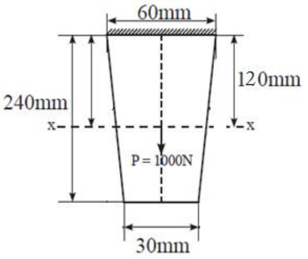
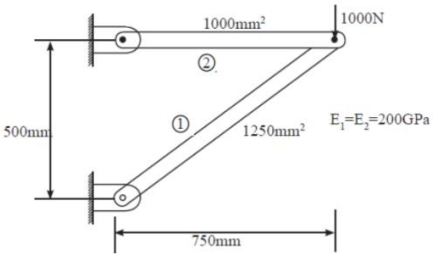
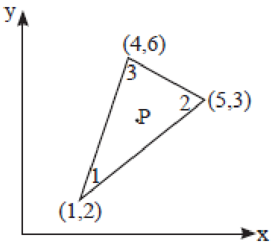
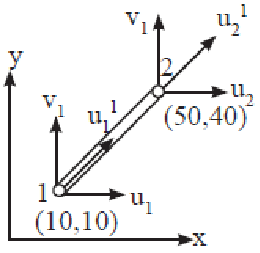
Internal Assessment Test II – April 2019

Sub: Finite Element Methods **Code:** 15ME61

Date: 15/ 04 / 2019 Duration: 90 mins Max Marks: 50 **Sem:** VI **Branch:** MECH

Note: Answer any **five** questions.

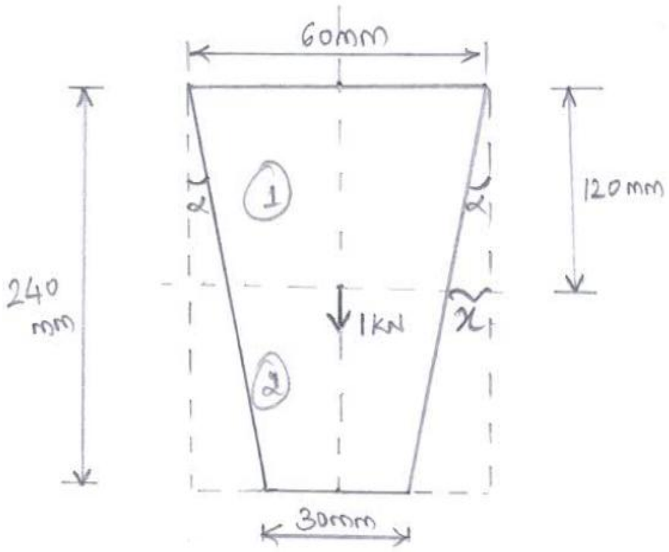
			Marks	OBE	
				CO	RBT
1		Figure 1 shows a thin plate of uniform thickness of 1mm, young's modulus E = 200GPa, weight density of the plate 76.6 × 10^{-6} N/mm^3. In addition to its weight, it is subjected to a point load of 1000N at its mid-point. Model the plate with 2 bar element.  Figure 1  Figure 2	10	CO2	L2
2		Derive shape function of Quadratic Bar element in natural co-ordinates.	10	CO3	L3
3		Derive the stiffness matrix for truss element in terms of direction cosines.	10	CO4	L3
4		For the Pin Jointed Configuration shown in below Figure 2. Formulate the stiffness matrix also determine the nodal displacements.	10	CO4	L4
5		Derive shape functions for QUAD 4 element using Lagrangian element.	10	CO3	L3
6	a	The nodal co-ordinates of a triangular element are shown in Figure 3. The x - coordinates of an interior point P is 3.3 and shape function. $N_1 = 0.3$. Determine N_2, N_3 and y - coordinate point P.  Figure 3  Figure 4	04	CO3	L2
	b	For the truss element shown in below Figure 4 (x, y) co-ordinates of the element are indicated near nodes 1, 2. The element displacement dof vector is given by $[u] = [1.5, 1.0, 2.1, 4.3]^T \times 10^{-2}$ mm Take E = 300×10^3 N/mm^2 A = 100 mm^2 Determine the following a) Element displacement dof in local co-ordinates b) Stress in the element c) Stiffness matrix of the element	06	CO4	L2

