CBCS SCHEME

USN 1CR18EC013 **18MAT21**

Second Semester B.E. Degree Examination, June/July 2019
Advanced Calculus and Numerical Methods

Time: 3 hrs. Max. Marks: 100

Note: Answer any FIVE full questions, choosing ONE full question from each module.

Module-1

1 a. If $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$, find div $\vec{F}$ and curl $\vec{F}$. **(06 Marks)**

b. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, −1, 2). **(07 Marks)**

c. Find the value of a, b, c such that $\vec{F} = (axy + bz^3)\hat{i} + (3x^2 - cz)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational, also find the scalar potential ϕ such that $\vec{F} = \nabla\phi$. **(07 Marks)**

OR

2 a. Find the total work done in moving a particle in the force field $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from t = 1 to t = 2. **(06 Marks)**

b. Using Green's theorem, evaluate
$\int_C (xy + y^2)dx + x^2dy$, where C is bounded by $y = x$ and $y = x^2$. **(07 Marks)**

c. Using Divergence theorem, evaluate $\int_S \vec{F}ds$, where $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. **(07 Marks)**

Module-2

3 a. Solve $(D^2 - 3D + 2)y = 2x^2 + \sin 2x$. **(06 Marks)**

b. Solve $(D^2 + 1)y = \sec x$ by the method of variation of parameter. **(07 Marks)**

c. Solve $x^2y'' - 4xy' + 6y = \cos(2 \log x)$ **(07 Marks)**

OR

4 a. Solve $(D^2 - 4D + 4)y = e^{2x} + \sin x$. **(06 Marks)**

b. Solve $(x+1)^2y'' + (x+1)y' + y = 2\sin[\log_e(x+1)]$ **(07 Marks)**

c. The current i and the charge q in a series containing an inductance L, capacitance C, emf E, satisfy the differential equation $L\frac{d^2q}{dt^2} + \frac{q}{C} = E$, Express q and i interms of 't' given that L, C, E are constants and the value of i and q are both zero initially. **(07 Marks)**

Module-3

5 a. Form the partial differential equation by elimination of arbitrary function from $\phi(x + y + z,\ x^2 + y^2 + z^2) = 0$ **(06 Marks)**

b. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$ **(07 Marks)**

c. Derive one dimensional heat equation in the standard form as $\frac{\partial U}{\partial t} = C^2\frac{\partial^2 U}{\partial x^2}$. **(07 Marks)**

OR

6 a. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ such that $z = e^y$ where $x = 0$ and $\frac{\partial z}{\partial x} = 1$ when $x = 0$. **(06 Marks)**

b. Solve $(mz - ny)\frac{\partial z}{\partial x} + (nx - \ell z)\frac{\partial z}{\partial y} = \ell y - mx$ **(07 Marks)**

c. Find all possible solutions of one dimensional wave equation $\frac{\partial^2 U}{\partial t^2} = C^2 \frac{\partial^2 U}{\partial x^2}$ using the method of separation of variables. **(07 Marks)**

Module-4

7 a. Discuss the nature of the series $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} x^n$. **(06 Marks)**

b. With usual notation prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ **(07 Marks)**

c. If $x^3 + 2x^2 - x + 1 = aP_3 + bP_2\ cP_1 + dP_0$, find a, b, c and d using Legendre's polynomial. **(07 Marks)**

OR

8 a. Discuss the nature of the series

$$\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{3.4} + \ldots\ldots\ldots$$ **(06 Marks)**

b. Obtain the series solution of Legendre's differential equation in terms of $P_n(x)$

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$ **(07 Marks)**

c. Express $x^4 - 3x^2 + x$ interms of Legendre's polynomial. **(07 Marks)**

Module-5

9 a. Find the real root of the equation $x\sin x + \cos x = 0$ near $x = \pi$ using Newton-Raphson method. Carry out 3 iterations. **(06 Marks)**

b. From the following data, find the number of students who have obtained (i) less than 45 marks (ii) between 40 and 45 marks.

