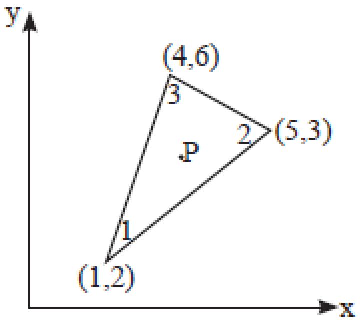
CMR INSTITUTE OF TECHNOLOGY

Improvement Test – June 2017

Sub:	Finite Element Methods	**Code:**	10ME64
Date: 01/06/2017	**Duration:** 90 mins	**Max Marks:** 50	**Sem:** VI
Branch:	MECH		

Note: Answer all questions.

		Marks	OBE	
			CO	RBT
1	a. Explain Simplex, Complex and Multiplex elements.	6	CO4	L1
	b. The nodal co-ordinates of a triangular element are shown in figure. The x - coordinates of an interior point P is 3.3 and shape function. $N_1 = 0.3$. Determine N_2, N_3 and y - coordinate point P. 	4	CO4	L2
2	Derive Shape function and Jacobian matrix for CST element	10	CO4	L3
3	Derive shape functions for QUAD 4 element using Lagrangian element	10	CO4	L3
4	Derive the expression for potential energy functional of a three dimensional body subjected to general loading.	10	CO2	L3
5	A cantilever beam of span 'L' is subjected to a point load at free end. Derive an equation for the deflection at free end by using RR method. Assume polynomial displacement function.	10	CO2	L3
6	Using Rayleigh Ritz method, find the maximum deflection of simply supported beam with point load at centre	10	CO2	L3

Improvement Test Solution

1.a **SIMPLEX, COMPLEX AND MULTIPLEX ~~Comp~~ ELEMENTS**

SIMPLEX

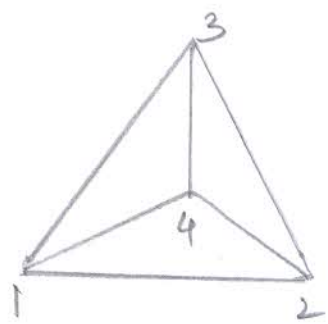
Simplex elements are those for which approximating polynomial consists of constant & linear terms.

Example:

1D line element

2D triangular element

3D



Polynomial Eqns.

1D $\quad u(x) = a_1 + a_2 x$

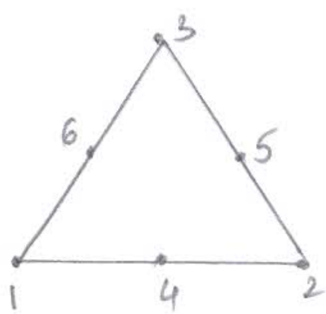
2D $\quad u(x,y) = a_1 + a_2 x + a_3 y$

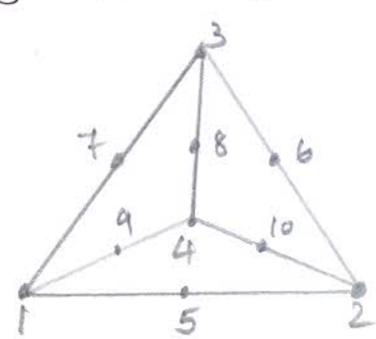
3D $\quad u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z$

COMPLEX

Complex elements are those for which approximating polynomial consists of quadratic, cubic & higher order terms in addition to constant & linear terms

Examples:





Polynomial eqns

For Quadratic

1D $u(x) = a_1 + a_2 x + a_3 x^2$

2D $u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$

3D $u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 yz + a_{10} zx$

For Cubic

1D $u(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$

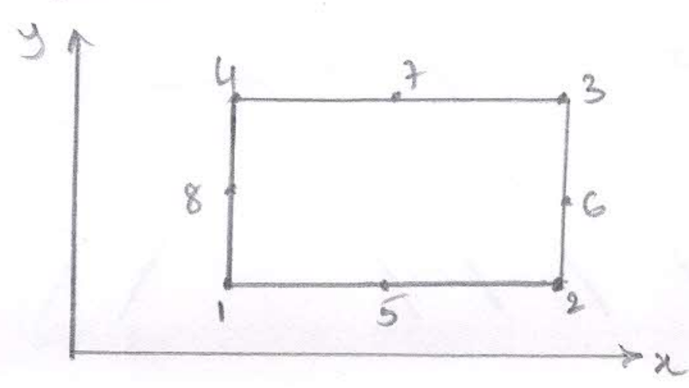
2D $u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy + a_7 x^3 + a_8 y^3 + a_9 x^2 y + a_{10} x y^2$

