DSP IAT 1 SOLUTION

1. Derive definition of DFT and its inverse(IDFT).

Let us consider a discrete time signal $x(n)$ having a finite duration, say in the range $0 \leqslant n \leqslant N-1$. The DTFT of this signal is

$$X(\omega) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n} \tag{3.1}$$

Let us sample $X(\omega)$ using a total of N equally spaced samples in the range: $\omega \in (0, 2\pi)$, so the sampling interval is $\frac{2\pi}{N}$. That is, we sample $X(\omega)$ using the frequencies

$$\omega = \omega_k = \frac{2\pi k}{N}, \quad 0 \leqslant k \leqslant N-1$$

The result is, by definition the DFT.

That is,

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j\omega_k n} \\ &= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} \end{aligned} \tag{3.2}$$

Equation (3.2) is known as N-point DFT analysis equation. Fig. 3.1 shows the Fourier transform of a discrete-time signal and its DFT samples.

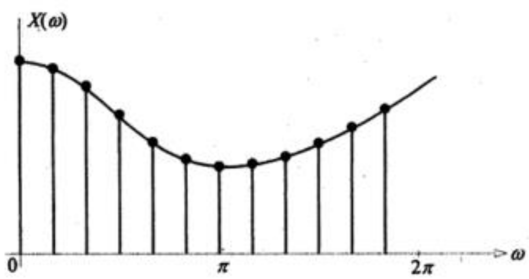

Fig. 3.1 Sampling of $X(\omega)$ to get $X(k)$. Solid line: $X(\omega)$; dots: DFT samples (shown for N=12).

While working with DFT, it is customary to introduce a complex quantity:

$$W_N = e^{-j\frac{2\pi}{N}}$$

Also, it is very common to represent the DFT operation for a sequence $x(n)$ of length N by DFT $\{x(n)\}$. As a consequence of this notation, we can rewrite equation (3.2) as

$$X(k) = \text{DFT}\{x(n)\} = \sum_{0}^{N-1} x(n)W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

The DFT values ($X(k),\ 0 \leqslant k \leqslant N-1$), uniquely define the sequence $x(n)$ through the inverse DFT formula (IDFT):

$$x(n) = \text{IDFT}\{X(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \quad 0 \leqslant n \leqslant N-1 \tag{3.4}$$

The above equation is known as the synthesis equation.

Proof:

$$\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x(m)W_N^{km}\right]W_N^{-kn}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left[\sum_{k=0}^{N-1} W_N^{-(n-m)k}\right]$$

2. Compute the N-point DFT of the signal, $x(n)=a^n$; $0 \le n \le N-1$. Also find the DFT of the sequence $x(n)=0.5^n u[n]$; $0 \le n \le 3$.

$$\begin{aligned} X(k) &= \text{DFT}\{x(n)\} \\ &= \sum_{n=0}^{N-1} x(n)W_N^{kn} \\ &= \sum_{n=0}^{N-1} a^n W_N^{kn} \\ &= \sum_{n=0}^{N-1} (aW_N^k)^n \end{aligned}$$

We know that $$\sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b-1}, \quad b \neq 1$$

Hence, $$\begin{aligned} X(k) &= \frac{a^N W_N^{kN} - 1}{aW_N^k - 1} \\ &= \frac{a^N - 1}{aW_N^k - 1}, \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

5) compute the 4 point DFT of the sequence

$x(n) = 0.5^n u(n)$; $0 \le n \le 3$.

Sol

w.k.t

$$a^n \xleftrightarrow{\text{DFT}} \frac{1 - a^N \cancel{W_N^{kN}}^{\,1}}{1 - aW_N^k}$$

$N = 4$,

$$\therefore X(k) = \frac{1-(0.5)^4 W_4^{4k}}{1 - 0.5\, W_4^k}$$

$$= \frac{1-(0.5)^4 \cdot 1}{1-(0.5)e^{-j\pi/2 k}}$$

$$\boxed{X(k) = \frac{0.9375}{1-(0.5)e^{-j\pi/2 k}}}$$

$W_4^{4k} = e^{-j2\pi 4k/4}$
$= 1$

$W_4^k = e^{-j2\pi k/4}$

$= \cos \pi/2\, k - j \sin \pi/2\, k$

$X(k) = \{1.875,\ 0.75 - j0.375,\ 0.625,\ 0.75 + j0.375\}$, $0 \le k \le 3$

3. Compute N-point DFT of the sequence $x(n)=\cos(n\omega_0)$ where $\omega_0=2\pi k_0/N$ and $0\leq n\leq N-1$.

Given,
$$x(n) = \cos(n\omega_0)$$
$$\Rightarrow \quad x(n) = \frac{1}{2}e^{jn\omega_0} + \frac{1}{2}e^{-jn\omega_0}$$
$$= \frac{1}{2}e^{jn\frac{2\pi}{N}k_0} + \frac{1}{2}e^{-jn\frac{2\pi}{N}k_0}$$
$$= \frac{1}{2}e^{-j\frac{2\pi}{N}(-k_0n)} + \frac{1}{2}e^{-j\frac{2\pi}{N}(k_0n)}$$
$$= \frac{1}{2}W_N^{-k_0n} + \frac{1}{2}W_N^{k_0n}$$