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 - 80
No. of Students	31	42	51	35	31

(07 Marks)

c. Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ using Simpson's $\frac{3}{8}^{th}$ rule by taking 7 ordinates. **(07 Marks)**

OR

10 a. Find the real root of the equation $x \log_{10} x = 1.2$ which lies between 2 and 3 using Regula-Falsi method. **(06 Marks)**

b. Using Lagrange's interpolation formula, find y at x = 4, for the given data:

x	0	1	2	5
y	2	3	12	147

(07 Marks)

c. Evaluate $\int_4^{5.2} \log_e x\, dx$ using Weddle's rule by taking six equal parts. **(07 Marks)**

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Advanced Calculus and Numerical Methods. (18MAT21)

II Semester BE, June/July - 2019

1(a) $\vec{F} = (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k$

$\text{Div}\vec{F} = \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$

$= 6x + 6y + 6z = 6(x+y+z)$

$$\text{Curl}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$= i[-3x+3x] - j[-3y+3y] + k(-3z+3z)$

$\underline{\text{Curl}\vec{F} = 0}$

(b) $\phi = x^2+y^2+z^2$, $\nabla\phi = 2xi+2yj+2zk$

$\nabla\phi_{(2,-1,2)} = 4i-2j+4k$

$\psi = x^2+y^2-z$, $\nabla\psi = 2xi+2yj-k$, $\nabla\psi_{(2,-1,2)} = 4i-2j-k$.

$\cos\theta = \frac{\nabla\phi \cdot \nabla\psi}{|\nabla\phi||\nabla\psi|} = \frac{8}{3\sqrt{21}}$ or $\theta = \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$

(c) $\text{curl}\vec{F} = \vec{0}$

$$\Rightarrow \text{Curl}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy+bz^3 & 3x^2-cz & 3xz^2-y \end{vmatrix} = 0$$

$\Rightarrow i(-1-c) - j(3z^2-3z^2b) + k(6x-ax) = 0i+0j+0k$

$-1-c=0$ $\qquad -(3z^2-3z^2b)=0$ $\qquad 6x-ax=0$

$\boxed{c=1}$ $\qquad \boxed{b=1}$ $\qquad \boxed{a=6}$

Given $\vec{F} = \nabla\phi$

$(6xy+z^3)i + (3x^2-z)j + (3xz^2-y)k = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$\Rightarrow \frac{\partial\phi}{\partial x} = 6xy+z^3 \Rightarrow \phi = 3x^2y + xz^3 + f_1(y,z)$ — (1)

$\Rightarrow \frac{\partial\phi}{\partial y} = (3x^2-z) \Rightarrow \phi = 3x^2y - yz + f_2(x,z)$ — (2)

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y \Rightarrow \phi = xz^3 - yz + f_3(x,y) \quad \text{--- } (3)$$

From (1) (2) & (3), $f_1(y,z) = -yz$, $f_2(x,z) = xz^3$

$f_3(x,y) = 3x^2y$ $\quad \therefore \boxed{\phi = 3x^2y + xz^3 - yz}$

2(a) Total work done $W = \int_C \vec{F}\cdot d\vec{r}$

$$= \int_C 3xy\,dx - 5z\,dy + 10x\,dz$$

put $x = t^2+1$, $y = 2t^2$, $z = t^3$ and limit t from 1 to 2.

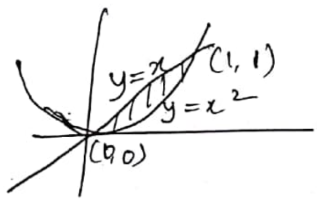
$$W = \int_{t=1}^{2} \{3(t^2+1)\,2t^2\,(2t) - 5(3t^2)\,4t + 10(t^2+1)\,3t^2\}\,dt$$

$$= \int_{t=1}^{2} (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$$

$$= 303$$

2(b) $M = xy + y^2 \qquad N = x^2$

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$



$$= \int_{x=0}^{1} \int_{y=x^2}^{x} [2x - (x+2y)]\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

(c) $\text{div}\,\vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left((x^2 - yz) i + (y^2 - xz) j + (z^2 - xy) k\right)$