3D $u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 yz + a_{10} zx + a_{11} x^3 + a_{12} y^3 + a_{13} z^3 + a_{14} x^2 y + a_{15} y^2 x + a_{16} y^2 z + a_{17} y z^2 + a_{18} z^2 x + a_{19} z x^2 + a_{20} xyz.$

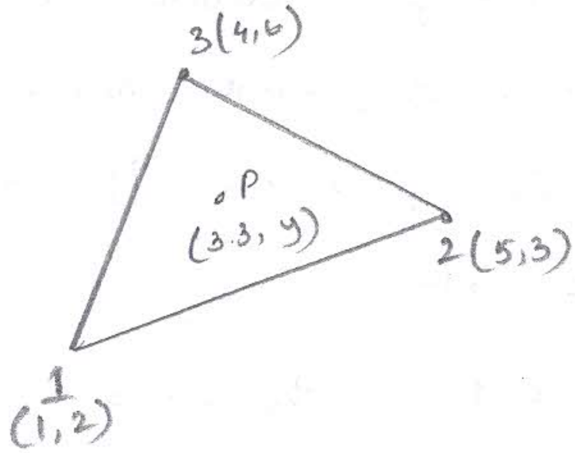
MULTIPLEX

Multiplex elements are those whose boundaries are parallel to coordinate axes to achieve inter element continuity

Ex:- Rectangular element.



1.b



$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$3.3 = 0.3(1) + \eta(5) + (1-\xi-\eta)4$$

$$3.3 = 0.3 + 5\eta + 4 - 4\xi - 4\eta$$

$$-4\xi + \eta = -1.$$

$$-4(0.3) + \eta = -1$$

$$\eta = 0.2$$

$$\therefore N_2 = \eta = 0.2 //$$

$$N_3 = 1-\xi-\eta = 1-0.3-0.2 = 0.5$$

$$N_3 = 0.5 //$$

<u>y Coordinate</u>

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$= 0.3(2) + 0.2(3) + 0.5(6)$$

$$y = 4.2 //$$

Shape function for CST

Let the Shape function N_1 be

$$N_1 = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta$$

At node 1, $N_1 = 1$, $\xi = 1$, $\eta = 0$

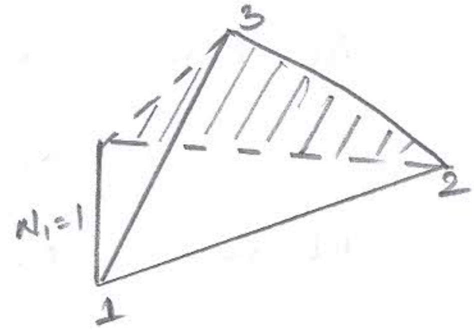
$$1 = \alpha_1 + \alpha_2 \rightarrow (1)$$

At node 2, $N_1 = 0$, $\xi = 0$, $\eta = 1$

$$0 = \alpha_1 + \alpha_3 \rightarrow (2)$$

At node 3, $N_1 = 0$, $\xi = 0$, $\eta = 0$

$$\alpha_1 = 0 \rightarrow (3)$$



$\alpha_3 = 0$; $\alpha_2 = 1$

$$\boxed{N_1 = \xi}$$

Let the Shape function N_2 be

$$N_2 = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta$$

At node 1, $N_2 = 0$, $\xi = 1$, $\eta = 0$

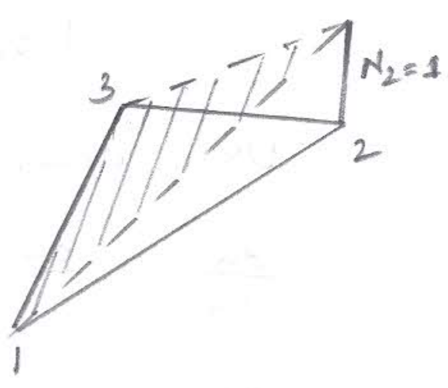
$$0 = \alpha_1 + \alpha_2 \rightarrow (1)$$

At node 2, $N_2 = 1$, $\xi = 0$, $\eta = 1$

$$1 = \alpha_1 + \alpha_2 \rightarrow (2)$$

At node 3, $N_2 = 0$, $\xi = 0$, $\eta = 0$

$$\alpha_1 = 0 \rightarrow (3)$$



$\alpha_2 = 0$, $\alpha_3 = 1$.

