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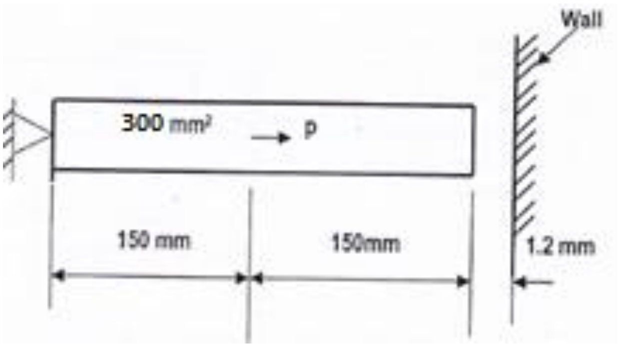
Internal Assessment Test 1 – March 2018

Sub:	Finite Element Methods					**Code:**	15ME61
Date:	12/ 03 / 2018	Duration:	90 mins	Max Marks:	50	**Sem:** VI	**Branch:** MECH

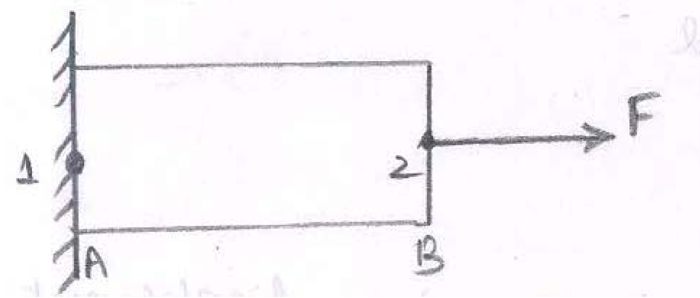
Note: Answer any **five** questions.

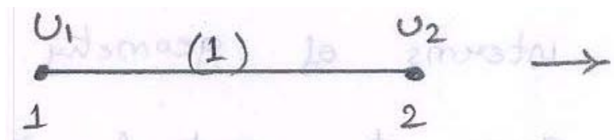
			Marks	OBE CO	RBT
1	**a**	Explain the basic steps involved in FEM, with the help of an example involving a structural member subjected to axial loads.	**8**	CO1	L2
	b	Write the equilibrium equation for 2D state of stress.	**2**	CO1	L1
2	**a**	Explain Simplex, Complex and Multiplex elements.	**3**	CO3	L1
	b	Derive stiffness matrix for one dimensional bar element.	**7**	CO3	L2
3		A cantilever beam of span 'L' is subjected to a point load at free end. Derive an equation for the deflection at free end by using RR method. Assume polynomial displacement function.	**10**	CO2	L3
4		A simply supported beam is subjected to a uniformly distributed load of P N/m for the entire span. Derive an equation for maximum deflection at the centre using trigonometric function by Rayleigh Ritz Method. Assume the trigonometry function $Y = C_1 \sin\left(\frac{\pi x}{L}\right) + C_2 \sin\left(\frac{3\pi x}{L}\right)$	**10**	CO2	L4
5		A compound bar 800 mm long is made of steel of 500 mm length with an area of 400 mm^2 with the remaining length made of brass having an area of 300 mm^2. At the junction it is subjected to an axial load of 200 KN which is in compression to steel. Both the ends are fixed. E_{steel}=200 GPa, E_{Brass}=70 GPa. Find nodal displacements, stress in each material.	**10**	CO4	L3
6		A load of P = 60 KN is applied as shown. Determine the following, a) Nodal Displacement, b) Stress in each member. Given Data: E = 200 GPa	**10**	CO4	L3



Steps in FEM

1. Discretization of given continuum

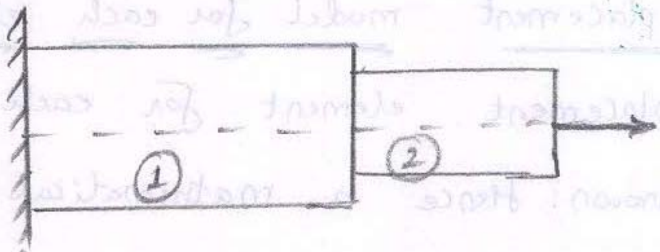




In this step the entire body is divided into small parts. Each part is called an element. The selection of type of element depends on the complexity and structure of the element.

