CMR
INSTITUTE OF
TECHNOLOGY

**Internal Assessment Test III – May 2018**

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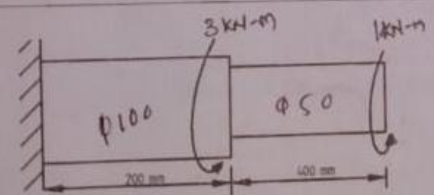
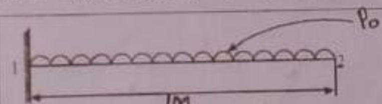
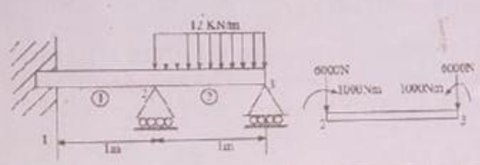
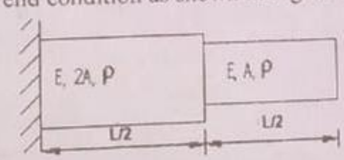
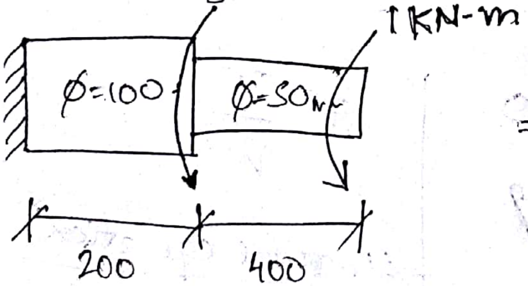
| **Sub:** | Finite Element Methods | **Code:** | 15ME61 |
| **Date:** 21/05/2018 **Duration:** 90 mins | Max Marks: 50 **Sem:** VI | **Branch:** | MECH |

**Note:** Answer all questions.

| | | Marks | OBE CO | OBE RBT |
|---|---|---|---|---|
| | PART- A (ANSWER ANY **ONE**) | | | |
| 1 | A solid stepped bar of circular cross section is subjected to a torque of 1 kN-m at its free end and 3 kN-m of torque at its change in cross-section as shown in Figure 1. Determine the angle of twist and shear stress in bar. Take E = 200 GPa & G = $7\times 10^4$ N/mm$^2$ <br>   Figure 1 <br>   $P_o$ = 5KN/m <br> E = 200GPa <br> I = $10^9$mm$^4$ <br> Figure 2 | 10 | CO4 | L2 |
| 2 | For the Cantilever beam subjected to UDL as shown in Figure 2, determine the deflections of the free end. Consider one element. | 10 | CO4 | L2 |
| | PART- B (BOTH COMPULSORY) | | | |
| 3. | For the beam and loading shown in Figure 3, determine <br> i. Slopes at 2 and 3 <br> ii. The vertical deflection at the midpoint of the distributed load. <br> Take E = 200 GPa, I = $4\times 10^6$ mm$^4$. <br> $200\times10^9$ <br>   Figure 3 | 20 | CO3 | L4 |
| 4 | Find eigen values and eigen vector for stepped bar when it is subjected to axial vibration, with fixed free end condition as shown in Figure 4. Draw mode shapes <br>   Figure 4 | 20 | CO5 | L4 |

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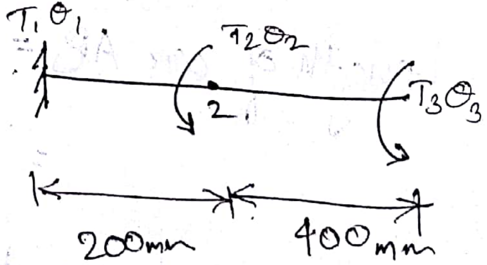



**Elemental stiffness matrix:**

$$k = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow J = \frac{\pi d^4}{32} = \frac{\pi \times 100^4}{32} \text{ mm}^4$$

$$= \frac{7\times10^4 \times 9.817\times10^6}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad = 9.817\times10^6$$

$$= 10^9 \begin{bmatrix} 3.435 & -3.435 \\ -3.435 & 3.435 \end{bmatrix}$$

$$K_2 = \frac{7\times10^9 \times 6.135\times10^5}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad J = \frac{\pi \times 50^4}{32}$$

$$K_2 = 10^9 \begin{bmatrix} 0.107 & -0.107 \\ -0.107 & 0.107 \end{bmatrix}$$

Global stiffness matrix:

$$K = 10^9 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3.43 & -3.43 & 0 \\ -3.43 & 3.537 & -0.107 \\ 0 & -0.107 & 0.107 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Equillibrium

$$[K][\theta] = [T]$$

$$\begin{bmatrix} \cancel{3.43} & \cancel{-3.43} & \cancel{0} \\ \cancel{-3.43} & 3.537 & -0.107 \\ \cancel{0} & -0.107 & 0.107 \end{bmatrix} \begin{bmatrix} \cancel{\theta_1} \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} \cancel{0} \\ 3\times10^6 \\ 1\times10^6 \end{bmatrix}$$

Angle of twist $\Rightarrow \theta_2 = 1.16\times10^{-3}$ rad $\qquad \theta_3 = 10.615\times10^{-3}$ rad

