Model Question Paper-1 with effect from 2018-19
(CBCS Scheme)

USN | | | | | | | | | | |

18MAT21

Second Semester B.E. Degree Examination
Advanced Calculus and Numerical Methods
(Common to all Branches)

Time: 3 Hrs | Max.Marks: 100

Note: Answer any FIVE full questions, choosing at least ONE question from each module.

Module-1

1. (a) Find the angle between the surfaces $x^2+y^2-z^2=4$ and $z=x^2+y^2-13$ at $(2,1,2)$ **(06 Marks)**

(b) If $\vec{F}=\nabla(xy^3z^2)$, find $div\vec{F}$ and $curl\vec{F}$ at the point $(1,-1,1)$. **(07 Marks)**

(c) Find the value of a,b,c such that $\vec{F}=(axy+bz^3)\vec{i}+(3x^2-cz)\vec{j}+(3xz^2-y)\vec{k}$ is a conservative force field. Hence find the scalar potential ϕ such that $\vec{F}=\nabla\phi$. **(07 Marks)**

OR

2. (a) Use Green's theorem to find the area between the parabolas $x^2=4y$ and $y^2=4x$. **(06 Marks)**

(b) Using Gauss divergence theorem, evaluate $\iint_S \vec{F}\cdot\hat{n}dS$ over the entire surface of the region above xy-plane bounded by the cone $z^2=x^2+y^2$ and the plane $z=4$, where $\vec{F}=4xz\vec{i}+xyz^2\vec{j}+3z\vec{k}$. **(07 Marks)**

(c) Find the work done by the force $\vec{F}=3x^2\vec{i}+(2xz-y)\vec{j}+z\vec{k}$, when it moves a particle from the point $t=0$ to $t=2$ along the curve $x=t, y=t^2/4, z=3t^3/8$. **(07 Marks)**

Module-2

3. (a) Solve: $(D^3+D^2-4D-4)y=3e^{-x}-4x-6$, where $D=\frac{d}{dx}$. **(06 Marks)**

(b) Solve: $\frac{d^2y}{dx^2}+y=\frac{1}{1+\sin x}$, using the method of variation of parameters. **(07 Marks)**

(c) Solve: $(x^2D^2-3xD+4)y=(1+x)^2$, where $D=\frac{d}{dx}$. **(07 Marks)**

OR

4. (a) Solve: $(D^3+8)y=x^4+2x+1$. where $D=\frac{d}{dx}$. **(06 Marks)**

(b) Solve: $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=8x^2+4x+1$ **(07 Marks)**

(c) The differential equation of the displacement $x(t)$ of a spring fixed at the upper end and a weight at its lower end is given by $10\frac{d^2x}{dt^2}+\frac{dx}{dt}+200x=0$. The weight is pulled down 0.25 cm, below the equilibrium position and then released. Find the expression for the displacement of the weight from its equilibrium position at any time t during its first upward motion. **(07 Marks)**

Module-3

5. (a) Form the partial differential equation by eliminating the arbitrary constants from $(x-a)^2+(y-b)^2+z^2=c^2$ **(06 Marks)**

(b) Solve $\frac{\partial^2 z}{\partial y^2}=z$, given that when $y=0, z=e^x$ and $z=e^{-x}$ **(07 Marks)**

(c) Derive one-dimensional wave equation in the standard form. **(07 Marks)**

OR

6. (a) Form the partial differential equation by eliminating the arbitrary function from $f(x^2+y^2, z-xy)=0$ **(06 Marks)**

(b) Solve: $(x^2-yz)p+(y^2-zx)q=z^2-xy$ **(07 Marks)**

(c) Solve one dimensional heat equation, using the method of separation of variables. **(07 Marks)**

Module-4

7. (a) Test for the convergence or divergence of the series : $\sum_{n=1}^{\infty}\frac{n!}{(n^n)^2}$ **(06 Marks)**

(b) Solve Bessel's differential equation leading to $J_n(x)$. **(07 Marks)**

(c) Express $f(x)=x^4+3x^3-x^2+5x-2$ in terms of Legendre polynomials. **(07 Marks)**

OR

8. (a) Test for the convergence or divergence of the series : $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$ **(06 Marks)**

(b) If α and β are two distinct roots of $J_n(x)=0$, prove that $\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx=0$ if $\alpha\neq\beta$. **(07 Marks)**