2) Selection of displacement model for each element

The displacement element for each element is unknown. Hence a mathematical model is represented for each finite element. They can be either a polynomial or trignometrical function. A polynomial function is adopted due to simplicity of mathematical calculation.



For element 1, stiffness matrix is K_1

For element 2, stiffness matrix is K_2

Global stiffness matrix $= K = K_1 + K_2$.

5. Equilibrium equation

$$[K][Q] = [F]$$

6. Enforcing boundary conditions
   i) Using Elimination Method
   ii) Using Penalty Method

7. Determination of unknowns
   Stresses, strains etc

$$\sigma = EBq$$

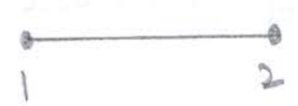
SIMPLEX, COMPLEX AND MULTIPLEX ~~COMP~~ ELEMENTS

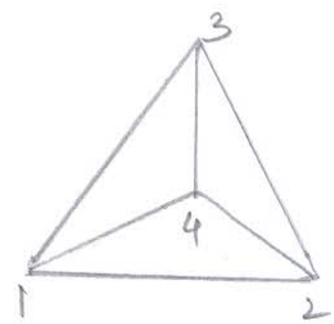
SIMPLEX

Simplex elements are those for which approximating polynomial consists of constant & linear terms.

Example:

1D line element | 2D triangular element | 3D





Polynomial Eqns.

1D $\quad u(x) = a_1 + a_2 x$

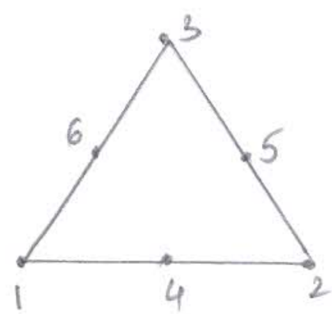
2D $\quad u(x,y) = a_1 + a_2 x + a_3 y$

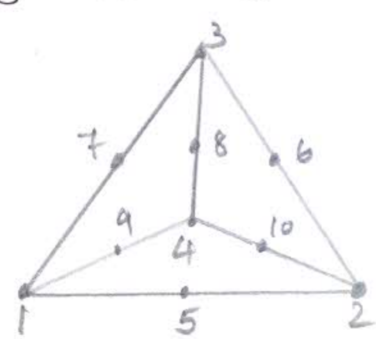
3D $\quad u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z$

COMPLEX

Complex elements are those for which approximating polynomial consists of quadratic, cubic & higher order terms in addition to constant & linear terms

Examples:





Polynomial eqns

For Quadratic

1D $\quad u(x) = a_1 + a_2 x + a_3 x^2$

2D $\quad u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$

3D $\quad u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2$
$\quad + a_8 xy + a_9 yz + a_{10} zx$

For Cubic

1D $\quad u(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$

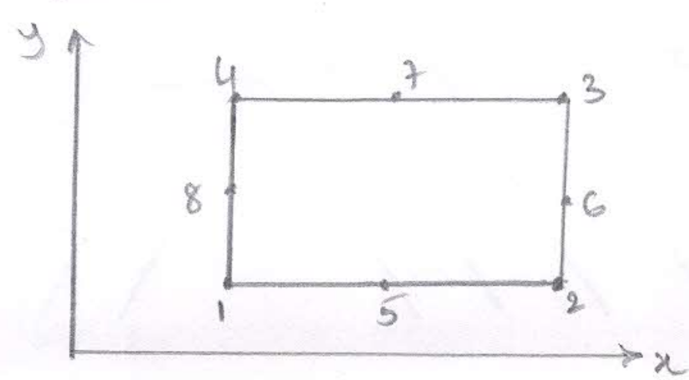
2D $\quad u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy + a_7 x^3$
$\quad + a_8 y^3 + a_9 x^2 y + a_{10} xy^2$

3D $\quad u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2$
$\quad + a_8 xy + a_9 yz + a_{10} zx + a_{11} x^3 + a_{12} y^3$
$\quad + a_{13} z^3 + a_{14} x^2 y + a_{15} y^2 x + a_{16} y^2 z + a_{17} yz^2$
$\quad + a_{18} z^2 x + a_{19} zx^2 + a_{20} xyz.$

MULTIPLEX

Multiplex elements are those whose boundaries are parallel to Coordinate axes to achieve inter element Continuity

Ex:- Rectangular element.