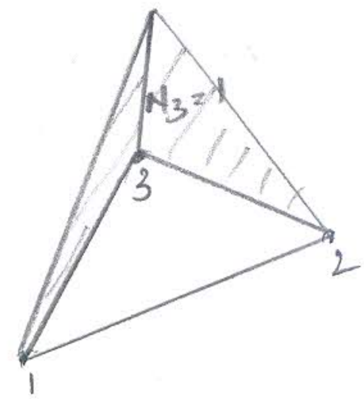
$$\boxed{N_2 = \eta}$$

Let the shape function N_3 be

$$N_3 = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta$$

At node 1, $N_3 = 0$, $\xi = 1$, $\eta = 0$

$$0 = \alpha_1 + \alpha_2 \rightarrow (1)$$



At node 2, $N_3 = 0$, $\xi = 0$, $\eta = 1$.

$$0 = \alpha_1 + \alpha_3 \rightarrow (2)$$

At node 3, $N_3 = 1$, $\xi = 0$, $\eta = 0$

$$\alpha_1 = 1 \rightarrow (3)$$

$$\alpha_2 = -1, \quad \alpha_3 = -1$$

$$\boxed{N_3 = 1 - \xi - \eta}$$

Jacobian Matrix

Strain can be expressed as

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \rightarrow (1)$$

Using chain rule of partial differentiation

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta}$$

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} \rightarrow (2)$$

Where $J \rightarrow$ Jacobian

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

Jacobian is a scaling matrix which Corelates b/w Cartesian Coordinate & natural Coordinate w.r.t the differential of the displacements.

Using Isoparametric formulation

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$x = \xi x_1 + \eta x_2 + (1-\xi-\eta) x_3$$

$$\frac{\partial x}{\partial \xi} = x_1 - x_3 = x_{13}$$

$$\frac{\partial x}{\partial \eta} = x_2 - x_3 = x_{23}$$

|||^{ly} $y = N_1 y_1 + N_2 y_2 + N_3 y_3$

$$y = \xi y_1 + \eta y_2 + (1-\xi-\eta) y_3$$

$$\frac{\partial y}{\partial \xi} = y_1 - y_3 = y_{13}$$

$$\partial y/\partial \eta = y_2 - y_3 = y_{23}$$

3 $N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$

$i = 1, 2, 3, 4$

$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$

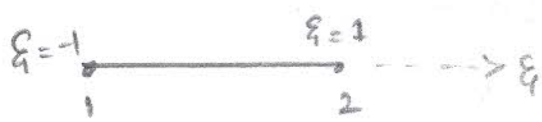
$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$

$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$

LAGRANGEAN METHOD

$$N_1(\xi, \eta) = L_1(\xi)\, L_1(\eta)$$



$$L_1(\xi) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{\xi - 1}{-1-1}$$

$$L_1(\xi) = \frac{1-\xi}{2}$$

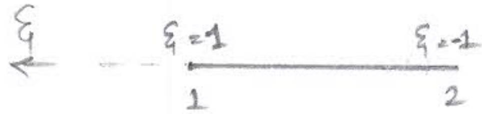
$$L_1(\eta) = \frac{\eta - \eta_4}{\eta_1 - \eta_4} = \frac{\eta - 1}{-1-1} = \frac{1-\eta}{2}$$

$$\therefore \quad N_1 = L_1(\xi)\; L_1(\eta)$$

$$= \frac{1-\xi}{2} \cdot \frac{1-\eta}{2}$$

$$\boxed{N_1 = \frac{1}{4}(1-\xi)(1-\eta)}$$

$$N_2(\xi, \eta) = L_2(\xi)\; L_2(\eta)$$



$$L_2(\xi) = \frac{\xi - \xi_1}{\xi_2 - \xi_1} = \frac{\xi - 1}{-1-1} = \frac{1+\xi}{2}$$

$$L_2(\eta) = \frac{\eta - \eta_3}{\eta_2 - \eta_3} = \frac{\eta + 1}{-1-1} = \frac{1-\eta}{2}$$

$$N_2 = \frac{1+\xi}{2} \; \frac{1-\eta}{2}$$

$$\boxed{N_2 = \frac{1}{4}(1+\xi)(1-\eta)}$$

$$N_3(\xi, \eta) = L_3(\xi)\; L_3(\eta)$$

$$L_3(\xi) = \frac{\xi - \xi_4}{\xi_3 - \xi_4} = \frac{\xi + 1}{+1+1} = \frac{1+\xi}{2}$$

