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## Internal Assessment Test III– May 2019

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|---|---|---|---|---|---|---|---|
| Sub: | Engineering Mathematics-IV | | | | Sub Code: | 17MAT41 | |
| Date: | 13th May 2019 | Duration: | 90 mins | Max Marks: 50 | Sem / Sec: | CS-A,B,C;IS-A,B;EC-A,B,C;EE-A;CV-A;ME-A;TC-A | OBE |

**Question 1 is compulsory and answer any SIX questions from the rest.**

| No. | Question | MARKS | CO | RBT |
|---|---|---|---|---|
| 1. | Discuss the conformal transformation $w = e^z$. | [08] | CO3 | L3 |
| 2. | Find the conformal transformation which transforms $z = \infty, i, 0$ into $w = -1, -i, 1$. Also find fixed points. | [07] | CO3 | L3 |
| 3. | Evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)} dz$, where C: \|Z\| = 3. | [07] | CO3 | L3 |
| 4. | Find the analytic function whose real part is $u = \frac{x^4y^4 - 2x}{x^2+y^2}$. | [07] | CO3 | L3 |
| 5 | State and prove Cauchy's integral formula. | [07] | CO3 | L3 |


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6. Show that $f(z) = z^n$, where n is a positive integer is analytic and hence find its derivative. [07] CO3 L3

7. Five dice were thrown 96 times and the numbers 1, 2 or 3 appearing on the face of the dice follows the frequency distribution as below. [07] CO5 L3

| No. of dice showing 1, 2 or 3 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Frequency | 3 | 8 | 24 | 35 | 19 | 7 |

Test the hypothesis that the data follows a binomial distribution.( $\chi^2_{0.05} = 11.07$ for 5d.f.)

8. Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws to C. C is just as likely to throw the ball to A as to B. If C was the first person to throw the ball, find the probability that for the fourth throw,**(a)** A has the ball (b) B has the ball (c) C has the ball. [07] CO6 L3

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IAT-3 M4 (13-5-2019) Regular (A-Sections).

SOLUTION MANUAL- 17MAT 41

1. Given $w = e^z \Rightarrow \frac{dw}{dz} = e^z \neq 0$ for every $z$ in the finite complex plane.

$\therefore e^z$ is conformal at all finite 'z'

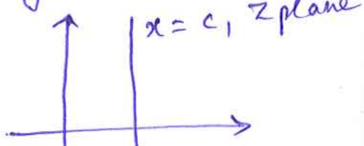
Further $w = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$\Rightarrow u = e^x \cos y \; ; \; v = e^x \sin y$

$\Rightarrow \quad u^2 + v^2 = (e^x \cos y)^2 + (e^x \sin y)^2 = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$

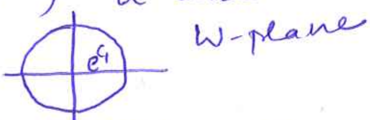
IIIly $\frac{u}{v} = \cot y$ or $v = u\tan y$

$\therefore$ we've $u^2 + v^2 = e^{2x} \rightarrow (A)$ & $v = u\tan y \rightarrow (B)$.



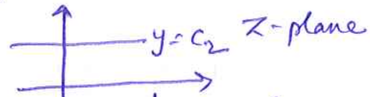

Case i: Let $x = c_1$ where $c_1$ is a constant

$\Rightarrow (A)$ can be written as $u^2 + v^2 = (e^{c_1})^2$ a circle with centre at '0' & radius $e^{c_1}$.

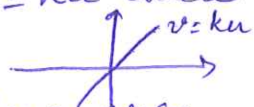



$\therefore$ Straight lines parallel to y axis in Z-plane are mapped to circles with centre at origin in the w-plane under $w = e^z$

Case ii Let $y = c_2$ where $c_2$ is a constant




$\Rightarrow (B)$ can be written as $v = u\tan c_2$ or $v = ku$ where $k$ is a constant.




$\therefore$ straight lines parallel to x-axis in the z-plane are mapped to lines through origin in the w-plane under $w = e^z$.