2. Stiffness matrix for one dimensional bar element

Strain energy for 3 D element $U_e = \frac{1}{2}\int_V \sigma^T \epsilon \, dv$

Strain energy for 1D element $U_e = \frac{1}{2}\int_{l_e} \sigma^T \epsilon \, A \, dx$

Strain $\in = Bq$

$$where\ B = \frac{1}{l_e}[-1 \quad 1]$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Strain energy for 1D element $U_e = \frac{1}{2}\int_{l_e} [EBq]^T \, Bq \, A \, dx$

Relation between natural and Cartesian coordinate is

$$\xi = \frac{x-x_1}{x_2-x_1} - 1$$

$$\frac{d\xi}{dx} = \frac{2}{l_e}$$

$$U_e = \frac{1}{2}\int_{l_e} q^T \, B^T E \, Bq \, A \, \frac{l_e}{2} \, d\xi$$

$$U_e = \frac{1}{2} q^T [EA \, \frac{l_e}{2} \int_{l_e} B^T \, Bd\xi \,] \, q$$

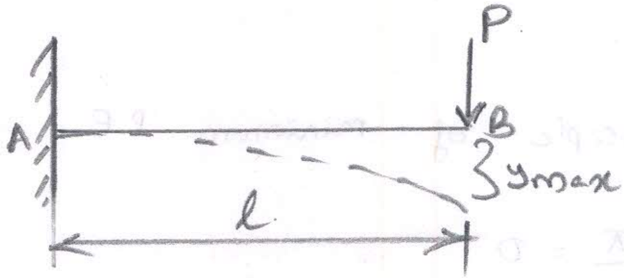
$$U_e = \frac{1}{2} q^T k_e \, q$$

$$Where\ k_e = stiffness\ matrix$$

$$k_e = EA \, \frac{l_e}{2} \int_{l_e} B^T \, Bd\xi$$

$$K = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(P) A Cantilever beam Subjected to point load at free end. Derive an eqn for max' deflection using R.R method.



Sol 1) Potential Energy functional

$$\pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$$

$$S.E = \frac{EI}{2} \int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$W.P = P \cdot y_{max}$$

2) Assume displacement function

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

B.C At A, $x=0$, $y=0$

$$\boxed{a_0 = 0}$$

At A, $x=0$, $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2$$

$$\boxed{a_1 = 0}$$

$$y = a_2 x^2 + a_3 x^3$$

$$\frac{dy}{dx} = 2a_2x + 3a_3x^2$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x$$

$y = y_{max}$ at $x = l$

$$y = a_2x^2 + a_3x^3$$

$$y_{max} = a_2l^2 + a_3l^3$$

3) P.E functional

$$\pi = \frac{EI}{2}\int_0^l (2a_2 + 6a_3x)^2\,dx - P(a_2l^2 + a_3l^3)$$

$$= \frac{EI}{2}\int_0^l \left[4a_2^2 + 36a_3^2x^2 + 24a_2a_3x\right]dx - P(a_2l^2 + a_3l^3)$$

$$= \frac{EI}{2}\left[4a_2^2x + 36a_3^2\frac{x^3}{3} + 24a_2a_3\frac{x^2}{2}\right]_0^l - P(a_2l^2 + a_3l^3)$$

$$\pi = \frac{EI}{2}\left[4a_2^2l + 36a_3^2\frac{l^3}{3} + 24a_2a_3\frac{l^2}{2}\right] - P(a_2l^2 + a_3l^3)$$

4) Using Principle of minimum P.E

$$\frac{\partial\pi}{\partial a_2} = 0 \;;\; \frac{EI}{2}\left[8a_2l + 12a_3l^2\right] - Pl^2 = 0 \rightarrow (1)$$

$$\frac{\partial\pi}{\partial a_3} = 0 \;;\; \frac{EI}{2}\left[24a_3l^3 + 12a_2l^2\right] - Pl^3 = 0 \rightarrow (2)$$

