -\infty < x < \infty.$

a) $p(x) \geq 0 \quad \forall x \Rightarrow k \geq 0$

b) $\int_{-\infty}^{\infty} p(x)\,dx = 1 \Rightarrow k\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 1$

$\Rightarrow 2k \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = 1$

$\Rightarrow 2k \left. \tan^{-1}x \right)_0^{\infty} = 1$

$\Rightarrow 2k \cdot \frac{\pi}{2} = 1 \Rightarrow \boxed{k = \frac{1}{\pi}}$ — 3 marks

i) $P(x \geq 0) = \int_0^{\infty} p(x)\,dx = \int_0^{\infty} \frac{1}{\pi(1+x^2)}\,dx$

$= \frac{1}{\pi} \left. \tan^{-1}x \right)_0^{\infty} = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$ — 2 marks

ii) $P(0 < x < 1) = \int_0^1 p(x)\,dx = \frac{1}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx$

$= \frac{1}{\pi} \left. \tan^{-1}x \right)_0^1 = \frac{1}{\pi} \cdot \frac{\pi}{4} = \frac{1}{4}$ — 2 marks

5. $p = 0.02 \quad n = 200 \quad m = np = 4$ → 1 mark ; $P(x) = \dfrac{e^{-m} m^x}{x!}$ — 1 mark

$P(X \geq 1) = 1 - P(X < 1) = 1 - P(0)$

$= 1 - \left[\frac{e^{-4}4^0}{0!}\right] = 1 - e^{-4} = 1 - 0.0183 = 0.9817$

— 2 marks

$P(X \geq 3) = 1 - [P(0) + P(1) + P(2)]$

$= 1 - \left[e^{-4} + \frac{e^{-4}4}{1!} + e^{-4}\cdot\frac{4^2}{2!}\right]$

$= 1 - e^{-4}[1 + 4 + 8] = 1 - 13e^{-4}$

$= 0.7621$ — 3 marks

6: X : daily turnover

Average daily turnover = ₹ 10,000.

p.d.f of exp. distribution:

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$ — 1 mark

$$\text{Mean} = \frac{1}{\alpha} \Rightarrow \alpha = \frac{1}{10{,}000}$$

$$\therefore f(x) = \begin{cases} \frac{1}{10{,}000} e^{-\frac{1}{10{,}000}x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$ — 2 marks

$$\text{Profit} = \frac{8}{100} \times \text{Sales} \Rightarrow \text{Sales} = \text{Profit} \times \frac{100}{8}.$$

$$\text{Profit} > ₹\,3000 \Rightarrow \text{Sales} > 3000 \times \frac{100}{8} = ₹\,37{,}500$$ — 1 mark

$$P(X > 37{,}500) = \int_{37{,}500}^{\infty} \frac{1}{10{,}000} e^{-\frac{1}{10{,}000}x}\,dx = -e^{-\frac{1}{10{,}000}x}\Big)_{37{,}500}^{\infty}$$

$$= e^{-3.75}$$ — 2 marks

$$P(X > 37{,}500 \text{ on 2 Consecutive days}) = e^{-3.75} \cdot e^{-3.75}$$

$$= e^{-7.5} = \underline{\underline{0.000553}}$$ — 1 mark.

7. $\mu = 70, \quad \sigma = 5$

$$z = \frac{x - \mu}{\sigma} = \frac{x - 70}{5}$$ — 1 mark

$$P(X < 65) = P\left(z < \frac{65 - 70}{5}\right) = P(z < -1)$$

$$= P(z > 1) = 0.5 - \phi(1) = 0.5 - 0.3413$$

$$= 0.1587$$

$$P(X > 75) = P\left(z > \frac{75 - 70}{5}\right) = P(z > 1) = 0.1587$$

$$P(65 < X < 75) = P(-1 < z < 1) = 2\phi(1) = 0.6826$$

$\therefore$ No: of students who scored less than 65 = 159

— 2 marks

No: of students who scored greater than 75 = 159 — 2 marks

and No: who scored b/w 65 & 75 = 683, approx — 2 marks

8. a)

$X = x_i$	1	5	$Y = y_j$	-4	2	7
$f(x_i)$	$1/2$	$1/2$	$g(y_j)$	$3/8$	$3/8$	$1/4$

— 1 marks

b) $E(X) = \sum x_i f(x_i) = \frac{1}{2} + \frac{5}{2} = 3$

$E(Y) = \sum y_j g(y_j) = -\frac{12}{8} + \frac{6}{8} + \frac{7}{4} = 1$ — 1 mark

c) $E(XY) = \sum_i \sum_j x_i y_j \, p(x_i, y_j)$

$= -\frac{4}{8} + \frac{2}{4} + \frac{7}{8} - \frac{20}{4} + \frac{10}{8} + \frac{35}{8}$ — 1 mark

$= \frac{12}{8} = \frac{3}{2}$

d) $Cov(X, Y) = E(XY) - E(X)E(Y)$ — 1 mark

$= \frac{3}{2} - 3 \cdot 1 = -\frac{3}{2}$

e) $\rho(X, Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$

$Var(X) = E(X^2) - [E(X)]^2$

$= 1^2 \times \frac{1}{2} + 5^2 \times \frac{1}{2} - 9 = 13 - 9 = 4$

$\sigma_x = 2$

$Var(Y) = E(Y^2) - [E(Y)]^2$

$= 16 \times \frac{3}{8} + 4 \times \frac{3}{8} + 49 \times \frac{1}{4} - 1$

$= \frac{48 + 12 + 98 - 8}{8} = \frac{150}{8}$

$\sigma_y = 4.33$

$\rho(X, Y) = \frac{-3}{2 \times 2 \times 4.33} = \underline{-0.1732}$ — 2 marks

$\rho(X, Y) \neq 0 \implies X$ and Y are not independent — 1 mark

9. $P = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$P^2 = \begin{bmatrix} 0.38 & 0.63 & 0 \\ 0.25 & 0.63 & 0.13 \\ 0.5 & 0.5 & 0 \end{bmatrix}$ —— 1 mark

$P^3 = \begin{bmatrix} 0.31 & 0.59 & 0.09 \\ 0.31 & 0.63 & 0.06 \\ 0.25 & 0.63 & 0.13 \end{bmatrix}$ —— 1 mark

All elements > 0 $\therefore$ P is a regular stochastic matrix $\Rightarrow$ the Markov chain is irreducible —— 1 mark

To find a vector $v = (x, y, z)$ such that $x + y + z = 1$ —① and $vP = v$ —— 1 mark

ie $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$

$\frac{1}{2}y = x$; $\frac{3}{4}x + \frac{1}{2}y = y$; $\frac{1}{4}x = z$

$y = 2x$, ~~$x = 4z$~~ . $z = \frac{x}{4}$

Sub in ①, $x + 2x + \frac{x}{4} = 1$

$\frac{13}{4}x = 1 \Rightarrow x = \frac{4}{13}$

$\therefore$ $y = \frac{8}{13}$ and $z = \frac{1}{13}$

$\therefore$ the stationary probability vector is $\left(\frac{4}{13}, \frac{8}{13}, \frac{1}{13}\right)$ —— 3 marks