Also lines $x = c_1$ & $y = c_2$ represent family of orthogonal trajectories in Z-plane. Even their images $u^2 + v^2 = (e^{c_1})^2$ & $v = ku$ form orthogonal trajectories in the W-plane. $\therefore$ The conformal mapping $w = e^z$ transforms orthogonal trajectories in z-plane to orthogonal trajectories in the W plane.

2. To find the bilinear transformation that maps $\infty, i, 0$ into $w = -1, -i, 1$ respectively.

Let $w = \dfrac{az+b}{cz+d}$ be the desired bilinear transformation

$z = \infty$ & $w = -1 \Rightarrow w = \dfrac{z\left(a+\frac{b}{z}\right)}{z\left(c+\frac{d}{z}\right)} \Rightarrow -1 = \dfrac{a+0}{c+0}$ or $\boxed{a+c=0} \rightarrow (1)$

$z = i$ & $w = -i \Rightarrow -i = \dfrac{ai+b}{ci+d} \Rightarrow -i(ci+d) = ai+b \Rightarrow \boxed{ai+b-c+di=0} \rightarrow (2)$

$z = 0$ & $w = 1 \Rightarrow 1 = \dfrac{0+b}{0+d} \Rightarrow 1 = \dfrac{b}{d} \Rightarrow \boxed{b-d=0} \rightarrow (3)$

$(1)+(2) \Rightarrow \boxed{(1+i)a+b+di=0} \rightarrow (4)$.

from (3) & (4).

$$\frac{a}{\begin{vmatrix}1 & -1\\ 1 & i\end{vmatrix}} = \frac{b}{\begin{vmatrix}-1 & 0\\ i & 1+i\end{vmatrix}} = \frac{d}{\begin{vmatrix}0 & 1\\ 1+i & 1\end{vmatrix}} = k \text{ (say)}$$

$\Rightarrow \dfrac{a}{i+1} = \dfrac{b}{-1-i} = \dfrac{d}{-1-i} = k \Rightarrow$ $a = k(1+i) \Rightarrow c = -k(1+i)$

$b = -k(1+i)$

$d = -k(1+i)$

$\therefore w = \dfrac{k(1+i)z - k(1+i)}{-k(1+i)z - k(1+i)} = \dfrac{z-1}{-z-1} = \dfrac{1-z}{1+z}$

The fixed points are given by $w = z$ ie $\dfrac{1-z}{1+z} = z$

$\Rightarrow 1-z = z+z^2 \Rightarrow z^2+2z-1 = 0 \Rightarrow z = \dfrac{-2 \pm \sqrt{2^2-4(1)(-1)}}{2(1)} = \dfrac{-2\pm\sqrt{8}}{2}$

$= -1 \pm \sqrt{2}$

$\therefore$ The fixed (invariant) points are $-1+\sqrt{2}, -1-\sqrt{2}$

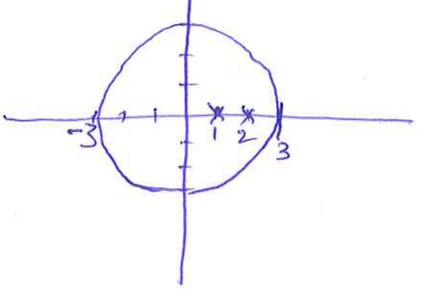
(3) Evaluate $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}\,dz$ where $C$ is $|z| = 3$