(c) Use Rodrigues's formula to show that $P_4(\cos\theta)=\frac{1}{8}(35\cos 4\theta+20\cos 2\theta+9)$ **(07 Marks)**

Module-5

9. (a) Find a real root of the equation $x\sin x+\cos x=0$, near $x=\pi$ correct to four decimal places, using Newton- Raphson method. **(06 Marks)**

(b) Use an appropriate interpolation formula to compute $f(2.18)$ using the following data: **(07 Marks)**

x	1.7	1.8	1.9	2.0	2.1	2.2
$f(x)$	5.474	6.050	6.686	7.389	8.166	9.025

(c) Use Weddle's rule to evaluate $\int_{-\pi/2}^{\pi/2}\cos x\,dx$, by dividing $[-\pi/2,\pi/2]$ into six equal parts. **(07 Marks)**

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OR

10. (a) Find a real root of $x\log_{10}x-1.2=0$, correct to three decimal places lying in the interval *(2,3)*, using Regula-Falsi method. **(06 Marks)**

(b) Using Lagrange's interpolation formula to fit a polynomial for the following data: **(07 Marks)**

x	2	10	17
y	1	3	4

(c) Using Simpson's $(3/8)^{th}$ rule, evaluate $\int_0^3 \frac{dx}{(1+x)^2}$ taking 4 equidistant ordinates. **(07 Marks)**

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Advanced Calculus and Numerical Methods
Model QP I – 18MAT21

Q.1 (a) wkt $\nabla\phi$ is a vector normal to the surface.

Let $\phi_1 = x^2+y^2-z^2$, $\phi_2 = x^2+y^2-z$

$\nabla\phi_1 = 2xi + 2yj - 2zk$, $\nabla\phi_2 = 2xi + 2yj - k$

$[\nabla\phi_1]_{(2,1,2)} = 4i+2j-4k$, $[\nabla\phi_2]_{(2,1,2)} = 4i+2j-k$

If θ is the angle between the two normals, we have

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{16+4+4}{\sqrt{4(9)}\sqrt{16+4+1}} = \frac{24}{2(3)\sqrt{21}} = \frac{4}{\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{21}}\right)$$

(b) $\vec{F} = \nabla(xy^3z^2)$

Let $\phi = xy^3z^2$

$$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = y^3z^2 i + 3xy^2z^2 j + 2xy^3z k$$

$\text{div}\,\vec{F} = \nabla\cdot\vec{F}$

$$= \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial}{\partial y}(3xy^2z^2) + \frac{\partial}{\partial z}(2xy^3z)$$

$$= 2xy(3z^2+y^2)$$

At $(1,-1,1)$, $\text{div}\,\vec{F} = -8$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix} = \vec{0}$$

(c) Since $\vec{F}$ is a conservative force field,

curl $\vec{F} = 0$

$\nabla \times \vec{F} = 0$

$$\begin{vmatrix} i & j & k & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} \\ axy+bz^3 & 3x^2-cz & 3xz^2-y & \end{vmatrix} = 0$$

$$i(-1+c) - j(3z^2 - 3bz^2) + k(6x - ax) = 0$$

$c - 1 = 0, \quad 3z^2 - 3bz^2 = 0, \quad 6x - ax = 0$

$c = 1 \qquad b = 1 \qquad a = 6$

$$\therefore \vec{F} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$$

$$\vec{F} = \nabla \phi$$

$$\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 6xy + z^3 \;①, \quad \frac{\partial \phi}{\partial y} = 3x^2 - z \;②, \quad \frac{\partial \phi}{\partial z} = 3xz^2 - y \;③$$

int ① w.r.to 'x'

$\phi = 3x^2y + z^3x + f_1(y,z)$ ④

int ② w.r.to 'y'