Hence,
$$X(k) = \text{DFT}\{x(n)\}, \quad 0 \leqslant k \leqslant N-1$$
$$= \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(k-k_0)n} + \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(k+k_0)n}$$
$$= \frac{1}{2}S_1 + \frac{1}{2}S_2$$

$$S_1 = \sum_{n=0}^{N-1} W_N^{(k-k_0)n}$$
$$= \frac{W_N^{(k-k_0)N} - 1}{W_N^{(k-k_0)} - 1} = \frac{W_N^{kN}\,W_N^{-k_0N} - 1}{W_N^{(k-k_0)} - 1}$$
$$= \frac{1\times 1 - 1}{W_N^{(k-k_0)} - 1} = 0, \qquad k \neq k_0$$

When $k = k_0$, we get

$$S_1 = \sum_{n=0}^{N-1}(1)^n = N$$

Hence,
$$S_1 = \sum_{n=0}^{N-1} W_N^{(k-k_0)n} = \begin{cases} 0, & k \neq k_0 \\ N, & k = k_0 \end{cases}$$

or
$$S_1 = N\delta(k-k_0)$$

Similarly,
$$S_2 = N\delta(k+k_0)$$
$$= N\delta\,[k-(N-k_0)]$$

Thus,
$$X(k) = \frac{1}{2}S_1 + \frac{1}{2}S_2$$
$$= \frac{N}{2}\delta(k-k_0) + \frac{N}{2}\delta\,[k-(N-k_0)]$$

4. Show that

$$\sum_{n=0}^{N-1} W_N^{kn} = N\,\delta(k) = \begin{cases} N, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

$$\sum_{n=0}^{N-1} W_N^{kn} = N\,\delta(k) = \begin{cases} N, & k=0 \\ 0, & k \neq 0 \end{cases}$$

***Proof*:**

We know that
$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}; \quad a \neq 1$$

Applying the above result to the left side of equation (3.3), we get

$$\begin{aligned} \sum_{n=0}^{N-1} \left(W_N^k\right)^n = \frac{1-W_N^{kN}}{1-W_N^k} &= \frac{1-e^{-j\frac{2\pi}{N}kN}}{1-e^{-j\frac{2\pi}{N}k}}; \quad k \neq 0 \\ &= \frac{1-1}{1-e^{-j\frac{2\pi k}{N}}} \\ &= 0, \quad k \neq 0 \end{aligned}$$

when $k=0$, the left side of equation (3.3) becomes

$$\sum_{n=0}^{N-1} W_N^{0\times n} = \sum_{n=0}^{N-1} 1 = N$$

Hence, we may write

$$\begin{aligned} \sum_{n=0}^{N-1} W_N^{kn} &= \begin{cases} N, & k=0 \\ 0, & k \neq 0 \end{cases} \\ &= N\delta(k), \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

5. Find the 4-point DFT and IDFT of the sequence x(n)={0,1,2,3}

Problems

1. Find the DFT and IDFT for the sequence $x(n) = \{0, 1, 2, 3\}$.

Sol. Step 1 → $N = 4$

Step 2 → Compute 4 Twiddle factors $[W_N^k = e^{-j\frac{2\pi k}{N}}]$

$W_4^0 = 1, \quad W_4^1 = -j, \quad W_4^2 = -1, \quad W_4^3 = +j$

Step 3 → DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad 0 \le k \le N-1$$

(Matrix relation $X = W_N x$)

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$\therefore \boxed{X(k) = \{6, -2+2j, -2, -2-2j\}}$$

IDFT (Matrix relation, $x = \frac{1}{N} W_N^* X$; where W_N^* = complex conjugate of W_N).

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}, \quad 0 \le n \le N-1$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ W_4^0 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ W_4^0 & W_4^{-3} & W_4^{-6} & W_4^{-8} \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & +1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\boxed{x(n) = \{\underset{\uparrow}{0}, 1, 2, 3\}}$$

6. Find the 4-point DFT and IDFT of the sequence $x(n)=\cos(n\pi/2)$, $0 \le n \le 3$

2. Find the 4 point DFT of the sequence $x(n) = \cos\left(\frac{n\pi}{2}\right)$, $0 \le n \le 3$ & IDFT for it. x_1

Sol. DFT

$$x(n) = \{1, 0, -1, 0\}, \quad N = 4$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad 0 \le k \le N-1$$

$$[X_k]_{4\times1} = [W_N]_{4\times4} [x_n]_{4\times1}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\boxed{X(k) = \{\underset{\uparrow}{0}, 2, 0, 2\}}$$

IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}, \quad 0 \le n \le N-1$$

$$[x_n]_{4\times1} = \frac{1}{4} [W_N^*]_{4\times4} [X_k]_{4\times1}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\boxed{x(n) = \{\underset{\uparrow}{1}, 0, -1, 0\}}$$