Multiply eqn (1) by $2l$

$$\frac{EI}{2}\left[16 a_2 l^2 + 24 a_3 l^3\right] - 2Pl^3 = 0$$

$$\frac{EI}{2}\left[12 a_2 l^2 + 24 a_3 l^3\right] - Pl^3 = 0$$

$$- \qquad - \qquad +$$

$$\frac{EI}{2}\left[4 a_2 l^2\right] - Pl^3 = 0$$

$$a_2 = \frac{2Pl^3}{4l^2 \cdot EI}$$

$$\boxed{a_2 = \frac{Pl}{2EI}}$$

Sub. a_2 in eqn (2)

$$\frac{EI}{2}\left[12 \cdot \frac{Pl}{2EI} \cdot l^2 + 24 a_3 l^3\right] - Pl^3 = 0$$

$$\frac{6Pl^3}{EI} + 24 a_3 l^3 = \frac{2Pl^3}{EI}$$

$$24 a_3 l^3 = \frac{2Pl^3}{EI} - \frac{6Pl^3}{EI} = \frac{-4Pl^3}{EI}$$

$$a_3 = \frac{-4Pl^3}{24 l^3 \cdot EI} = \frac{-P}{6EI}$$

$$\boxed{a_3 = \frac{-P}{6EI}}$$

Max' deflection

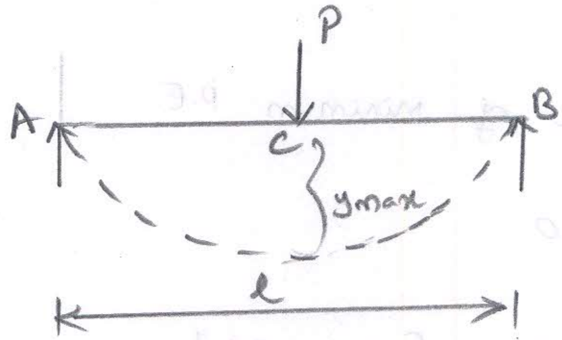
$$y_{max} = a_2 l^2 + a_3 l^3$$

$$= \frac{P.\,l^3}{2EI} - \frac{Pl^3}{6EI}$$

$$\boxed{y_{max} = \frac{Pl^3}{3EI}}$$

(P) Find the maximum deflection for the fig. shown using R.R method. Assume trignometric function

$$y = C_1 \sin\frac{\pi x}{l} + C_2 \sin\frac{3\pi x}{l}$$



Sol 1) P.E functional $\pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$

$$S.E = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \rightarrow (1)$$

$$W.P = P.\, y_{max}.$$

2) Assume displacement function.

$$y = C_1 \sin\frac{\pi x}{l} + C_2 \sin\frac{3\pi x}{l}$$

B.C At A, $x=0$, $y=0$

At B, $x=l$, $y=0$.

$$\frac{dy}{dx} = C_1 \frac{\pi}{l}\cos\frac{\pi x}{l} + C_2 \frac{3\pi}{l} \cdot \cos\frac{3\pi x}{l}$$

$$\frac{d^2y}{dx^2} = -C_1 \frac{\pi^2}{l^2}\sin\frac{\pi x}{l} - C_2 \frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}$$

Sub. $\frac{d^2y}{dx^2}$ in eqn (1)

$$S.E = \frac{EI}{2}\int_0^l \left[-C_1\frac{\pi^2}{l^2}\sin\frac{\pi x}{l} - C_2\frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]^2 dx$$

$$= \frac{EI}{2}\int_0^l C_1^2\frac{\pi^4}{l^4}\sin^2\frac{\pi x}{l} + C_2^2\frac{81\pi^4}{l^4}\sin^2\frac{3\pi x}{l}$$

$$+ 18C_1C_2\frac{\pi^4}{l^4}\sin\frac{\pi x}{l}.\sin\frac{3\pi x}{l}$$

$$= \frac{EI}{2}\left\{\int_0^l C_1^2\frac{\pi^4}{l^4}\left[\frac{1-\cos(2\pi x/l)}{2}\right]dx\right.$$

$$+\int_0^l 81C_2^2\frac{\pi^4}{l^4}\left[\frac{1-\cos(6\pi x/l)}{2}\right]dx$$

$$+\int_0^l 18C_1C_2\frac{\pi^4}{l^4}\left[\frac{\cos(2\pi x/l)-\cos(4\pi x/l)}{2}\right]$$