$\phi = 3x^2y - zy + f_2(x,z)$ ⑤

int ③ w.r.to z

$\phi = xz^3 - yz + f_3(x,y)$ ⑥

from ④, ⑤, & ⑥

$\phi = 3x^2y + xz^3 - zy$

$f_1(y,z) = -zy, \; f_2(x,z) = z^3x, \; f_3(x,y) = 3x^2y$

2(a) We have the area

$$A = \iint dx\,dy = \frac{1}{2} \int_C (x\,dy - y\,dx)$$

From Green's thm

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

Area enclosed by the curve is $\iint_R dx\,dy$

Taking $N = \frac{x}{2}$, $M = -\frac{y}{2}$,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$

$$\therefore \frac{1}{2}\int_C x\,dy - y\,dx = \iint_R dx\,dy = \text{Area } A$$

Let's find the points of intersection of

$y^2 = 4x$ and $x^2 = 4y$

i.e., $\left(\frac{x^2}{4}\right)^2 = 4x$ or $x(x^3 - 64) = 0$

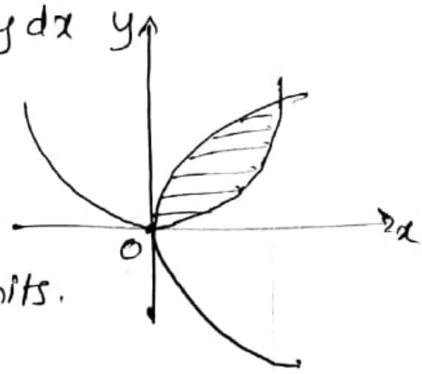
$\Rightarrow x = 0,\ x = 4 \quad \therefore y = 0,\ y = 4$

The points of intersection are $(0,0)$ and $(4,4)$.

C_1 is the curve $x^2 = 4y \;\therefore dy = \frac{x}{2}dx$ & $0 \le x \le 4$

C_2 is the curve $y^2 = 4x \;\therefore dx = \frac{y}{2}dy$ and $0 \le y \le 0$

Now, $A = \frac{1}{2}\int_{C_1} x\,dy - y\,dx + \frac{1}{2}\int_{C_2} x\,dy - y\,dx$



$= \frac{16}{3}$

Thus the required area is $16/3$ sq. units.

2(b): we have $\vec{F} = 4xz\,i + xyz^2\,j + 3z\,k$

We have $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dV$

$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = 4z + xz^2 + 3$

$$\iiint_V \text{div}\,\vec{F}\,dV = \iiint_V (4z + xz^2 + 3)\,dx\,dy\,dz$$

Putting $z=4$ in $z^2 = x^2+y^2$.

$$x^2+y^2=16$$

$\therefore$ y varies from $-\sqrt{16-x^2}$ to $\sqrt{16-x^2}$

If $y=0$: $x^2=16$ and x varies from -4 to 4.

$\therefore$ z varies from 0 to 4.

$$\iiint_V \operatorname{div}\vec{F}\, dV = \int_{z=0}^{4}\int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (4z+xz^2+3)\, dy\, dx\, dz$$

$$= 704\pi$$

Thus $\iint_S \vec{F}\cdot\hat{n}\, ds = 704\pi$.

2(c)

$x=t,\ y=t^2/4,\ z=3t^3/8$

$$\vec{dr} = dx\, i + dy\, j + dz\, k$$

$$= i + \frac{t}{2} j + \frac{9t^2}{8} k$$

$$\vec{F} = 3x^2 i + (2xz-y) j + zk$$

$$= 3t^2 i + \left(\frac{3t^4}{4} - \frac{t^2}{4}\right) j + \frac{3t^3}{8} k$$

$\therefore$ Required work done is

$$\int_C F\cdot dr = \int_{t=0}^{2} 3t^2 + \left(\frac{3t^4}{4} - \frac{t^2}{4}\right)\frac{t}{2} + \frac{9t^2}{8}\left(\frac{3t^3}{8}\right)\cdot dt$$

$$= \int_{t=0}^{2} 3t^2 + \frac{3t^5 - t^3}{8} + \frac{27t^5}{64}\cdot dt$$

$$= \left[\frac{8t^3}{3} + \frac{8t^6}{48} - \frac{t^4}{32} + \frac{27 \cdot t^6}{64 \cdot 6}\right]_0^2$$

$$= \left[8 + \frac{64}{16} - \frac{16}{32} + \frac{9 \cdot 64}{2 \cdot 64}\right]$$

$$= 8 + 4 - \frac{1}{2} + \frac{9}{2}$$

$$= \frac{32 + 16 - 2 + 9}{4} = \frac{55}{4}$$ NB

$$= \frac{16 + 8 - 1 + 9}{2} = \frac{32}{2} = 16$$

Modul - 2 (Paper - 1)