$\text{div}\,\vec{F} = 2x + 2y + 2z$

$$\int_S \vec{F}\cdot d\vec{s} = \iiint_V \text{div}\,\vec{F}\,dz\,dy\,dx = \int_0^a \int_0^b \int_0^c 2(x+y+z)\,dz\,dy\,dx$$

$$= 2\int_0^a \int_0^b \left(xz + yz + \frac{z^2}{2}\right)\Big|_{z=0}^{c} dy\,dx = 2\int_0^a \int_0^b \left(cx + cy + \frac{c^2}{2}\right) dy\,dx$$

$$= 2\int_0^b \left(ycx + \frac{cy^2}{2} + \frac{c^2y}{2}\right)\Big|_{y=0}^{b} dx = 2\int_0^b \left(bcx + \frac{b^2c}{2} + \frac{c^2b}{2}\right) dx$$

$$= 2\left[\frac{bcx^2}{2} + \frac{b^2cx}{2} + \frac{c^2bx}{2}\right]_{x=0}^{a}$$

$$= 2\left[\frac{a^2bc}{2} + \frac{b^2ca}{2} + \frac{c^2ba}{2}\right] = abc(a+b+c)$$

3(a) AE $m^2 - 3m + 2 = 0$

$m = 1, 2$

$y_c = c_1 e^x + c_2 e^{2x}$, $y_p = \frac{x^2}{D^2 - 3D + 2} + \frac{\sin 2x}{D^2 - 3D + 2} = P_1 + P_2$

$\therefore P_1 = \frac{2x^2}{2 - 3D + D^2}$

$$\begin{array}{r} x^2 - 3x + 7/2 \\ 2 - 3D + D^2 \,\big)\, 2x^2 \\ x^2 - 6x + 2 \\ (-)\ \ (+)\ \ \ (-) \\ \hline 6x - 2 \\ -6x + 9 \\ \hline 7 \\ 7 \\ \hline 0 \end{array}$$

$P_1 = x^2 - 3x + 7/2$

$P_2 = \frac{\sin 2x}{D^2 - 3D + 2}$ put $D^2 = -4$

$= \frac{\sin 2x}{-4 - 3D + 2} = \frac{\sin 2x}{-3D - 2} \times \frac{-3D+2}{-3D+2} = \frac{\sin 2x(-3D+2)}{9D^2 - 4}$

$= \frac{-6\cos 2x + 2\sin 2x}{-40} = \frac{\sin 2x - 3\cos 2x}{-20}$

$\therefore y_p = x^2 - 3x + 7/2 + \frac{3\cos 2x - \sin 2x}{20}$

$\therefore$ General soln is $y = y_c + y_p$

(b) AE is $m^2 + 1 = 0 \Rightarrow m = \pm i$, $\phi(x) = \sec x$

$y_c = C_1 \cos x + C_2 \sin x$

Assume $y_1 = \cos x$ $y_2 = \sin x$

$y_1' = -\sin x$ $y_2' = \cos x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \sin^2 x + \cos^2 x = 1$

$A = -\int \frac{\phi(x)\, y_2}{W} dx + K_1 = -\int \frac{\sec x \sin x}{1} dx + K_1 = -\int \tan x\, dx + K_1$

$A = -\log \sec x + K_1$

$$B = \int \frac{y_1 \phi(x)}{w} dx = \int \frac{\cos x \sec x}{1} dx = \int dx = x + k_2$$

$$\boxed{B = x + k_2}$$

$\therefore$ General soln is $y = Ay_1 + By_2$

(a) $y = (-\log \sec x + k_1)\cos x + (x + k_2)\sin x$

(c) Given $(x^2D^2 - 4D + 6)y = \cos(2\log x)$ --- ①

put $t = \log x \Rightarrow e^t = x$, $xy' = Dy$, $x^2y'' = D(D-1)y$

Eqn ① becomes

$[D(D-1) - 4D + 6]y = \cos 2t$

$[D^2 - D - 4D + 6]y = \cos 2t \Rightarrow (D^2 - 5D + 6)y = \cos 2t$

AE is $m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$

$y_c = c_1 e^{2t} + c_2 e^{3t}$

$y_p = \frac{\cos 2t}{D^2 - 5D + 6}$ put $D^2 = -4$

$= \frac{\cos 2t}{-4 - 5D + 6} = \frac{\cos 2t}{-5D + 2} = \frac{\cos 2t}{-5D + 2} \times \frac{-5D - 2}{-5D - 2}$

$= \frac{-5D(\cos 2t) - 2(\cos 2t)}{25D^2 - 4} = \frac{10\sin 2t - 2\cos 2t}{-100 - 4}$

$y_p = \frac{2\cos 2t - 10\sin 2t}{104}$

$\therefore$ General Soln is $\boxed{y = y_c + y_p}$.