** $\sin A.\sin B = \frac{\cos(A-B)-\cos(A+B)}{2}$

$$S.E = \frac{EI}{2}\left\{C_1^2\frac{\pi^4}{l^4}\left[\frac{l}{2}\right] + 81C_2^2\frac{\pi^4}{l^4}\left[\frac{l}{2}\right]\right\}$$

$W.P = P.\,y_{max}$

$y = y_{max}$ at $x = l/2$

$y_{max} = C_1 - C_2$

$$W.P = P\left[C_1 - C_2\right]$$

3) P.E functional $\pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$

$$\pi = \frac{EI}{2}\left[\frac{C_1^2\pi^4}{2l^3} + \frac{81\ C_2^2\pi^4}{2l^3}\right] - P(C_1 - C_2)$$

4) Using Principle of minimum P.E

$$\frac{\partial \pi}{\partial C_1} = 0.$$

$$\frac{EI}{2}\left[\frac{2C_1\pi^4}{2l^3}\right] - P = 0$$

$$\boxed{C_1 = \frac{2Pl^3}{\pi^4 EI}}$$

$$\frac{\partial \pi}{\partial C_2} = 0 \ ; \quad \frac{EI}{2}\left[\frac{81 \times 2\ C_2\ \pi^4}{2\ l^3}\right] + P = 0$$

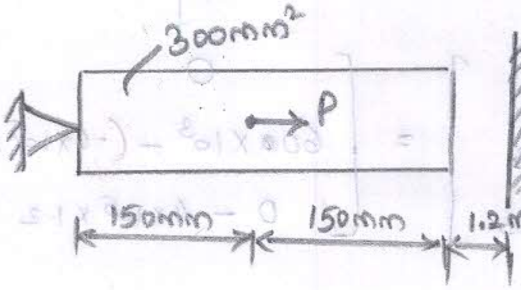
$$\boxed{C_2 = -\frac{2Pl^3}{81\pi^4 EI}}$$

5) $y_{max} = C_1 - C_2$

$$= \frac{2Pl^3}{\pi^4 EI} + \frac{2Pl^3}{81\pi^4 EI}$$

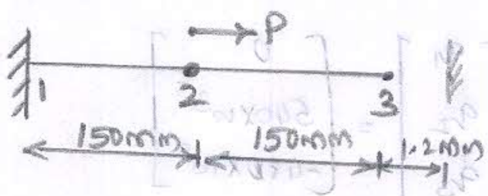
$$= \frac{162Pl^3 + 2Pl^3}{81EI\pi^4} = \frac{164Pl^3}{81EI\pi^4}$$

$$\boxed{y_{max} = \frac{Pl^3}{48.11EI}}$$



$E = 200\ GPa$

F.E. Model



Elemental stiffness matrix

$$K = \frac{EA}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

For element 1

$$K_1 = \frac{200\times10^3\times300}{150}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$K_1 = 10^5 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix}4 & -4\\ -4 & 4\end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

For element 2

$$K_2 = \frac{200\times10^3\times300}{150}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$K_2 = 10^5 \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix}4 & -4\\ -4 & 4\end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

Global stiffness matrix

$$K = 10^5 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix}4 & -4 & 0\\ -4 & 8 & -4\\ 0 & -4 & 4\end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Load Vector

$$F = \begin{bmatrix} F_1 - K_{13} a_3 \\ F_2 - K_{23} a_3 \\ F_3 - K_{33} a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 600 \times 10^3 - (-4\times10^5 \times 1.2) \\ 0 - 4\times10^5 \times 1.2 \end{bmatrix} = \begin{bmatrix} 0 \\ 540\times10^3 \\ -480\times10^3 \end{bmatrix}$$

Equilibrium eqn. $[K][q] = [F]$

$$10^5 \begin{bmatrix} 4 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 04 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 540\times10^3 \\ -480\times10^3 \end{bmatrix}$$

Applying B.C : Using Elimination method

$$10^5 \begin{bmatrix} 4 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 540\times10^3 \\ -480\times10^3 \end{bmatrix}$$

$q_1 = 0$; $q_2 = 0.675$ mm; $q_3 = 1.2$ mm.