3.a $(D^3 + D^2 - 4D - 4)y = 3e^{-x} - 4x - 6$

A.E is $m^3 + m^2 - 4m - 4 = 0$

ie $m^2(m+1) - 4(m+1) = 0$

$(m+1)(m^2-4) = 0$

$m = -1, \pm 2$

$y_c = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-2x}$

$$y_p = \frac{3e^{-x}}{D^3+D^2-4D-4} - \frac{4x+6}{D^3+D^2-4D-4} = P_1 - P_2$$

$$P_1 = \frac{3e^{-x}}{D^3+D^2-4D-4} = \frac{3e^{-x}}{-1+1+4-4} \quad (Dr = 0)$$

$$\therefore P_1 = x\left[\frac{3e^{-x}}{3D^2+2D-4}\right] = x\left[\frac{3e^{-x}}{3-2-4}\right] = -xe^{-x}$$

$$P_2 = \frac{4x+6}{-4-4D+D^2+D^3}$$

$-4-4D+D^2+D^3 \,)\; 4x+6 \;(\; -x - \frac{1}{2}$

$4x+4$

$\overline{\quad 2 \quad}$

2

$\overline{\quad 0 \quad}$

$P_2 = -x - \frac{1}{2}$

$$\therefore y = y_c + y_p = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-2x} - xe^{-x} + x + \frac{1}{2}$$

[3.b] $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$

We have $(D^2+1)y = \frac{1}{1+\sin x}$

A.E is $m^2+1=0$

$m = \pm i$

$\therefore y_c = c_1\cos x + c_2\sin x.$

Take $y_p = A(x)\cos x + B(x)\sin x$

We have $y_1 = \cos x$, $y_2 = \sin x$

$y_1' = -\sin x$, $y_2' = \cos x$

$W = y_1y_2' - y_2y_1' = 1$, $\phi(x) = \frac{1}{1+\sin x}$

$A = -\int \frac{y_2\,\phi(x)}{W}dx$

$A = -\int \frac{\sin x\left(\frac{1}{1+\sin x}\right)}{1}dx$

$= -\int \frac{1+\sin x - 1}{1+\sin x}dx$

$= \int \left(-1 + \frac{1}{1+\sin x}\right)dx$

$= -x + \int \left(\frac{1-\sin x}{1-\sin^2 x}\right)dx$

$= -x + \int \frac{1-\sin x}{\cos^2 x}dx$

$= -x + \int(\sec^2 x - \sec x\tan x)\,dx$

$= -x + \tan x - \sec x$

3

$$B = \int \frac{y_1 \phi(x)}{W} dx$$

$$= \int \cos x \left(\frac{1}{1+\sin x} \right) dx$$

$$= \int \frac{\cos x(1-\sin x)}{\cos^2 x} dx$$

$$= \int \frac{1-\sin x}{\cos x} dx$$

$$= \int (\sec x - \tan x) dx$$

$$= \log(\sec x + \tan x) + \log(\cos x)$$

$$= \log\left(\frac{1+\sin x}{\cos x}\right) + \log(\cos x)$$

$$= \log(1+\sin x) - \log(\cos x) + \log(\cos x)$$

$$= \log(1+\sin x)$$

$$\therefore y = c_1 \cos x + c_2 \sin x + (\tan x - \sec x - x)\cos x + \log(1+\sin x) \cdot \sin x.$$

3.C So

$(x^2D^2 - 3xD + 4)y = (1+x)^2$ where $D = \frac{d}{dx}$

We have $x^2y'' - 3xy' + 4y = (1+x)^2 \rightarrow (1)$

Put $t = \log x \Rightarrow x = e^t$

Then $xy' = Dy$, $x^2y'' = D(D-1)y$ where $D = \frac{d}{dt}$

Now eqⁿ ① becomes

$$[D(D-1)-3D+4]y = (1+e^t)^2$$

$$(D^2-D-3D+4)y = 1+2e^t+e^{2t}$$

$$(D^2-4D+4)y = 1+2e^t+e^{2t}$$

A.E is $m^2-4m+4=0$

$$(m-2)^2=0$$

$$m=2,2.$$

$$\therefore y_c = (c_1+c_2t)e^{2t}$$

$$y_p = \frac{1}{D^2-4D+4} + \frac{2e^t}{D^2-4D+4} + \frac{e^{2t}}{D^2-4D+4}$$

$$= \frac{e^{0t}}{4} + \frac{2e^t}{1-4+4} + \frac{e^{2t}}{4-8+4}$$

$$= \frac{1}{4} + \frac{2e^t}{1} + t\left[\frac{e^{2t}}{2D-4}\right]$$

$$y_p = \frac{1}{4} + 2e^t + t^2\left(\frac{e^{2t}}{2}\right)$$

$$\therefore y = (c_1+c_2t)e^{2t} + 2e^t + \frac{1}{2}t^2e^{2t} + \frac{1}{4}$$ ①