Shear stress :- $\tau_1 = \frac{GR}{L}(\theta_2 - \theta_1)$

$$= \frac{7\times10^4 \times 50}{200}(1.16\times10^{-3} - 0)$$

$$\boxed{\tau_1 = 20.3 \ N/mm^2}$$

$$\tau_2 = \frac{G R_2}{L_2}(\theta_3 - \theta_2)$$

$$= \frac{7\times10^4 \times 25}{400}(10.15\times10^{-3} - 1.16\times10^{-3})$$

$$\boxed{\tau_2 = 39.33 \ N/mm^2}$$

2 Elemental Stiffness matrix

$$K = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ +6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$K = \frac{200\times10^9\times10^{-3}}{1^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

Load vector

$$f = \begin{bmatrix} -2500 \\ -416.67 \\ -2500 \\ 416.67 \end{bmatrix} = \begin{bmatrix} Pl_e/2 \\ Pl_e^2/12 \\ Pl_e/2 \\ -Pl_e^2/12 \end{bmatrix}$$

Equilibrium eqn

$$[K][q] = [F]$$

$$2\times10^{12}\begin{bmatrix} \cancel{12} & \cancel{6} & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -2500 \\ -416.67 \\ -2500 \\ 416.67 \end{bmatrix}$$

Disp. at node 2 ; $v_2 = -3.12\times10^{-10}\ m$

$\theta_2 = -4.16\times10^{-6}\ rad$

3. Elemental Stiffness matrix

$$K = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$K_1 = \frac{200\times10^9 \times 4\times10^{-6}}{1^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$K_2 = 8\times10^5\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

Global Stiffness

$$K = 8\times10^5\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}$$

Load vector

$$f_e = \begin{bmatrix} 0 \\ 0 \\ -6\times10^3 \\ -1\times10^3 \\ -6\times10^3 \\ +1\times10^3 \end{bmatrix}$$

Equilibrium eqn

$$[K][q] = [F]$$

$$8\times10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6\times10^3 \\ -1\times10^3 \\ -6\times10^3 \\ 1\times10^3 \end{bmatrix}$$

Slope at node 2 ; $\theta_2 = -2.67\times10^{-4}$ rad

Slope at node 3 ; $\theta_3 = 4.46\times10^{-4}$ rad

Max' deflection

$$y = H.q$$

$$y = H_1 . v_2 + H_2 . \frac{l_e}{2} . \theta_2 + H_3 . v_3 + H_4 . \frac{l_e}{2} \theta_3$$

$\xi = 0$ at center

$$H_1 = \frac{1}{4}\left[2 - 3\xi + \xi^3\right] = \frac{1}{2}$$

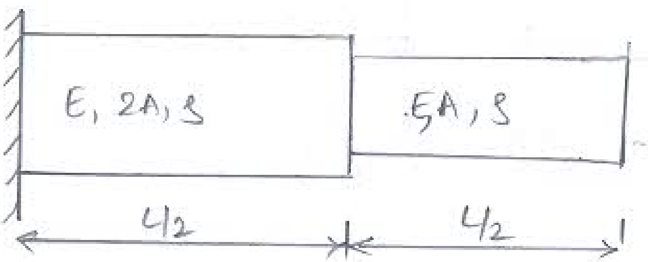
$$H_2 = \frac{1}{4}\left[1 - \xi - \xi^2 + \xi^3\right] = \frac{1}{4}$$

$$H_3 = \frac{1}{4}\left[2 + 3\xi - \xi^3\right] = \frac{1}{2}$$

$$H_4 = \frac{1}{4}\left(-1 - \xi - \xi^2 + \xi^3\right) = -\frac{1}{4}$$

$$y = 8.9\times10^{-5}\ \text{m}$$

(P) Find eigen values & eigen vectors for stepped bar when it is subjected to axial vibration, with fixed free end condition as shown. Draw mode shape.




Sol: F.E. Model


Elemental stiffness matrix

$$K = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_1 = \frac{4EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{EA}{L}\begin{array}{c} \begin{matrix} 1 & \quad 2 \end{matrix} \\ \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array}\begin{matrix} 1 \\ 2 \end{matrix}$$

$$K_2 = \frac{2EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{EA}{L}\begin{array}{c} \begin{matrix} 2 & \quad 3 \end{matrix} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}\begin{matrix} 2 \\ 3 \end{matrix}$$

Global Stiffness

$$K = \frac{EA}{L}\begin{array}{c} \begin{matrix} 1 & \quad 2 & \quad 3 \end{matrix} \\ \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{array}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

## Elemental Mass matrix

$$m^e = \frac{\rho A l_e}{6}\begin{bmatrix} 2 & +1 \\ +1 & 2 \end{bmatrix}$$

$$m^{e_1} = \frac{\rho . 2A . L}{2 . 6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{\rho A L}{12} \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ & \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$m^{e_2} = \frac{\rho . A L}{12} \begin{array}{cc} & \begin{array}{cc} 2 & 3 \end{array} \\ & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

## Global Mass matrix

$$m^e = \frac{\rho A L}{12} \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ & \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