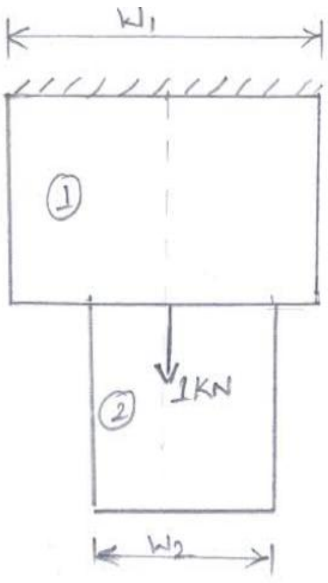
IAT 2 - Solution April 2019

Subject : FEM(15ME61)

1. **Given:**

   Thin plate of uniform thickness of 1mm, young's modulus E = 200GPa, weight density of the plate 76.6×10^{-6} N/mm^3





$E = 200 GPa = 200 \times 10^3$ Mpa

From fig.

$\tan\alpha = \frac{15}{240}$; $\tan\alpha = \frac{x}{120}$

$\alpha = 3.57°$ $x = 7.5$ mm

Volume of tappered plate ① = Volume of stepped plate ①

Volume $= \frac{h}{2}$ [Addition of two ‖'el sides] × thickness

$$\frac{120}{2}\left[60+45\right] \times 1 = W_1 \times 120 \times 1$$

$W_1 = 52.5$ mm//

Area $= A_1 = W_1 \times t = 52.5 \times 1$

$A_1 = 52.5$ mm²//

$$\frac{120}{2}\left[45+30\right] \times 1 = W_2 \times 120 \times 1.$$

$W_2 = 37.5$mm//

$A_2 = W_2 \times t = 37.5$ mm²//

Elemental Stiffness matrix

$$K = \frac{EA}{le}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For element 1

$$K_1 = \frac{200 \times 10^3 \times 52.5}{120} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^5 \begin{array}{c} \\ \\ \end{array}\begin{array}{cc|c} 1 & 2 & \\ \hline 0.875 & -0.875 & 1 \\ -0.875 & 0.875 & 2 \end{array}$$

For element 2

$$K_2 = \frac{200 \times 10^3 \times 37.5}{120} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^5 \begin{array}{cc|c} 2 & 3 & \\ \hline 0.625 & -0.625 & 2 \\ -0.625 & 0.625 & 3 \end{array}$$

Global Stiffness matrix

$$K = 10^5 \begin{array}{ccc|c} 1 & 2 & 3 & \\ \hline 0.875 & -0.875 & 0 & 1 \\ -0.875 & 1.5 & -0.625 & 2 \\ 0 & -0.625 & 0.625 & 3 \end{array}$$

Elemental body forces

$$F = \frac{\rho A l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For element 1

$$F_1 = \frac{76.6 \times 10^{-6} \times 52.5 \times 120}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{array}{c|c} 0.24129 & 1 \\ 0.24129 & 2 \end{array}$$

For element 2

$$F_2 = \frac{76.6 \times 10^{-6} \times 37.5 \times 120}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.17235 \\ 0.17235 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

Global force vector

$$F = \begin{bmatrix} 0.24129 \\ 1000.41364 \\ 0.17235 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

Equilibrium eqn $[K][q] = [F]$

$$10^5 \begin{bmatrix} 0.875 & -0.875 & 0 \\ -0.875 & 1.5 & -0.625 \\ 0 & -0.625 & 0.625 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0.24129 \\ 1000.41364 \\ 0.17235 \end{bmatrix}$$

Applying B.C :- Using Elimination method

$$10^5 \begin{bmatrix} 0.875 & -0.875 & 0 \\ -0.875 & 1.5 & -0.625 \\ 0 & -0.625 & 0.625 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0.24129 \\ 1000.41364 \\ 0.17235 \end{bmatrix}$$