Soln:- Integrand $f(z) = \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}$ has poles at 1 & 2

and both of them are inside C

Further 1 is a pole of order 2

2 " " " 1



$$\text{Res}(f(z))\Big|_{z=1} = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d}{dz} (z-1)^2 f(z) = \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz} (z-1)^2 \frac{(\sin \pi z^2 + \cos \pi z^2)}{(z-1)^2(z-2)}$$

$$= \lim_{z\to 1} \frac{d}{dz} \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2}$$

$$= \lim_{z\to 1} \left[ (\sin \pi z^2 + \cos \pi z^2) \cdot \left( \frac{-1}{(z-2)^2} \right) + \frac{(\cos \pi z^2 \cdot 2\pi z - \sin \pi z^2 \cdot 2\pi z)}{(z-2)} \right]$$

$$= -1 \left( \frac{-1}{1} \right) + \frac{(-2\pi - 0)}{-1} = 1 + 2\pi$$

$$\text{Res}(f(z))\Big|_{z=2} = \lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} (z-2) \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$$

$$= \lim_{z\to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = \frac{0 + 1}{1^2} = 1$$

$\therefore$ By Cauchy's Residue Thm $\int_C f(z)\,dz = 2\pi i$ (Sum of residues of $f(z)$ at poles inside C)

$$= 2\pi i \,(1 + 2\pi + 1)$$
$$= 4\pi i \,(1+\pi)$$

(4) Given real part $u$ of $f(z) = u + iv$; $u = \dfrac{x^4 y^4 - 2x}{x^2 + y^2}$

$$u_x = \frac{\partial u}{\partial x} = \frac{(x^2+y^2)(4x^3y^4 - 2) - (x^4y^4 - 2x)(2x+0)}{(x^2+y^2)^2}$$

$$= \frac{4x^5y^4 + 4x^3y^6 - 2x^2 - 2y^2 - 2x^5y^4 + 4x^2}{(x^2+y^2)^2}$$

$$= \frac{2x^5y^4 + 4x^3y^6 + 2x^2 - 2y^2}{(x^2+y^2)^2}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{(x^2+y^2)(4x^4y^3 - 0) - (x^4y^4 - 2x)(0+2y)}{(x^2+y^2)^2}$$

$$= \frac{4x^6y^3 + 4x^4y^5 - 2x^4y^5 + 4xy}{(x^2+y^2)^2}$$

But since $f(z) = u + iv$ is analytic, $u$ & $v$ satisfy CR eqns ie $u_x = v_y$ & $u_y = -v_x$

Also $f'(z) = u_x + i v_x = \dfrac{2x^5y^4 + 6x^3y^6 + 2x^2 - 2y^2 - i(4x^6y^3 + 2x^4y^5 + 4xy)}{(x^2+y^2)^2}$

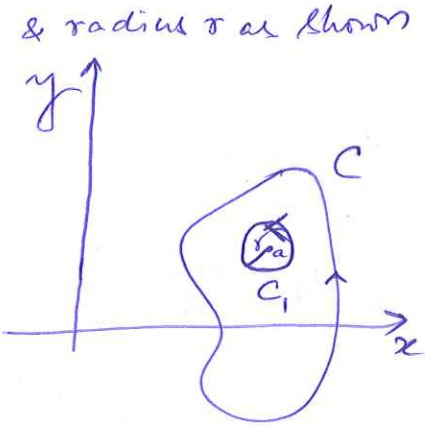
By Milne-Thompson method, put $x = z$ & $y = 0$ in the above

$\Rightarrow f'(z) = \dfrac{0 + 0 + 2z^2 - i(0)}{z^4} = \dfrac{2}{z^2} \Rightarrow f(z) = 2\left(\dfrac{-2}{z^3}\right) + c$

$$\boxed{f(z) = -\frac{4}{z^3} + c}$$

(5) Cauchy's Integral formula: If $f(z)$ is analytic within and on a simple closed curve $C$ and if '$a$' is any point inside $C$ then $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{z-a}\,dz$.