$$L_3(\eta) = \frac{\eta - \eta_2}{\eta_3 - \eta_2} = \frac{\eta + 1}{1+1} = \frac{1+\eta}{2}$$

$$\boxed{N_3 = \frac{1}{4}(1+\xi)(1+\eta)}$$

$$N_4(\xi,\eta) = L_4(\xi)\,L_4(\eta)$$

$$L_4(\xi) = \frac{\xi-\xi_3}{\xi_4-\xi_3} = \frac{\xi-1}{-1-1} = \frac{1-\xi}{2}$$

$$L_4(\eta) = \frac{\eta-\eta_1}{\eta_4-\eta_1} = \frac{\eta+1}{1+1} = \frac{1+\eta}{2}$$

$$\boxed{N_4 = \frac{1}{4}(1-\xi)(1+\eta)}$$

4 P.E functional for a 3D elastic body subjected to point load, traction & body forces

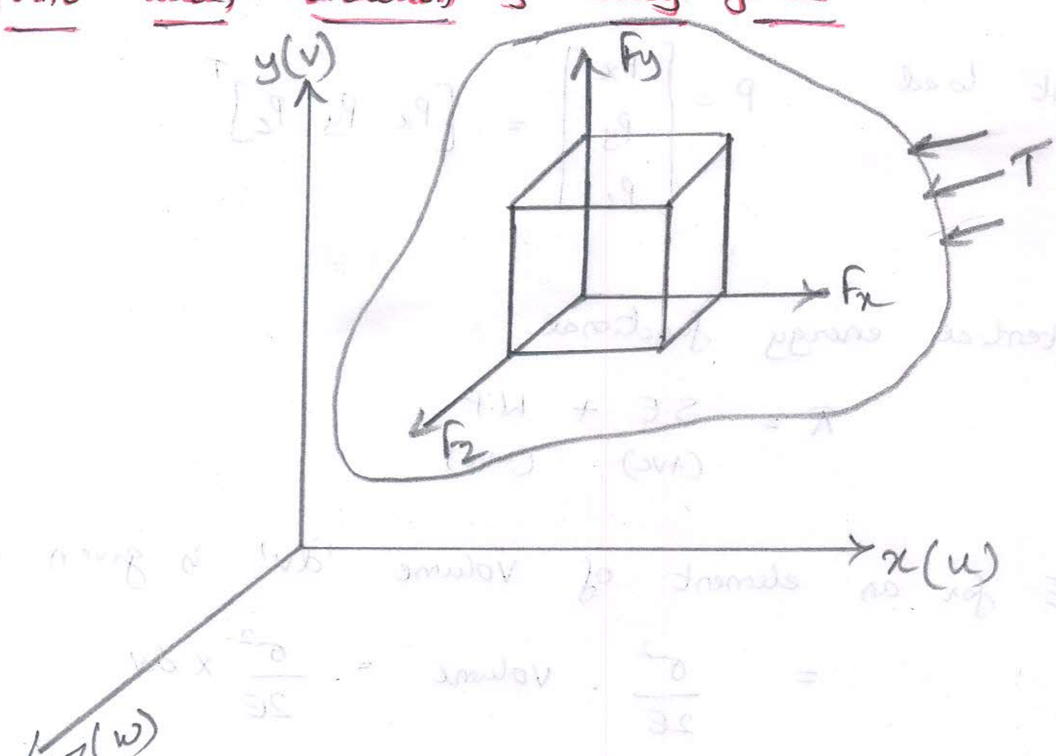


Fig. shows the body lying on a 3 coordinate axis x, y, z subjected to traction 'T' on the surface & a point load 'P_i' at the i^{th} point.

Consider an element of volume 'dv' having body forces (density) ρ, u, v, w be the displacements along x, y, z directions.

Displacement $$U = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = [u \quad v \quad w]^T$$

Body force $$f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = [f_x \quad f_y \quad f_z]^T$$

Traction $$T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = [T_x \quad T_y \quad T_z]^T$$

Point load $$P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = [P_x \quad P_y \quad P_z]^T$$

Potential energy functional

$$\pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$$

S.E for an element of volume 'dv' is given by

$$= \frac{\sigma^2}{2E} \cdot \text{volume} = \frac{\sigma^2}{2E} \times dv$$

Using Hooke's law $\frac{\sigma}{\epsilon} = E \Rightarrow \sigma = E\epsilon$

$$\therefore\ S.E = \frac{(E\epsilon)^2}{2E} \cdot dv = \frac{1}{2}\epsilon^2 . E . dv$$

$$= \frac{1}{2}\epsilon . (E\epsilon)\, dv$$

$$S.E = \frac{1}{2}\epsilon . \sigma . dv$$

Stress is expressed as | Strain is expressed as

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \qquad \epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