$q_1 = 0$; $q_2 = 0.01144$ mm ; $q_3 = 0.01144$ mm

Displacement at Node 1 ; $q_1 = 0$

Node 2 ; $q_2 = 0.01144$ mm

Node 3 ; $q_3 = 0.01144$ mm

QUADRATIC ELEMENT OR 3 NODED BAR ELEMENT

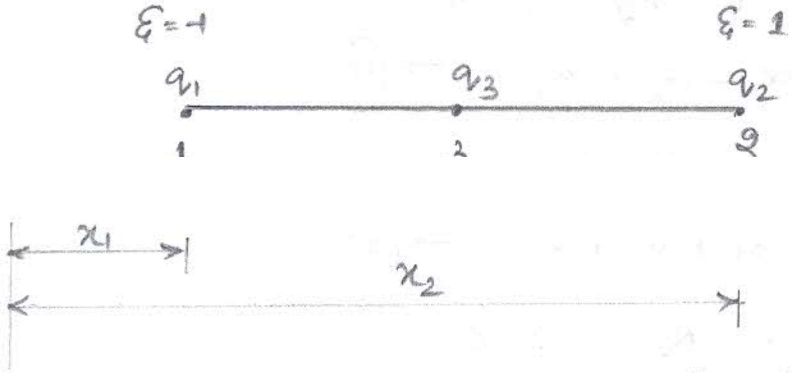


Fig. shows a 3 noded bar element having end nodes 1, 2 & midside node 3. At each node 1 DOF.

DERIVATION OF SHAPE FUNCTION

Let $N_1 = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2$

At node 1, $N_1 = 1$, $\xi = -1$.

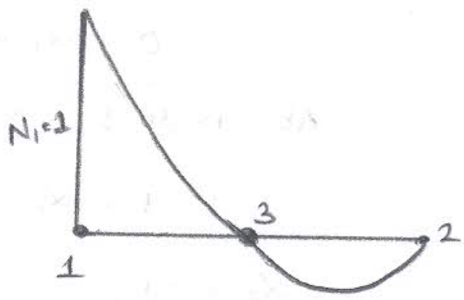
$$1 = \alpha_1 - \alpha_2 + \alpha_3 \rightarrow ①$$

At node 2, $N_1 = 0$, $\xi = 1$.

$$0 = \alpha_1 + \alpha_2 + \alpha_3 \rightarrow ②$$

At node 3, $N_1 = 0$, $\xi = 0$.

$$\alpha_1 = 0 \rightarrow ③$$



$\alpha_2 = -1/2$; $\alpha_3 = 1/2$

$$\boxed{N_1 = \frac{\xi}{2}(\xi - 1)}$$

Let $N_2 = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2$

At node 1, $N_2 = 0$, $\xi = -1$.

$$0 = \alpha_1 - \alpha_2 + \alpha_3 \rightarrow ④$$

At node 2, $N_2 = 1$, $\xi = 1$

$$1 = \alpha_1 + \alpha_2 + \alpha_3 \rightarrow ⑤$$

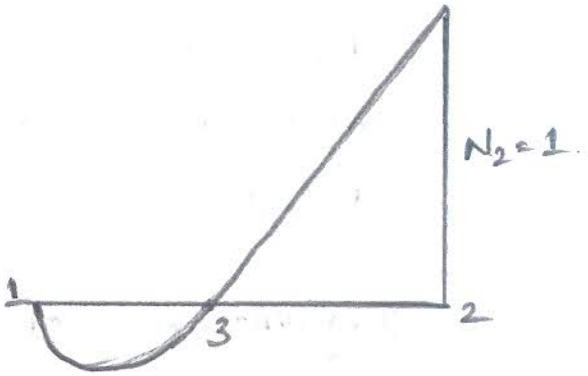
At node 3, $N_2 = 0$, $\xi = 0$

$$0 = \alpha_1 \rightarrow ⑥$$

$\alpha_2 = 1/2$; $\alpha_3 = 1/2$

$$N_2 = \frac{1}{2}\xi + \frac{1}{2}\xi^2$$

$$\boxed{N_2 = \frac{\xi}{2}(\xi + 1)}$$



Let $N_3 = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2$

At node 1, $N_3 = 0$, $\xi = -1$.

$$0 = \alpha_1 - \alpha_2 + \alpha_3 \rightarrow ⑦$$

At node 2, $N_3 = 0$, $\xi = 1$.