## Eqn of motion

$$|K - \lambda M| = 0$$

$$\frac{EA}{L}\begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \frac{\rho A L}{12} = 0$$

$$\left| \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} - \frac{\rho L^2}{12 E} \lambda \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} - b \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \right| = 0 \qquad \because b = \frac{\rho L^2}{12E}\lambda$$

$$\begin{vmatrix} 6-6b & -(2+b) \\ -(2+b) & 2-2b \end{vmatrix} = 0$$

$$(6-6b)(2-2b) - (2+b)^2 = 0$$

$$12 - 12b - 12b + 12b^2 - 4 - b^2 - 4b = 0$$

$$8 - 28b + 11b^2 = 0$$

$b_1 = 0.3279$ $\qquad$ $b_2 = 2.217$

$$\frac{\rho L^2}{12E} \cdot \lambda = 0.3279$$

$$\lambda_1 = \frac{3.9348\, E}{\rho L^2} = \omega_1^2$$

$$\omega_1 = \frac{1.98}{L}\sqrt{\frac{E}{\rho}} \text{ r/s}$$

$$\frac{\rho L^2}{12E} \cdot \lambda = 2.217$$

$$\lambda_2 = \frac{26.604\, E}{\rho L^2} = \omega_2$$

$$\omega_2 = \frac{5.16}{L}\sqrt{\frac{E}{\rho}} \text{ r/s}$$

<u>For first mode shape</u>

$$|K - \lambda M|[x] = 0$$

$$\left(\frac{EA}{L}\begin{bmatrix}6 & -2\\ -2 & 2\end{bmatrix} - \left(3.9348\,\frac{E}{8L^2}\right)\cdot\frac{8AL}{12}\begin{bmatrix}6 & 1\\ 1 & 2\end{bmatrix}\right)\begin{bmatrix}x_2^{(1)}\\ x_3^{(1)}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\left(\begin{bmatrix}6 & -2\\ -2 & 2\end{bmatrix} - \begin{bmatrix}1.98 & 0.33\\ 0.33 & 0.66\end{bmatrix}\right)\begin{bmatrix}x_2^{(1)}\\ x_3^{(1)}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

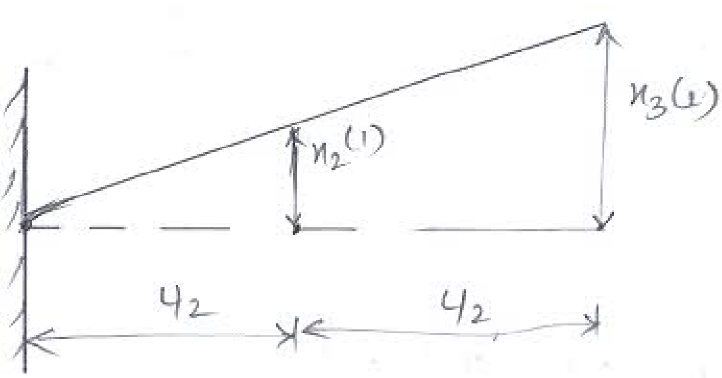
$$\begin{bmatrix}4.02 & -2.33\\ -2.33 & 1.34\end{bmatrix}\begin{bmatrix}x_2^{(1)}\\ x_3^{(1)}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$4.02\,x_2^{(1)} - 2.33\,x_3^{(1)} = 0$$

$$x_3^{(1)} = 1.72\,x_2^{(1)}$$

Eigen vector or 1st mode shape

$$\left[x^{(1)}\right] = \begin{bmatrix}x_2^{(1)}\\ x_3^{(1)}\end{bmatrix} = \begin{bmatrix}x_2^{(1)}\\ 1.72\,x_2^{(1)}\end{bmatrix} = x_2^{(1)}\begin{bmatrix}1\\ 1.72\end{bmatrix}$$



For <u>Second</u> <u>mode</u> <u>shape</u>

put $\omega_2^2 = \lambda_2$ in $|K - \lambda M| [x] = 0$.

$$\left(\frac{\cancel{EA}}{\cancel{L}}\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} - \left[\frac{26.604 \cancel{E}}{8\cancel{L}^2}\right]\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \frac{\cancel{8}A\cancel{L}}{12}\right)\begin{bmatrix} x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} - 2.217\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{bmatrix} x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 13.302 & 2.217 \\ 2.217 & 4.434 \end{bmatrix}\right)\begin{bmatrix} x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 - 13.302 & -2 - 2.217 \\ -2 - 2.217 & 2 - 4.434 \end{bmatrix}\begin{bmatrix} x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7.302 & -4.217 \\ -4.217 & -2.434 \end{bmatrix}\begin{bmatrix} x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-7.302\, x_2^{(2)} - 4.217\, x_3^{(2)} = 0$$

$$x_3^{(2)} = -1.72\, x_2^{(2)}$$

$$\left[x^{(2)}\right] = \begin{bmatrix} x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} x_2^{(2)} \\ -1.72\, x_3^{(2)} \end{bmatrix}$$

$$= x_2^{(2)} \begin{bmatrix} 1 \\ -1.72 \end{bmatrix} \quad //$$