(a). $y = c_1 e^{2t} + c_2 e^{3t} + \frac{2\cos 2t - 10\sin 2t}{104}$

Replace $t = \log x$ (or) $e^t = x$

$$y = c_1 x^2 + c_2 x^3 + \frac{2\cos(2\log x) - 10\sin(2\log x)}{104}$$

4 (a) AE is $m^2 - 4m + 4 = 0$

$m = 2, 2$, $y_c = (c_1 + c_2 x)e^{2x}$, Let $y_p = P_1 + P_2$

$P_1 = \frac{e^{2x}}{D^2 - 4D + 4}$ put $D = 2$ $\Rightarrow P_1 = \frac{e^{2x}}{4 - 8 + 4}$ $(Dr = 0)$

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$$P_1 = \frac{x e^{2x}}{2D-4} \Rightarrow P_1 = \frac{x^2 e^{2x}}{2}$$

$$P_2 = \frac{\sin x}{D^2-4D+4} \quad \text{put } D^2=-1 \Rightarrow P_2 = \frac{\sin x}{-4D+3} \times \frac{-4D-3}{-4D-3}$$

$$P_2 = \frac{-4\,D(\sin x) - 3\sin x}{16D^2-9} = \frac{-4\cos x - 3\sin x}{-25}$$

$$P_2 = \frac{4\cos x - 3\sin x}{25} \qquad \therefore y_p = \frac{x^2 e^{2x}}{2} + \frac{4\cos x - 3\sin x}{25}$$

$\therefore y = y_c + y_p.$

(b) put $t = \log(x+1) \Rightarrow e^t = x+1.$

$(x+1)y' = D.y \quad (x+1)^2 y'' = D(D-1)y$

$[D(D-1) + D + 1]y = 2\sin t$

$[D^2 + \cancel{0}D + 1]y = 2\sin t$

$(D^2+1)y = 2\sin t$

AE is $m^2+1=0 \quad m = \pm i$

$y_c = c_1 \cos t + c_2 \sin t$

$$y_p = \frac{2\sin t}{D^2+0D+1} \quad \text{put } D^2=-1 \Rightarrow y_p = \frac{\cancel{2}\sin t}{\cancel{2}D} \quad (Dr=0)$$

$$\therefore y_p = \frac{\cancel{2}t\sin t}{\cancel{2}D.} = -t\cos t$$

$\therefore y = y_c + y_p = c_1\cos t + c_2 \sin t - t\cos t$

put $t = \log(x+1)$ ⓐ $e^t = x+1$

$y = c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1)) - \log(x+1)\cos\log(x+1)$

(c) Given DE $(D^2+A^2)q = B$ — ①

where $A^2 = \frac{1}{LC}$, $B = \frac{E}{L}$.

AE $m^2 + A^2 = 0 \Rightarrow m = \pm iA$; $y_c = c_1 \cos At + c_2 \sin At$

$$y_p = \frac{B}{D^2+A^2} = \frac{B}{A^2} \text{ (by putting } D=0)$$

$$\therefore q = y_c + y_p = c_1\cos At + c_2 \sin At + \frac{B}{A^2}$$

Diff. wrt t

$$\frac{dq}{dt} = -AC_1 \sin At + AC_2 \cos At$$

put $q = 0$ at $t = 0 \Rightarrow \boxed{C_1 = -\frac{B}{A^2}}$

put $\frac{dq}{dt} = 0$ at $t = 0 \Rightarrow C_2 = 0$

$$\therefore\ q(t) = EC\left[1 - \cos\frac{t}{\sqrt{C^2}}\right],\quad q'(t) = E\sqrt{\frac{C}{L}}\sin\sqrt{\frac{1}{CL}}\,t$$