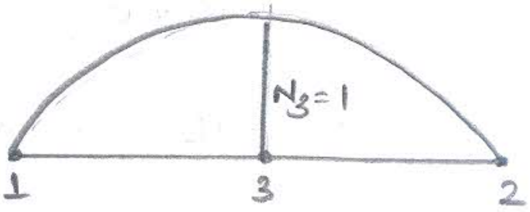
$$0 = \alpha_1 + \alpha_2 + \alpha_3 \rightarrow ⑧$$

At node 3, $N_3 = 1$, $\xi = 0$

$$1 = \alpha_1 \rightarrow ⑨$$

$\alpha_3 = -1$, $\alpha_2 = 0$

$$\boxed{N_3 = 1 - \xi^2}$$



3

DERIVATION OF ELEMENTAL STIFFNESS MATRIX FOR A TRUSS ELEMENT

Relation b/w nodal displacement of a truss element in local coordinates & global coordinate is expressed as

$$q' = L q$$

Where $q' = \begin{bmatrix} q_1' \\ q_2' \end{bmatrix}$; $L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$; $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$

A truss element in local coordinate is equivalent to 1D bar element having the stiffness matrix

$$K' = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Strain energy for a truss element in local coordinate is given by

$$U_e = \frac{1}{2} q'^T K' q' \rightarrow ①$$

It is required to determine strain energy of truss element in global coordinates.

$$q' = L q$$

Sub. this in eqn ① we get

$$U_e = \frac{1}{2} q^T L^T K' L q$$

$$U_e = \frac{1}{2} q^T \left(L^T k' L \right) q$$

$$U_e = \frac{1}{2} q^T k_e q$$

Where k_e - elemental stiffness matrix in global coordinates

$$K_e = L^T K' L$$

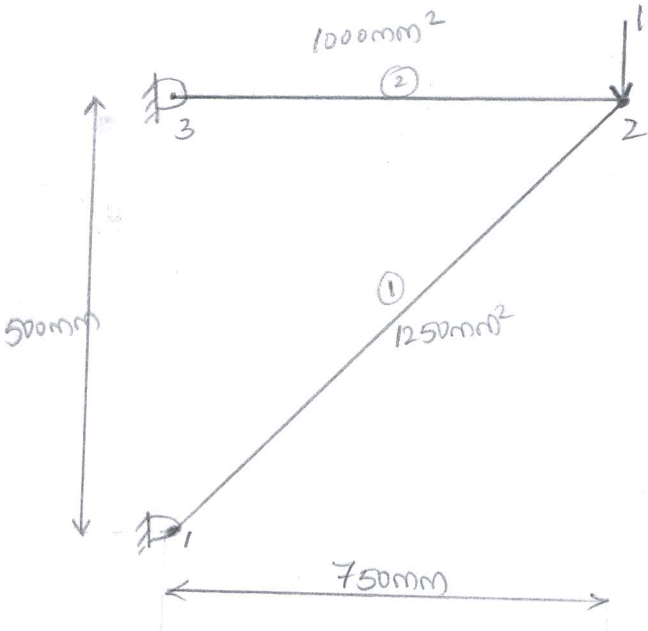
$$L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} ; \quad L^T = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} ; \quad K' = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$K_e = \frac{EA}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Where l & m are direction cosines.

4.



$E_1 = E_2 = 200 GPa$

Sol

Nodal Data

Node No	x (mm)	y (mm)
1	0	0
2	750	500
3	0	500

Element Connectivity table

Element No	Initial Node (IN)	Final Node (FN)	length of element l_e	l	m.
1	1	2	901.387	0.832	0.554
2	2	3	750	-1	0

$$l_e = \sqrt{(x_{FN} - x_{IN})^2 + (y_{FN} - y_{IN})^2}$$

$$l = \frac{x_{FN} - x_{IN}}{l_e} \quad ; \quad m = \frac{y_{FN} - y_{IN}}{l_e}$$

$$K = \frac{EA}{le} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

For element 1

$$K = \frac{2 \times 10^5 \times 1250}{901.387} \begin{bmatrix} 0.69 & 0.46 & -0.69 & -0.46 \\ 0.46 & 0.31 & -0.46 & -0.31 \\ -0.69 & -0.46 & 0.69 & 0.46 \\ -0.46 & -0.31 & 0.46 & 0.31 \end{bmatrix}$$