Displacement at node 1 : $q_1 = 0$

node 2 : $q_2 = 0.675$ mm

node 3 : $q_3 = 1.2$ mm //

Stress in each element

$$\sigma = E \cdot B \cdot q = E \cdot \frac{1}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Element 1

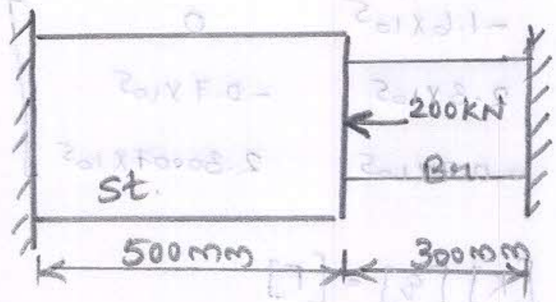
$$\sigma_1 = 200\times10^3 \times \frac{1}{150} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.675 \end{bmatrix}$$

$\sigma_1 = 900 \ N/mm^2$ //

Element 2

$$\sigma_2 = 200\times10^3 \times \frac{1}{150} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.675 \\ 1.2 \end{bmatrix}$$

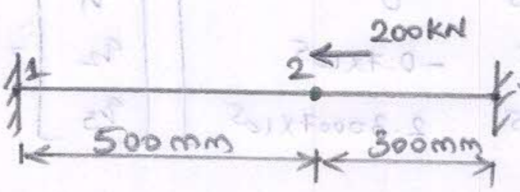
$\sigma_2 = 700 \ N/mm^2$ //



$E_{St} = 200 GPa$ $E_{Br} = 70 GPa$

$A_{St} = 400 mm^2$ $A_{Br} = 300 mm^2$

F.E. Model



Stiffness matrix for each element

$$K = \frac{EA}{le}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For element 1

$$K_1 = \frac{200\times10^3\times400}{500}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^5 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1.6 & -1.6 \\ -1.6 & 1.6 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array}$$

For element 2

$$K_2 = \frac{70\times10^3\times300}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 0.7 & -0.7 \\ -0.7 & 0.7 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

Global Stiffness matrix

$$K = 10^5 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.6 & -1.6 & 0 \\ -1.6 & 2.3 & -0.7 \\ 0 & -0.7 & 0.7 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Using Penalty method $C = |\max k_{ij}| \times 10^4$

$$C = 2.3\times10^5\times10^4 \Rightarrow C = 2.3\times10^9$$

$$K = 10^5\begin{bmatrix} 1.6+C & -1.6 & 0 \\ -1.6 & 2.3 & -0.7 \\ 0 & -0.7 & 0.7+C \end{bmatrix}$$

$$K = \begin{bmatrix} 2.30016\times10^{9} & -1.6\times10^{5} & 0 \\ -1.6\times10^{5} & 2.3\times10^{5} & -0.7\times10^{5} \\ 0 & -0.7\times10^{5} & 2.30007\times10^{5} \end{bmatrix}$$

Equilibrium eqn $[K][q] = [F]$

$$\begin{bmatrix} 2.30016\times10^{9} & -1.6\times10^{5} & 0 \\ -1.6\times10^{5} & 2.3\times10^{5} & -0.7\times10^{5} \\ 0 & -0.7\times10^{5} & 2.30007\times10^{5} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -200\times10^{3} \\ 0 \end{bmatrix}$$

Solving $q_1 = -6.049\times10^{-5}$ mm ; $q_2 = -0.8696$ mm,
$q_3 = -2.6466\times10^{-5}$ mm

Displacement at node 1 : $q_1 = -6.049\times10^{-5}$ mm
node 2 : $q_2 = -0.8696$ mm
node 3 : $q_3 = -2.6466\times10^{-5}$ mm //

Stresses in each element

Element 1 $\sigma = E.B.q = E.\frac{1}{l_e}[-1 \;\; 1]\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

$$\sigma_1 = 200\times10^{3}\times\frac{1}{500}[-1 \;\; 1]\begin{bmatrix} -6.049\times10^{-5} \\ -0.8696 \end{bmatrix}$$

$$\sigma_1 = -347.82 \text{ N/mm}^2 //$$

Element 2

$$\sigma_2 = 70\times10^{3}\times\frac{1}{300}[-1 \;\; 1]\begin{bmatrix} -0.8696 \\ -2.6466\times10^{-5} \end{bmatrix}$$

$$\sigma_2 = 202.88 \text{ N/mm}^2 //$$