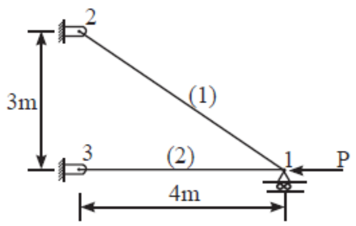
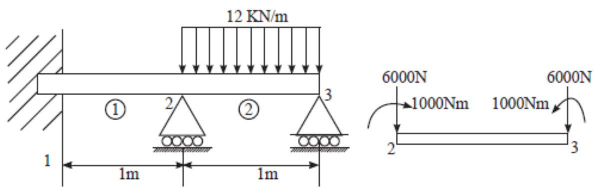
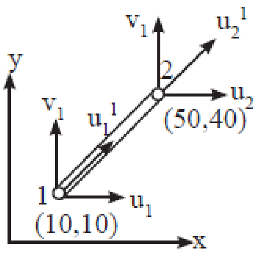
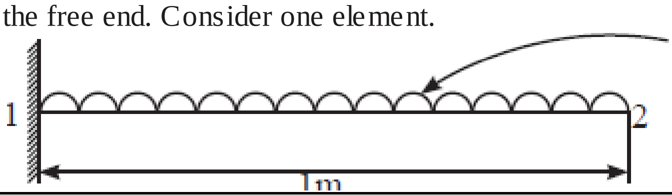
CMR INSTITUTE OF TECHNOLOGY

Internal Assessment Test II – May 2017

Sub:	Finite Element Methods		**Code:**	10ME64
Date: 09/ 05 / 2017	**Duration:** 90 mins	**Max Marks:** 50	**Sem:** VI	**Branch:** MECH

Note: Answer all questions.

		Marks	OBE	
			CO	RBT
1	a. Derive the stiffness matrix for 2D truss element.	08	CO4	L3
	b. For the two-bar truss shown in Figure 1. Determine the nodal displacements element stresses and support reactions. A force of P = 1000 KN is applied at node 1. Assume E = 210GPa and A = 600mm^2 for each element.  Figure 1  Figure 2	12	CO3	L3
2	a. Derive the Hermite shape functions for a beam element.	08	CO3	L3
	b. For the beam and loading shown in Figure 2, determine i. Slopes at 2 and 3 ii. The vertical deflection at the midpoint of the distributed load. Take E = 200 GPa, I = 4 × 10^6 mm^4.	12	CO4	L3
3	For the truss element shown in below figure (x, y) co-ordinates of the element are indicated near nodes 1, 2. The element displacement dof vector is given by [u] = [1.5, 1.0, 2.1, 4.3] $^T \times 10^{-2}$ mm Take E = 300×10^3 N/mm^2 A = 100 mm^2 Determine the following a) Element displacement dof in local co-ordinates b) Stress in the element c) Stiffness matrix of the element  **OR**	10	CO4	L2
4	For the Cantilever beam subjected to UDL as shown in figure, determine the deflections of the free end. Consider one element.  P_o = 5KN/m E = 200GPa I = 10^{9}mm^4	10	CO4	L2

SUB:- FEM (10ME64)

SOLUTION FOR 2^{ND} INTERNAL - MAY 2017

1.a. Stiffness matrix for truss

Relation b/w nodal displacement of a truss element in local coordinate & global coordinate is expressed as

$$q' = L\, q$$

where q' - Local coordinate ; L = Transformation matrix

$$= \begin{bmatrix} q_1' \\ q_2' \end{bmatrix} \qquad L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$l, m \rightarrow$ direction cosines.

q = Nodal disp. in global coordinate

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

A truss in local coordinate is equivalent to one dimensional bar element having stiffness matrix

$$k' = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Strain energy for truss element in local coordinate is given by

$$U_e = \frac{1}{2} q'^T k' q' \rightarrow (1)$$

It is required to determine strain energy of truss element in global coordinate

$$q' = L q$$

Sub. this in eqn (1) we get

$$U_e = \frac{1}{2} q^T L^T K' L q$$

$$= \frac{1}{2} q^T \left(L^T K' L \right) q$$

$$U_e = \frac{1}{2} q^T K_e q$$

where K_e = Elemental stiffness matrix in global coordinate

$$K_e = L^T K' L$$

$$L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} ; \quad K' = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$L^T = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}$$

$$K_e = \frac{EA}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

1.b. Nodal Data

Node No.	x (mm)	y (mm)
1	4000	0
2	0	3000
3	0	0

Element Connectivity Table

Element NO	Initial Node	Final Node	length of element l_e (mm)	l	m
1	2	1	5000	−0.8	−0.6
2	1	3	4000	−1	0