5(a) $u = x + y + z \qquad v = x^2 + y^2 + z^2$

$$\frac{\partial u}{\partial x} = 1 + p \qquad \frac{\partial u}{\partial y} = 1 + q,\quad \frac{\partial v}{\partial x} = 2x + 2zp,\quad \frac{\partial v}{\partial y} = 2y + 2zq$$

$$\Rightarrow \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\partial v/\partial x}{\partial v/\partial y} \Rightarrow \frac{1+p}{1+q} = \frac{2x + 2zp}{2y + 2zq} \Rightarrow \frac{1+p}{1+q} = \frac{x + zp}{y + zq}$$

$$\therefore (z - y)p + (x - z)q = x - y.$$

5(b) $$\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y) \Rightarrow \frac{\partial}{\partial x}\left[\frac{\partial^2 z}{\partial x \partial y}\right] = \cos(2x + 3y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x + 3y)}{2} + f(y)$$

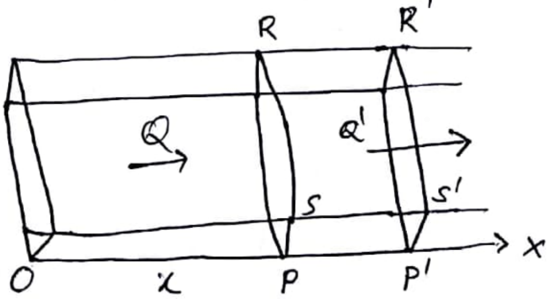
$$\frac{\partial z}{\partial y} = -\frac{\cos(2x + 3y)}{4} + x f(y)\ g(y).$$

$$z = \frac{-\sin(2x + 3y)}{12} + x\int f(y)\,dy + \int g(y)\,dy + h(x)$$

$$z = -\frac{\sin(2x + 3y)}{12} + xF(y) + G(y) + h(x).$$

where $F(y) = \int f(y)\,dy,\ G(y) = \int g(y)\,dy.$

5(C) Heat equation derivation



Consider a nonhomogeneous bar of constant cross-sectional area A. Let ρ be the density, s be the specific heat and K be the thermal conductivity of the meterial.

Let the sides be insulated so that heat flow is parallel and perpendicular to the area A. Heat flow is along the positive x-axis.

Let $u = u(x, t)$ be the temprature of the slab at a distance x from the origin. consider an element of bar between the planes PQRS & P'Q'R'S' at a distance x and $x + \delta x$ from the end O. Let δu be the change in temprature in a slab of thickness δx of the bar.

The mass of the element $= A\rho\,\delta x$

The quantity of heat stored in this slab element $= A\rho s\,\delta x\,\delta u$

Hence the rate of increase of heat in this slab element is

$$R = (A\rho s\,\delta x)\frac{\partial u}{\partial t} \quad \text{—— (1)}$$

If R_I is the rate of inflow of heat and R_o is the rate of outflow of heat we have

$$R_I = -KA\left[\frac{\partial u}{\partial x}\right]_x \quad \text{and} \quad R_o = -KA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} \quad \text{—— (2)}$$

where the negative sign is due to emperical law (4)

Hence we have from (i)

$$R = R_I - R_o$$

i.e $$A\rho s\,\delta x\frac{\partial u}{\partial t} = KA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - KA\left[\frac{\partial u}{\partial x}\right]_x$$

$$\frac{\partial u}{\partial t} = \frac{K}{\rho s}\left\{\frac{\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - \left[\frac{\partial u}{\partial x}\right]_x}{\delta x}\right\} \quad \text{—— (3)}$$

Taking limit as $\delta x \to 0$, RHS of (3) is equal to

$$\frac{K}{\rho s}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{K}{\rho s}\cdot\frac{\partial^2 u}{\partial x^2}$$

Further denoting $c^2 = \frac{K}{\rho s}$ which is called the diffusity of the substance (3) becomes

$$\boxed{\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}}$$

6(a) Suppose z is function of x only. Then given PDE assumes the form of ODE.

$$\frac{d^2z}{dx^2}+z=0 \text{ or } (D^2+1)z=0 \text{ where } D=\frac{d}{dx}$$

AE is $m^2+1=0$ or $m^2=-1$ $\therefore m=\pm\sqrt{-1}=\pm i$

Solution of ODE is $z=c_1\cos x+c_2\sin x$

Replace c_1 & c_2 by functions of y.