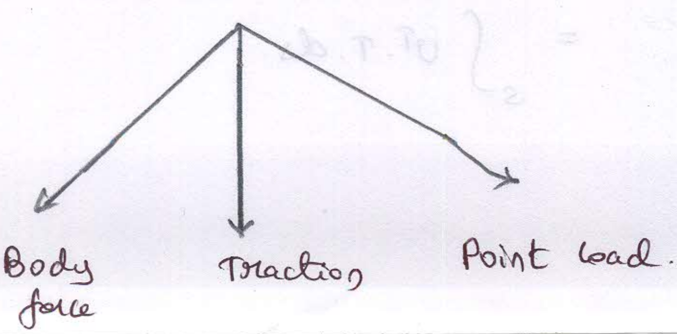
Where $\sigma \rightarrow$ Normal stress $\epsilon \rightarrow$ Normal strain

$\tau \rightarrow$ shear stress $\gamma \rightarrow$ shear strain

$\therefore$ Strain energy (S.E) of an element $= \frac{1}{2}\sigma^T\epsilon . dv$

S.E for 3D body $= \frac{1}{2}\int_V \sigma^T\epsilon . dv$

<u>Work</u> <u>Potential</u>



Work potential due to body force

W.P due to body force for an element of volume 'dv'

$$= f_x \cdot u \cdot dv + f_y \cdot v \cdot dv + f_z \cdot w \cdot dv$$

$$= \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

$$= U^T \cdot f \cdot dv$$

∴ W.P for a 3D body due to body force

$$= \int_V U^T \cdot f \cdot dv$$

Work potential due to traction

W.P due to traction for elemental surface 'ds'

$$= T_x \cdot U \cdot ds + T_y \cdot V \cdot ds + T_z \cdot W \cdot ds$$

$$= \begin{bmatrix} U & V & W \end{bmatrix} \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$= U^T \cdot T \cdot ds$$

W.P due to traction for entire Surface

$$= \int_S U^T \cdot T \cdot ds$$

Work potential due to point load

W.P. due to point load 'P_i' is given by

$$= P_x u + P_y v + P_z w.$$

$$= [u \quad v \quad w] \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$= u^T . P$$

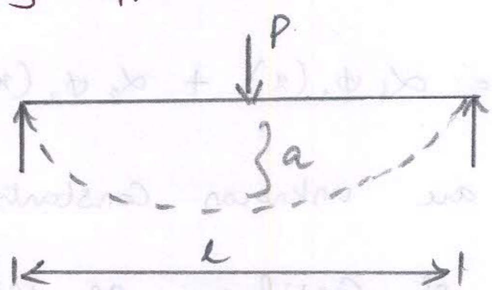
If there are 'n' no of Point loads

$$W.P = \sum U_i^T P_i$$

$$\text{P.E functional } \pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$$

$$\boxed{\pi = \frac{1}{2}\int_V \sigma^T \epsilon . dv - \int_V u^T . f . dv - \int_S u^T . T . ds - \sum u_i^T P_i}$$

5) Find the Central deflection of a Simply Supported beam with Centrally applied load using R.R method



Sol 1) Potential Energy functional

$$\pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$$

$$S.E \text{ for beams } = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \longrightarrow (1)$$

$$W.P = P.y$$

2) Assume the displacement function

$$y = a \sin \frac{\pi x}{l}$$

Boundary condition At $x=0$, $y=0$

$x=l$, $y=0$

$\frac{dy}{dx} \neq 0$ at $x=0$, $y=0$

$$y = a \sin \frac{\pi x}{l}$$

$$\frac{dy}{dx} = a . \cos \frac{\pi x}{l} . \frac{\pi}{l}$$

$$\frac{d^2y}{dx^2} = -a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}$$

Sub. this in eqn 1

$$S.E = \frac{EI}{2}\int_0^l \left[-\frac{a\pi^2}{l^2}\sin\frac{\pi x}{l}\right]^2 dx$$

$$= \frac{EI}{2} \cdot -a^2\frac{\pi^4}{l^4}\int_0^l \sin^2\frac{\pi x}{l}\, dx$$