$$K_1 = 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 1.9 & 1.27 & -1.9 & -1.27 \\ 1.27 & 0.86 & -1.27 & -0.86 \\ -1.9 & -1.27 & 1.9 & 1.27 \\ -1.27 & -0.86 & 1.27 & 0.86 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$K_2 = 10^5 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} 2.66 & 0 & -2.66 & 0 \\ 0 & 0 & 0 & 0 \\ -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Global Stiffness

$$K = 10^5 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1.9 & 1.27 & -1.9 & -1.27 & 0 & 0 \\ 1.27 & 0.86 & -1.27 & 0.86 & 0 & 0 \\ -1.9 & -1.27 & 4.56 & 1.27 & -2.66 & 0 \\ -1.27 & 0.86 & 1.27 & 0.86 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Equilibrium eqn

$$[K][q] = [F]$$

$$10^5 \begin{bmatrix} 1.9 & 1.27 & -1.9 & -1.27 & 0 & 0 \\ 1.27 & 0.86 & -1.27 & 0.86 & 0 & 0 \\ -1.9 & -1.27 & 4.56 & 1.27 & -2.66 & 0 \\ -1.27 & 0.86 & 1.27 & 0.86 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1000 \\ 0 \\ 0 \end{bmatrix}$$

Applying B.C. Using elimination method.

$$10^5 \begin{bmatrix} 4.56 & 1.27 \\ 1.27 & 0.86 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1000 \end{bmatrix}$$

$q_3 = -5.5 \times 10^{-3}$ mm

$q_4 = 0.01975$ mm.

Nodal displacements.

At nodes 1, 3 = $q_1 = q_2 = q_5 = q_6 = 0$.

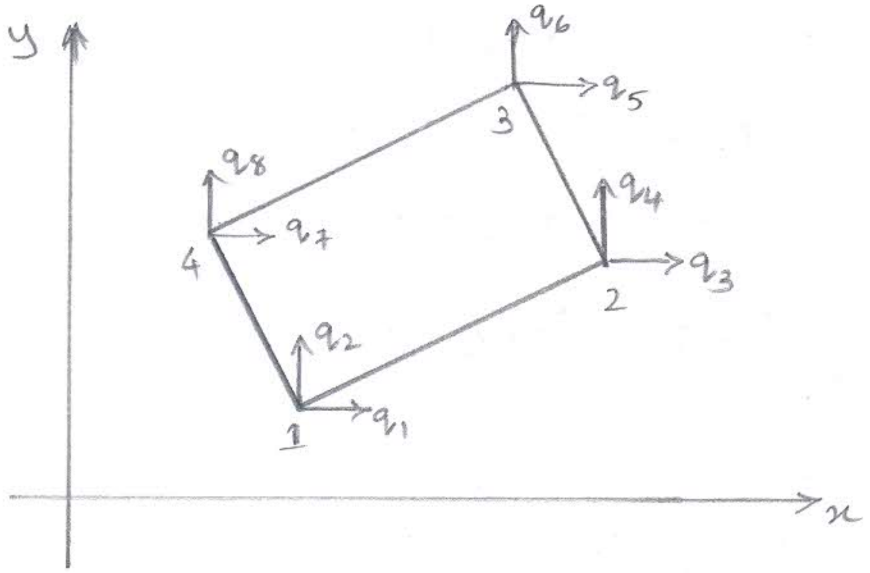
At node 2 ; $q_3 = -5.5 \times 10^{-3}$ mm //

$q_4 = 0.01975$ mm //

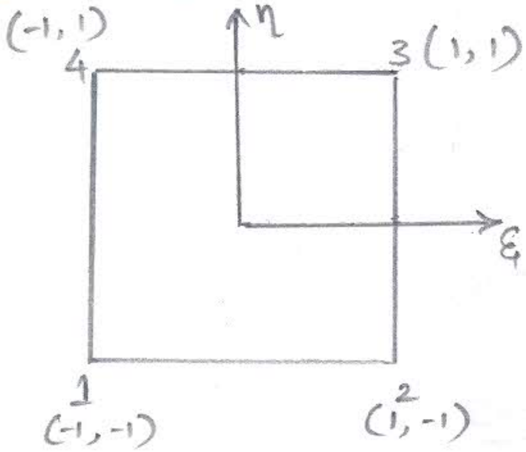
5

4 NODED QUADRILATERAL ELEMENT [QUAD 4 ELEMENT]

Fig. shows a quadrilateral element having 4 Nodes at each node 2 DOF.