$$z=f(y)\cos x+g(y)\sin x \quad \text{——} (1)$$

Now, we shall apply the given conditions to find $f(y)$ and $g(y)$. By data, when $x=0$, $z=e^y$. Hence (1) becomes

$$e^y=f(y)\cos 0+g(y)\sin 0$$

$$e^y=f(y)\cdot 1+g(y)\cdot 0 \qquad \therefore \boxed{f(y)=e^y}$$

Also by data, when $x=0$, $\frac{\partial z}{\partial x}=1$

Differentiating (1) wrt x partially we get

$$\frac{\partial z}{\partial x}=-f(y)\sin x+g(y)\cos x$$

Applying the conditions we get

$$1=-f(y)\sin 0+g(y)\cos 0 \qquad \therefore g(y)=1$$

We substitute $f(y)=e^y$ and $g(y)=1$ in (1)

Thus $\boxed{z=e^y\cos x+\sin x.}$

6(b) Given equation is of the form $Pp+Qq=R$

The auxilary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using the multipliers l, m, n

$$\frac{l\,dx+m\,dy+n\,dz}{lmz-nly+mnx-lmz+nly-mnx}=\frac{l\,dx+m\,dy+n\,dz}{0}$$

$$\therefore l\,dx+m\,dy+n\,dz=0$$

Integrating we get $lx+my+nz=c_1$.

Chooshing multipliers x, y, z each ratio in eqn (1) is equal to

$$\frac{xdx + ydy + zdz}{mxz - nxy + nxy - lyz - mxz} = \frac{xdx + ydy + zdz}{0}$$

$\therefore xdx + ydy + zdz = 0$

Integrating we get $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$ (or) $x^2 + y^2 + z^2 = 2c_2$

$\therefore$ general soln is

$$\boxed{\phi(lx + my + nz, x^2 + y^2 + z^2) = 0}$$

6(a) Consider $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Let $u = XT$ where $X = X(x), T = T(t)$ be the solution of the PDE. Hence $\frac{\partial^2 (XT)}{\partial t^2} = c^2 \frac{\partial^2 (XT)}{\partial x^2}$ (or) $X \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 X}{dx^2}$

Dividing by $c^2 XT$ we have $\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2}$

Equating both sides to a common constant K, we have

$\frac{1}{X} \frac{d^2 X}{dx^2} = K$ and $\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = K$

$\frac{d^2 X}{dx^2} - KX = 0$ and $\frac{d^2 T}{dt^2} - c^2 KT = 0$

$(D^2 - K)X = 0$ and $(D^2 - c^2 K)T = 0$

where $D^2 = \frac{d^2}{dx^2}$ in the first equation and $D^2 = \frac{d^2}{dt^2}$ in the second equation

Case(i): Let $K = 0$

The equations become $D^2 X = 0$ and $D^2 T = 0$

In both the equations AE is $m^2 = 0$, $m = 0, 0$

Solutions are given by

$X = (c_1 + c_2 x)e^{0x}$ and $T = (c_3 + c_4 t)e^{0t}$

i.e $X = (c_1 + c_2 x)$ and $T = (c_3 + c_4 t)$

Soln of PDE is $u = XT = (c_1 + c_2 x)(c_3 + c_4 t)$

Case(ii): Let K be positive say, $K = +p^2$

The equations become

$(D^2-p^2)X=0$ and $(D^2-c^2p^2)T=0$

AE's are $m^2-p^2=0$ and $m^2-c^2p^2=0$

$\therefore m^2=p^2$ or $m=\pm p$ and $m^2=c^2p^2$ or $m=\pm cp$

Solutions are given by

$X=c_1e^{px}+c_2e^{-px}$ and $T=c_3e^{cpt}+c_4e^{-cpt}$

Hence the solution of PDE is

$u=XT=(c_1'e^{px}+c_2'e^{-px})(c_3'e^{cpt}+c_4'e^{-cpt})$

Case(iii): Let k be negative, say $k=-p^2$

The equations become

$(D^2+p^2)X=0$ and $(D^2+c^2p^2)T=0$

AES are $m^2+p^2=0$ & $m^2+c^2p^2=0$

or $m^2=-p^2$ & $m^2=-c^2p^2$

$m=\pm ip$ and $m=\pm icp$

Solutions are given by

$X=c_1\cos px+c_2\sin px$ and $T=c_3\cos cpt+c_4\sin cpt$

Hence solution of PDE is

$u=XT=(c_1''\cos px+c_2'\sin px)(c_3''\cos cpt+c_4''\sin cpt)$

7(a) $u_n=\dfrac{(n+1)^n x^n}{n^{n+1}}$ by data

$$(u_n)^{1/n}=\frac{\{(n+1)^n\}^{1/n}(x^n)^{1/n}}{(n^{n+1})^{1/n}}=\frac{(n+1)x}{n^{1+1/n}}=\frac{(n+1)x}{n\,n^{1/n}}$$