Thus $y = (c_1+c_2\log x)x^2 + 2x + \frac{1}{2}x^2(\log x)^2 + \frac{1}{4}$

4.a $(D^3+8)y = x^4+2x+1$, $D = \frac{d}{dx}$

A.E is $m^3+8=0$

$(m+2)(m^2-2m+4)=0$

$m=-2$, $m = 1 \pm i\sqrt{3}$

$y_c = c_1 e^{-2x} + e^x \{c_2 \cos(\sqrt{3})x + c_3 \sin(\sqrt{3}x)\}$

$y_p = \frac{x^4+2x+1}{8+D^3}$

$$\begin{array}{l|l} & \frac{x^4}{8} - \frac{x}{8} + \frac{1}{8} \\ \hline 8+D^3 & x^4+2x+1 \\ & x^4+3x \\ \hline & -x+1 \\ & -x-0 \\ \hline & 1 \\ & 1 \\ \hline & 0 \end{array}$$

$\therefore y_p = \frac{x^4}{8} - \frac{x}{8} + \frac{1}{8}$

$\therefore y = c_1 e^{-2x} + e^x[c_2 \cos(\sqrt{3}x) + c_3 \sin(\sqrt{3}x)] + \frac{1}{8}(x^4 - x + 1)$

4.b $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 8x^2+4x+1$ → (1)

Put $t = \log(3x+2) \Rightarrow e^t = 3x+2 \Rightarrow x = \frac{1}{3}(e^t-2)$

Then $(3x+2)\frac{dy}{dx} = 3Dy$

$(3x+2)^2 \frac{d^2y}{dx^2} = 3^2 D(D-1)y = 9D(D-1)y$ $\boxed{D = \frac{d}{dt}}$

Now eq^n (1) becomes

$$\left[9D(D-1)+9D-36\right]y = 8\tfrac{1}{9}\left(e^{t}-2\right)^{2}+4\cdot\tfrac{1}{3}\left(e^{t}-2\right)+1$$

$$9(D^2-4)y = \frac{8}{9}\left[e^{2t}-4e^{t}+4\right]+\frac{4}{3}\left(e^{t}-2\right)+1$$

$$9(D^2-4)y = \frac{8}{9}e^{2t}-\frac{20}{9}e^{t}+\frac{17}{9}$$

$$(D^2-4)y = \frac{1}{81}\left(8e^{2t}-20e^{t}+17\right)$$

A.E is $m^2-4=0$

$m=\pm 2$

$\therefore y_c = c_1e^{2t}+c_2e^{-2t}$

$$y_p = \frac{1}{81}\left[\frac{8e^{2t}}{D^2-4}-\frac{20e^{t}}{D^2-4}+\frac{17e^{0t}}{D^2-4}\right]$$

$$y_p = \frac{1}{81}\left[t\,\frac{8e^{2t}}{2D}-\frac{20e^{t}}{-3}+\frac{17}{-4}\right]$$

$$y_p = \frac{1}{81}\left[t\,2e^{2t}+\frac{20}{3}e^{t}-\frac{17}{4}\right]$$

$\therefore y = y_c + y_p$

$$y = c_1(3x+2)^2+\frac{c_2}{(3x+2)^2}+\frac{1}{81}\left[2\log(3x+2)\,(3x+2)^2+\frac{20}{3}(3x+2)-\frac{17}{4}\right]$$