Element Stiffness matrix

$$K = \frac{EA}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

For element 1

$$K_1 = \frac{210\times 10^3 \times 600}{5000}\begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

$$
K_1 = 10^3 \begin{array}{c} \\ \end{array}
\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}
$$

$$
K_1 = 10^3 \begin{bmatrix} 16.128 & -12.096 & -16.12 & 12.096 \\ -12.096 & 9.072 & 12.096 & -9.072 \\ -16.128 & 12.096 & 16.128 & -12.096 \\ 12.096 & -9.072 & -12.096 & 9.072 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}
$$

For element 2

$$
K_2 = \frac{210 \times 10^3 \times 600}{4000} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\begin{array}{cccc} 1 & 2 & 5 & 6 \end{array}
$$

$$
= 10^3 \begin{bmatrix} 31.5 & 0 & -31.5 & 0 \\ 0 & 0 & 0 & 0 \\ -31.5 & 0 & 31.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}
$$

Global stiffness matrix

$$
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}
$$

$$
K = 10^3 \begin{bmatrix} 47.628 & -12.096 & -16.128 & 12.096 & -31.5 & 0 \\ -12.096 & 9.072 & 12.096 & -9.072 & 0 & 0 \\ -16.128 & 12.096 & 16.128 & -12.096 & 0 & 0 \\ 12.096 & -9.072 & -12.096 & 9.072 & 0 & 0 \\ -31.5 & 0 & 0 & 0 & 31.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}
$$

Equilibrium eqn.

$$[K][q] = [F]$$

$$10^3 \begin{bmatrix} 47.628 & -12.096 & -16.128 & 12.096 & -31.5 & 0 \\ 12.096 & 9.072 & 12.096 & -9.072 & 0 & 0 \\ -16.128 & 12.096 & 16.128 & -12.096 & 0 & 0 \\ 12.096 & -9.072 & -12.096 & 9.072 & 0 & 0 \\ -31.5 & 0 & 0 & 0 & 31.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} -10^6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Nodal displacement

$$q_1 = -20.996 \text{ mm}$$

$$q_2 = q_3 = q_4 = q_5 = q_6 = 0$$

Stress

$$\sigma = \frac{E}{l_e} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$= \frac{210 \times 10^3}{5000} \begin{bmatrix} 0.8 & -0.6 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} -20.996 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_1 = -705.46 \text{ N/mm}^2$$

$$\sigma_2 = \frac{210\times10^3}{4000}\begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}-20.996\\0\\0\\0\end{bmatrix}$$

$$= -1102.96\ N/mm^2$$

Reaction

$R_2 = K_{21}\,q_1 = -12.096\times10^3\times -20.996 = 253.96\ kN$

$R_3 = K_{31}\,q_1 = -16.126\times10^3\times -20.996 = 338.62\ kN$

$R_4 = K_{41}\,q_1 = -253.96\ kN$

$R_5 = K_{51}\,q_1 = -31.5\times10^3\times -20.996 = 661.37\ kN$

2.a. Hermite Shape function

Consider $H_1 = a_1 + b_1\xi + c_1\xi^2 + d_1\xi^3$

At node 1, $H_1 = 1$, $\xi = -1$

$1 = a_1 - b_1 + c_1 - d_1 \rightarrow (1)$

At node 2, $H_1 = 0$, $\xi = 1$.

$0 = a_1 + b_1 + c_1 + d_1 \rightarrow (2)$

$H_1' = b_1 + 2c_1\xi + 3d_1\xi^2$

At node 1, $H_1' = 0$, $\xi = -1$

$0 = b_1 - 2c_1 + 3d_1 \rightarrow (3)$

At node 2; $H_1' = 0$, $\xi = 1$

$0 = b_1 + 2c_1 + 3d_1 \rightarrow (4)$

Solving (1), (2), (3) & (4)

$a_1 = \frac{1}{2}$; $b_1 = \frac{-3}{4}$; $c_1 = 0$; $d_1 = \frac{1}{4}$

$\therefore \quad H_1 = \frac{1}{4}\left[2 - 3\xi + \xi^3\right]$

Consider Shape function $H_2 = a_2 + b_2\xi + c_2\xi^2 + d_2\xi^3$

At node 1, $H_2 = 0$; $\xi = -1$

$$0 = a_2 - b_2 + c_2 - d_2 \rightarrow (5)$$

At node 2, $H_2 = 0$, $\xi = 1$

$$0 = a_2 + b_2 + c_2 + d_2 \rightarrow (6)$$

$$H_2' = b_2 + 2c_2\xi + 3d_2\xi^2$$

At node 1 ; $H_2' = 1$, $\xi = -1$

$$1 = b_2 - 2c_2 + 3d_2 \rightarrow (7)$$

At node 2 ; $H_2' = 0$; $\xi = 1$

$$0 = b_2 + 2c_2 + 3d_2 \rightarrow (8)$$

$a_2 = \frac{1}{4}$; $b_2 = \frac{-1}{4}$; $c_2 = -\frac{1}{4}$; $d_2 = \frac{1}{4}$