Now, $\lim\limits_{n\to\infty}(u_n)^{1/n}=\lim\limits_{n\to\infty}\dfrac{(n+1)x}{n\,n^{1/n}}=\lim\limits_{n\to\infty}\dfrac{n(1+1/n)x}{n\,n^{1/n}}$

$=\lim\limits_{n\to\infty}\dfrac{(1+1/n)x}{n^{1/n}}=x$

$\therefore \Sigma u_n$ is convergent if $x<1$ & divergent if $x>1$

& Test fails if $x=1$.

(b) $$J_n(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)r!}$$

put $n=1/2$ in (1) we have

$$J_{1/2}(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{1/2+2r} \frac{1}{\Gamma(r+3/2)\, r!}$$

$$= \sqrt{\frac{x}{2}} \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{\Gamma(r+3/2)\, r!}$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma 3/2} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma 5/2} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma 7/2\; 2!} - - - \right] \quad (2)$$

WKT $\Gamma 1/2 = \sqrt{\pi}$ and $\Gamma n = (n-1)\,\Gamma n-1$

$\Gamma 3/2 = \frac{\sqrt{\pi}}{2}$ $\quad \Gamma 5/2 = \frac{3\sqrt{\pi}}{4}$ $\quad \Gamma 7/2 = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{15\sqrt{\pi}}{8}$

$$\therefore J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\cdot\frac{8}{15\sqrt{\pi}\cdot 2} - - - \right]$$

$$= \sqrt{\frac{x}{2\pi}} \left[2 - \frac{x^2}{3} + \frac{x^4}{60} - - - \right]$$

$$= \sqrt{\frac{x}{2\pi}} \cdot \frac{2}{x} \left[x - \frac{x^3}{6} + \frac{x^5}{120} - - - \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - - - \right]$$

$$\boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \sin x}$$

(c) $x^3 + 2x^2 - x + 1 = aP_3 + bP_2 + cP_1 + dP_0$ find the values of a, b, c, d.

Let $f(x) = x^3 + 2x^3 - x + 1$, by substituting Legendre polynomials we have

$$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] + 2\left[\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)\right] - P_1(x) + P_0(x)$$

$$f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) + \frac{5}{3}P_0(x)$$

Hence we have

$$aP_3(x) + bP_2(x) + cP_1(x) + dP_0(x) = \frac{2}{5}P_3 + \frac{4}{3}P_2 - \frac{2}{5}P_1 + \frac{5}{3}P_0$$

$\therefore a = \frac{2}{5}$, $b = \frac{4}{3}$ $\quad c = -\frac{2}{5}$ $\quad d = 5/3$

8(a) By the data $u_n = \frac{x^n}{n(n+1)}$, $\quad u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)(n+2)} \times \frac{(n+1)n}{x^n} = \frac{n}{n+2}\, x$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+2}\, x = x$$

Thus by D'Alembert's ratio test $\sum u_n$ is convergent if $x<1$, $\sum u_n$ is divergent if $x>1$ and test fails if $x=1$

8(b.) consider $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Let $y = \sum_{r=0}^{\infty} a_r x^r$ be the solution of above DE.