$$= \frac{EI}{2} \cdot a^2\frac{\pi^4}{l^4}\int_0^l \frac{1-\cos(2\pi x/l)}{2} \cdot dx$$

$$= \frac{EI}{4} \cdot a^2\frac{\pi^4}{l^4}\left[1 - \sin\frac{2\pi x}{l} \cdot \frac{2\pi}{l}\right]_0^l$$

$$= \frac{EI}{4} \cdot \frac{a^2\pi^4}{l^4} \cdot l$$

$$S.E = \frac{EI}{4} \cdot \frac{a^2\pi^4}{l^3}$$

$$W.P = P \cdot y_{max}$$

$$y = y_{max} \text{ at } x = l/2$$

$$y_{max} = a\sin\left[\frac{\pi}{\cancel{l}} \cdot \frac{l}{2}\right] = a\sin\left(\frac{\pi}{2}\right) = a$$

$$\therefore W.P = Pa$$

3) P.E functional $\pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$

$$\pi = \frac{EI}{4} \cdot \frac{a^2 \pi^4}{l^3} - Pa$$

4) Using principle of minimum P.E

$$\frac{\partial \pi}{\partial a} = 0$$

$$\frac{\partial \pi}{\partial a} = \frac{2EI\, a \pi^4}{4 l^3} - P = 0$$

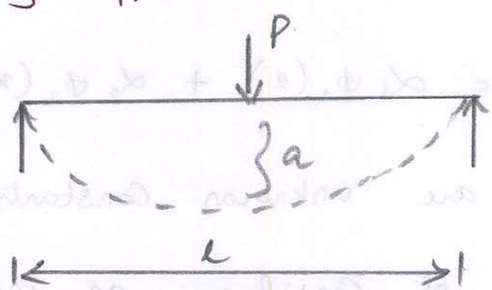
$$\frac{2EI\, a \pi^4}{4 l^3} = P$$

$$a = \frac{4Pl^3}{2EI\, \pi^4}$$

$$= \frac{Pl^3}{\frac{EI}{2} \pi^4}$$

$$\boxed{a = \frac{Pl^3}{48.7\, EI}}$$

Find the Central deflection of a Simply Supported beam with Centrally applied load using R.R method



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Boundary Condition At $x=0$, $y=0$

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$\frac{dy}{dx} \neq 0$ at $x=0$, $y=0$

$$y = a \sin\frac{\pi x}{l}$$

$$\frac{dy}{dx} = a . \cos\frac{\pi x}{l} . \frac{\pi}{l}$$

$$\frac{d^2y}{dx^2} = -a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}$$

Sub. this in eqn (1)

$$S.E = \frac{EI}{2}\int_0^l \left[-\frac{a\pi^2}{l^2}\sin\frac{\pi x}{l}\right]^2 dx$$

$$= \frac{EI}{2} \cdot -a^2\frac{\pi^4}{l^4}\int_0^l \sin^2\frac{\pi x}{l}\, dx$$

$$= \frac{EI}{2} \cdot a^2\frac{\pi^4}{l^4}\int_0^l \frac{1-\cos(2\pi x/l)}{2} \cdot dx$$

$$= \frac{EI}{4} \cdot a^2\frac{\pi^4}{l^4}\left[1 - \sin\frac{2\pi x}{l} \cdot \frac{2\pi}{l}\right]_0^l$$

$$= \frac{EI}{4} \cdot \frac{a^2\pi^4}{l^4} \cdot l$$

$$S.E = \frac{EI}{4} \cdot \frac{a^2\pi^4}{l^3}$$

$$W.P = P \cdot y_{max}$$

$$y = y_{max} \quad \text{at} \quad x = l/2$$

$$y_{max} = a \sin\left[\frac{\pi}{\cancel{l}} \cdot \frac{l}{2}\right] = a\sin\left(\frac{\pi}{2}\right) = a$$

$$\therefore W.P = Pa$$

3) P.E functional $\pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$

$$\pi = \frac{EI}{4} \cdot \frac{a^2\pi^4}{l^3} - Pa$$

4) Using principle of minimum P.E

$$\frac{\partial \pi}{\partial a} = 0$$

$$\frac{\partial \pi}{\partial a} = \frac{2EI\,a\,\pi^4}{4l^3} - P = 0$$

$$\frac{2EI\,a\pi^4}{4l^3} = P$$

$$a = \frac{4Pl^3}{2EI\,\pi^4}$$

$$= \frac{Pl^3}{\frac{EI}{2}\pi^4}$$

$$\boxed{a = \frac{Pl^3}{48.7\,EI}}$$