Proof:- Given '$a$' is a point inside $C$. Let us consider a circle $C_1$ lying entirely inside $C$ with centre at '$a$' & radius $r$ as shown




Clearly $f(z)$ is analytic in the region between $C$ & $C_1$. So by a consequence of Cauchy's Thm, $\displaystyle\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz$

$$\Rightarrow \int_C \frac{f(z)}{z-a}dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta$$

$$= i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\, d\theta$$

$\because$ $|z-a| = r$
$z - a = re^{i\theta}$
$z = a + re^{i\theta} \Rightarrow dz = ire^{i\theta}d\theta$
$\theta$ varies from 0 to $2\pi$

As $r \to 0$ on both sides we get

$$\lim_{r\to 0}\int_C \frac{f(z)}{z-a}dz = \lim_{r\to 0} i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

$$\Rightarrow \int_C \frac{f(z)}{z-a}dz = i\int_{\theta=0}^{2\pi} f(a)\,d\theta = i f(a)\cdot 2\pi = 2\pi i\, f(a).$$

Hence the proof.

(6) Given $f(z) = z^n \Rightarrow u + iv = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$

$\Rightarrow u = r^n \cos n\theta$ & $v = r^n \sin n\theta$.

$\Rightarrow u_r = \dfrac{\partial u}{\partial r} = nr^{n-1}\cos n\theta$ & $v_r = \dfrac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$.

Also $u_\theta = \dfrac{\partial u}{\partial \theta} = -nr^n \sin n\theta$ & $v_\theta = \dfrac{\partial v}{\partial \theta} = nr^n \cos n\theta$

$\Rightarrow ru_r = v_\theta$ & $rv_r = -u_\theta$ $\therefore$ CR eqns are satisfied. (in polar form)

$\therefore f(z) = z^n$ is analytic.

Also $f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(nr^{n-1}\cos n\theta + inr^{n-1}\sin n\theta)$

$= nr^{n-1}e^{-i\theta}(\cos n\theta + i\sin n\theta) = nr^{n-1}e^{-i\theta}(e^{in\theta})$

$= nr^{n-1}e^{i(n-1)\theta} = n(re^{i\theta})^{n-1} = nz^{n-1}$

$\therefore f'(z) = nz^{n-1}$

(7) From the given data, we can write

the Probability of a single dice showing 1, 2 or 3 is $p = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

$\therefore q = 1 - p = \frac{1}{2}$.

To fit a binomial distribution to the given data each frequency $f$ for $x = 0, 1, 2, 3, 4, 5$ is to be replaced by

$N \cdot {}^nC_x\, p^x q^{n-x}$ where $n = 5$

$\therefore$ we have

| No. of dice showing 1, 2 or 3 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| frequency (observed) $O_i$ | 3 | 8 | 24 | 35 | 19 | 7 |
| frequency (Expected using Binomial Distribution) $E_i$ | $96 \times \frac{1}{2^5} = 3$ | $\frac{96 \times 5}{2^5} = 15$ | $\frac{96 \times 10}{2^5} = 30$ | $\frac{96 \times 10}{2^5} = 30$ | $96 \times \left(\frac{5}{2^5}\right) = 15$ | $\frac{96 \times 1}{2^5} = 3$ |

$\therefore \chi^2 = \sum \dfrac{(O_i - E_i)^2}{E_i} = \dfrac{(3-3)^2}{3} + \dfrac{(8-15)^2}{15} + \dfrac{(30-24)^2}{30} + \dfrac{(30-35)^2}{30} + \dfrac{(15-19)^2}{15} + \dfrac{(7-3)^2}{3}$

$= 11.7$

This $\chi^2 = 11.7$ is greater than $11.07 = \chi^2_{0.05}$ for 5 degrees of freedom.

$\because \chi^2 > \chi^2_\alpha$, we reject the null hypothesis that the data follows a Binomial Distribution

(8) The transition probability matrix for the state space $\{A, B, C\}$ is given by

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

as per the data given.

$\because$ C has the ball initially, the corresponding initial probability vector is $p^{(0)} = (0, 0, 1)$.

$\because$ We need probabilities after three throws, we need to obtain $p^{(3)} = p^{(0)} P^3$

but $P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$P^3 = PP^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$.

Now $p^{(3)} = p^{(0)} P^3 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$

$\Rightarrow$ Prob. that A has the ball after 3 throws $= \frac{1}{4}$

" B " " " $= \frac{1}{4}$

" C " " " $= \frac{1}{2}$