$y' = \sum_0^{\infty} a_r r x^{r-1}$, $y'' = \sum_0^{\infty} a_r r(r-1) x^{r-2}$

Hence eqn ① becomes

$(1-x^2) \sum_0^{\infty} a_r r(r-1) x^{r-2} - 2x \sum_0^{\infty} a_r r x^{r-1} + n(n+1) \sum_0^{\infty} a_r x^r = 0$

$\sum_0^{\infty} a_r r(r-1) x^{r-2} - \sum_0^{\infty} a_r r(r-1) x^r - \sum_0^{\infty} 2a_r r x^r + n(n+1) \sum_0^{\infty} a_r x^r = 0$

Equating coefficients of x^{-2} & x^{-1} we get $a_0 \neq 0$, $a_1 \neq 0$

Now equate coefficient of x^r $(r \geq 0)$ to zero

$a_{r+2}(r+2)(r+1) = a_r[r(r-1) + 2r - n(n+1)]$

$a_{r+2} = \frac{-[n(n+1) - r^2 - r]}{(r+2)(r+1)} a_r$

Putting $r = 0, 1, 2, 3 \dots$ in (3) we obtain

$a_2 = \frac{-n(n+1)}{2} a_0$, $a_3 = \frac{-(n^2+n-2)}{6} a_1 = \frac{-(n-1)(n+2)}{6} a_1$

$a_4 = \frac{n(n+1)(n-2)(n+3)}{24} a_0$ & so on.

Substitute these values in extended form of eqn ①

$y = (a_0 + a_2 x^2 + a_4 x^4 + \dots) + (a_1 x + a_3 x^3 + a_5 x^5 + \dots)$

$y = a_0\left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n-2)(n+3)}{4!} x^4 - \dots\right] +$

$a_1\left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!} x^5 - \dots\right]$

$\therefore y = a_0 u(x) + a_1 v(x)$ is the soln of given DE.

8(c) $f(x) = x^4 - 3x^2 + x$

$\therefore x^4 = \frac{8}{35} P_4 + \frac{4}{7} P_2 + \frac{1}{5} P_0$, $x^2 = \frac{1}{3} P_0 + \frac{2}{3} P_2$, $x = P_1(x)$

$f(x) = \left[\frac{8}{35} P_4 + \frac{4}{7} P_2 + \frac{1}{5} P_0\right] - 3\left[\frac{1}{3} P_0 + \frac{2}{3} P_2\right] + P_1$

$f(x) = \frac{8}{35} P_4(x) - \frac{10}{7} P_2(x) + P_1(x) - \frac{4}{5} P_0(x)$

9(a) $f(x) = x\sin x + \cos x$

$f'(x) = x\cos x + \sin x - \sin x = x\cos x$, $x_0 = \pi$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{f(\pi)}{f'(\pi)} = \pi - \frac{(\pi\sin\pi + \cos\pi)}{\pi\cos\pi}$

$x_1 = 2.8233$, $x_2 = 2.7986$, $x_3 = 2.7984$

9(b) Less than 40 marks $= 31$

<50 marks $= 31+42 = 73$

<60 marks $= 124$, <70 marks $124+35 = 159$

<80 marks $= 159+31 = 190$

x	$f(x)=y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$y_0 = 31$				
		$\Delta y_0 = 42$			
50	73		$\Delta^2 y_0 = 9$		
		51		$\Delta^3 y_0 = -25$	
60	124		-16		$\Delta^4 y_0 = 37$
		35		12	
70	159		-4		
		31			
80	190				

Newton's forward interpolation formula

$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$

where $r = \frac{x - x_0}{h}$, $r = \frac{45-40}{10} = 0.5$

$\therefore f(45) = 47.86 \approx 48$, No. of students obtaining less than 45 marks is 48.

To find $f(45) - f(40) = 48 - 31 = 17$

The number of students scoring marks between 40 & 45 is 17

9(c) $y = \frac{1}{1+x^2}$, $a = 0$, $b = 6$, $n = 6$, $h = 1$

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.0588	0.0385	0.0270

$I = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right] = 1.3571$

10(a) Regula falsi method $x = \frac{b f(a) - a f(b)}{f(b) - f(a)}$

$a = 2$, $b = 3$ then

$x_1 = 2.7210$, $x_2 = 2.7402$, $x_3 = 2.7402$

10(b)

x	0	1	2	5
y	2	3	12	147

at $x = 4$, $y = ?$

Lagrange's formula.

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \cdots$$

$y(4) = 78$

10(c) $y = \log x$ $a = 4$, $b = 5.2$, $n = 6$, $h = 0.2$

x	4	4.2	4.4	4.6	4.8	5	5.2
y	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

weddle's rule $I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$

$I = 1.8279$