DERIVATION OF SHAPE FUNCTION



$$N_i = \frac{1}{4}\left(1+\xi\xi_i\right)\left(1+\eta\eta_i\right)$$

$i = 1, 2, 3, 4$

$$N_1 = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)$$

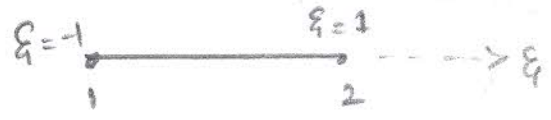
$$N_2 = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)$$

$$N_3 = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)$$

$$N_4 = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)$$

LAGRANGEAN METHOD

$$N_1\left(\xi, \eta\right) = L_1\left(\xi\right) L_1\left(\eta\right)$$



$$L_1\left(\xi\right) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{\xi - 1}{-1-1}$$

$$L_1\left(\xi\right) = \frac{1-\xi}{2}$$

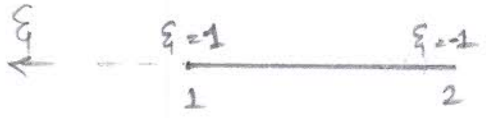
$$L_1(\eta) = \frac{\eta-\eta_4}{\eta_1-\eta_4} = \frac{\eta-1}{-1-1} = \frac{1-\eta}{2}$$

$$\therefore \quad N_1 = L_1(\xi)\; L_1(\eta)$$

$$= \frac{1-\xi}{2} \cdot \frac{1-\eta}{2}$$

$$\boxed{N_1 = \frac{1}{4}(1-\xi)(1-\eta)}$$

$$N_2(\xi,\eta) = L_2(\xi)\; L_2(\eta)$$



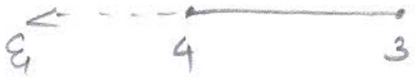
$$L_2(\xi) = \frac{\xi-\xi_1}{\xi_2-\xi_1} = \frac{\xi-1}{-1-1} = \frac{1+\xi}{2}$$

$$L_2(\eta) = \frac{\eta-\eta_3}{\eta_2-\eta_3} = \frac{\eta+1}{-1-1} = \frac{1-\eta}{2}$$

$$N_2 = \frac{1+\xi}{2}\;\frac{1-\eta}{2}$$

$$\boxed{N_2 = \frac{1}{4}(1+\xi)(1-\eta)}$$

$$N_3(\xi,\eta) = L_3(\xi)\; L_3(\eta)$$



$$L_3(\xi) = \frac{\xi-\xi_4}{\xi_3-\xi_4} = \frac{\xi+1}{+1+1} = \frac{1+\xi}{2}$$

$$L_3(\eta) = \frac{\eta-\eta_2}{\eta_3-\eta_2} = \frac{\eta+1}{1+1} = \frac{1+\eta}{2}$$

$$\boxed{N_3 = \frac{1}{4}(1+\xi)(1+\eta)}$$

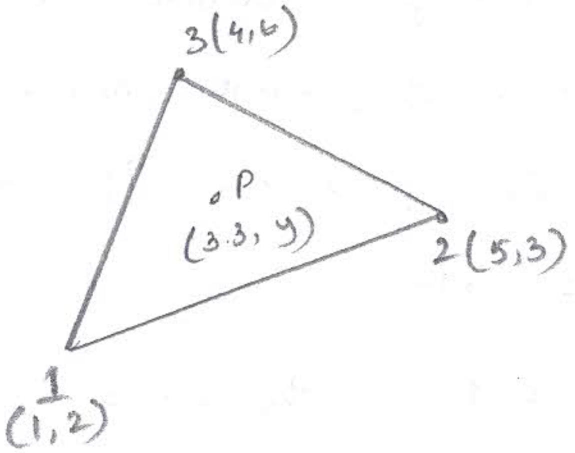
$$N_4(\xi,\eta) = L_4(\xi)\ L_4(\eta)$$

$$L_4(\xi) = \frac{\xi-\xi_3}{\xi_4-\xi_3} = \frac{\xi-1}{-1-1} = \frac{1-\xi}{2}$$