$$H_2 = \frac{1}{4}\left[1 - \xi - \xi^2 + \xi^3\right]$$

$|||^{ly}$ $H_3 = \frac{1}{4}\left[2 + 3\xi - \xi^3\right]$

$H_4 = \frac{1}{4}\left[-1 - \xi + \xi^2 + \xi^3\right]$

2.b. <u>Elemental</u> <u>Stiffness</u> <u>matrix</u>

$$K = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$K_1 = \frac{200\times10^9 \times 4\times10^{-6}}{1^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$K_2 = 8\times10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

<u>Global</u> <u>Stiffness</u>

$$K = 8\times10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}$$

<u>Load vector</u>

$$f_e = \begin{bmatrix} 0 \\ 0 \\ -6\times10^3 \\ -1\times10^3 \\ -6\times10^3 \\ +1\times10^3 \end{bmatrix}$$

Equilibrium eqn

$$[K][q] = [F]$$

$$8\times10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6\times10^3 \\ -1\times10^3 \\ -6\times10^3 \\ 1\times10^3 \end{bmatrix}$$

Slope at node 2 ; $\theta_2 = -2.67\times10^{-4}$ rad

Slope at node 3 ; $\theta_3 = 4.46\times10^{-4}$ rad

Max' deflection

$$y = H \cdot q$$

$$y = H_1 \cdot v_2 + H_2 \cdot \frac{l_e}{2} \cdot \theta_2 + H_3 \cdot v_3 + H_4 \cdot \frac{l_e}{2}\,\theta_3$$

$\xi = 0$ at center

$$H_1 = \frac{1}{4}\left[2 - 3\xi + \xi^3\right] = \frac{1}{2}$$

$$H_2 = \frac{1}{4}\left[1 - \xi - \xi^2 + \xi^3\right] = \frac{1}{4}$$

$$H_3 = \frac{1}{4}\left[2 + 3\xi - \xi^3\right] = \frac{1}{2}$$

$$H_4 = \frac{1}{4}\left(-1 - \xi - \xi^2 + \xi^3\right) = -\frac{1}{4}$$

$$y = 8.9\times10^{-5}\ \text{m}$$

3. <u>Nodal Data</u>

Node No	x (mm)	y (mm)
1	10	10
2	50	40

Element Connectivity Table

Element No	Initial Node	Final Node	l_e	ℓ	m
1	1	2	50	0.8	0.6

i) <u>Element displacement dof in local coordinate</u>

$$q' = L q$$

$$\begin{bmatrix} q_1' \\ q_2' \end{bmatrix} = \begin{bmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1 \\ 2.1 \\ 4.3 \end{bmatrix} \times 10^{-2}$$

$$q_1' = 0.018 \text{ mm}$$

$$q_2' = 0.0426 \text{ mm}$$

ii) Stress

$$\sigma = E \cdot \frac{1}{l_e} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$= 300 \times 10^3 \times \frac{1}{50} \begin{bmatrix} -0.8 & -0.6 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1 \\ 2.1 \\ 4.3 \end{bmatrix} \times 10^{-2}$$

$$\sigma = 147.6 \ N/mm^2$$

Stiffness matrix

$$K = \frac{EA}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & m^2 & lm & m^2 \end{bmatrix}$$

$$= \frac{300 \times 10^3 \times 100}{500} \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}$$

$$= 10^5 \begin{bmatrix} 3.84 & 2.88 & -3.84 & -2.88 \\ 2.88 & 2.16 & -2.88 & -2.16 \\ -3.84 & -2.88 & 3.84 & 2.88 \\ -2.88 & -2.16 & 2.88 & 2.16 \end{bmatrix}$$

4. <u>Elemental</u> <u>Stiffness</u> <u>matrix</u>

$$K = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ +6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$K = \frac{200\times10^9\times10^{-3}}{1^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

<u>Load</u> <u>vector</u>

$$f = \begin{bmatrix} -2500 \\ -416.67 \\ -2500 \\ 416.67 \end{bmatrix} = \begin{bmatrix} Pl_e/2 \\ Pl_e^2/12 \\ Pl_e/2 \\ -Pl_e^2/12 \end{bmatrix}$$

<u>Equilibrium</u> <u>eqn</u>

$$[K][q] = [F]$$

$$2\times10^{12}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -2500 \\ -416.67 \\ -2500 \\ 416.67 \end{bmatrix}$$

Disp. at node 2 ; $V_2 = -3.12\times10^{-10}$ m

$\theta_2 = -4.16\times10^{-6}$ rad