$$L_4(\eta) = \frac{\eta-\eta_1}{\eta_4-\eta_1} = \frac{\eta+1}{1+1} = \frac{1+\eta}{2}$$

$$\boxed{N_4 = \frac{1}{4}(1-\xi)(1+\eta)}$$

6 a



$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$3.3 = 0.3(1) + \eta(5) + (1-\xi-\eta)4$$

$$3.3 = 0.3 + 5\eta + 4 - 4\xi - 4\eta$$

$$-4\xi + \eta = -1.$$

$$-4(0.3) + \eta = -1$$

$$\eta = 0.2$$

$$\therefore N_2 = \eta = 0.2 //$$

$$N_3 = 1-\xi-\eta = 1-0.3-0.2 = 0.5$$

$$N_3 = 0.5 //$$

<u>y Coordinate</u>

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$= 0.3(2) + 0.2(3) + 0.5(6)$$

$$y = 4.2 //$$

6 .b

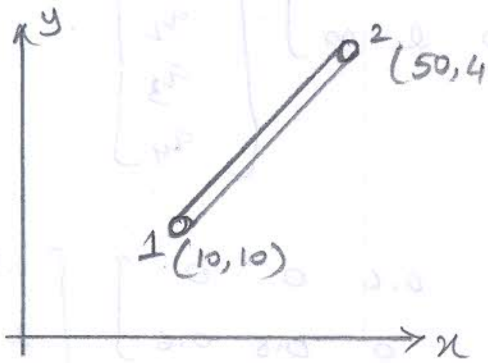
For the truss element shown (x, y) coordinates are indicated near nodes 1, 2. The element displacement dof vector is given by $\{q] = [1.5, 1.0, 2.1, 4.3]^T \times 10^{-2}$ mm.

Take $E = 300 \times 10^3$ N/mm². $A = 100$ mm². Determine

i) Element displacement dof in local coordinate (q_1', q_2')

ii) Stress in element

iii) Stiffness matrix of element



Sol

Node Data

Node No	x (mm)	y (mm)
1	10	10
2	50	40

Element Connectivity Table

Element No	Initial Node	Final Node	Length of element	l	m
1	1	2	50	0.8	0.6

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(50-10)^2 + (40-10)^2}$$

$$l_e = 50\text{mm}$$

$$l = \frac{x_{FN} - x_{IN}}{l_e} = \frac{40}{50} = 0.8$$

$$m = \frac{y_{FN} - y_{SN}}{l_e} = \frac{30}{10} = 0.6$$

i) Element displacement dof in local coordinate

$$q' = L\,q$$

$$\begin{bmatrix} q_1' \\ q_2' \end{bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1 \\ 2.1 \\ 4.3 \end{bmatrix} \times 10^{-2}$$

$$\begin{bmatrix} q_1' \\ q_2' \end{bmatrix} = \begin{bmatrix} 0.018 \\ 0.0426 \end{bmatrix} \text{mm} //$$

ii) Stress

$$\sigma = E \cdot \frac{1}{l_e} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$= 300 \times 10^{3} \times \frac{1}{50} \begin{bmatrix} -0.8 & -0.6 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1 \\ 2.1 \\ 4.3 \end{bmatrix} \times 10^{-2}$$

$$\sigma = 147.6 \text{ N/mm}^2 //$$

Stiffness matrix

$$K = \frac{EA}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= \frac{300\times10^3\times 100}{50}\begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}$$

$$K = 10^5\begin{bmatrix} 3.84 & 2.88 & -3.84 & -2.88 \\ 2.88 & 2.16 & -2.88 & -2.16 \\ -3.84 & -2.88 & 3.84 & 2.88 \\ -2.88 & -2.16 & 2.88 & 2.16 \end{bmatrix} //$$