4.c Given $10\frac{d^2x}{dt^2}+\frac{dx}{dt}+200x=0 \longrightarrow$ (1)

and $x=0.25$cm, $\frac{dx}{dt}=0$ at $t=0$

using $D=\frac{d}{dt}$, eq^n (1) becomes

$$(10D^2+D+200)x=0$$

A.E $\quad 10m^2 + m + 200 = 0$

$$m = -0.05 \pm i\,4.47$$

$$x_c = e^{-0.05t}\left[c_1\cos(4.47t) + c_2\sin(4.47t)\right]$$

$$\therefore x = e^{-0.05t}\left[c_1\cos(4.47t) + c_2\sin(4.47t)\right] \rightarrow (1)$$

$$\frac{dx}{dt} = e^{-0.05t}\left[-4.47c_1\sin(4.47t) + 4.47c_2\cos(4.47t)\right] - 0.05\,e^{-0.05t}\left[c_1\cos(4.47t) + c_2\sin(4.47t)\right] \rightarrow (2)$$

using ~~x = 0.0~~ $x = 0.25$, $t = 0$ in (1)

$0.25 = c_1$

using $\frac{dx}{dt} = 0$, $t = 0$ in (2)

$$0 = 4.47c_2 - c_1(0.05)$$

$$\Rightarrow 4.47c_2 = 0.05c_1 = 0.05(0.25)$$

$$\Rightarrow c_2 = 0.0027$$

$$\therefore x = e^{-0.05t}\left[0.25\cos(4.47t) + 0.0027\sin(4.47t)\right]$$

Q 5(a) Given $(x-a)^2+(y-b)^2+z^2=c^2$ — ①

diff. eqⁿ ①, partially w.r.t. 'x'

$2(x-a)+0+2z\frac{\partial z}{\partial x}=0 \Rightarrow (x-a)+pz=0$

$\Rightarrow (x-a)=-pz$ — ②

diff eqⁿ ①, partially w.r.t. 'y'

$0+2(y-b)+2z\frac{\partial z}{\partial y}=0 \Rightarrow (y-b)+qz=0$

$\Rightarrow (y-b)=-qz$ — ③

using ② & ③ in ①

$(-pz)^2+(-qz)^2+z^2=c^2 \Rightarrow$

$(p^2+q^2+1)z^2=c^2$ is the required PDE.

5(b) Given $\frac{\partial^2 z}{\partial y^2}=z \Rightarrow (D^2-1)z=0$, $D\equiv \partial/\partial y$. [assuming z is f(y) only]

A.E. $\Rightarrow m^2-1=0 \Rightarrow m=\pm 1$

C.F. $= z_c = ~~c_1 c_2~~ c_1e^{y}+c_2e^{-y}=z$. — ①

Given that $z=e^x$, when $y=0$, from ①

$e^x=c_1+c_2$ — ②

$\frac{\partial z}{\partial y}=e^{-x}$, when $y=0$. differentiating eqⁿ ①

~~$e^{-x}=c_1+c_2$~~ $\frac{\partial z}{\partial y}=c_1e^{y}-c_2e^{-y}$

$\Rightarrow e^{-x}=c_1-c_2$ — ③

Solving ② & ③

$c_1=\frac{e^x+e^{-x}}{2}$, $c_2=\frac{e^x-e^{-x}}{2}$

$=\cosh x$ $=\sinh x$

$\Rightarrow$ $z = e^{+y}\cosh x + e^{-y}\sinh x$ is the required sol^n.

5(c) Derivation of One dimensional wave Equation

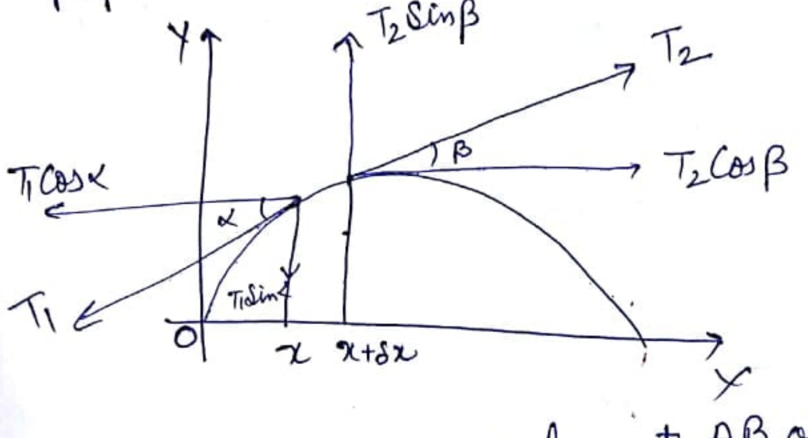
Consider a flexible string tightly streched between two fixed points at a distance 'l' apart. Let ρ be the mass per unit lenght of the string.

We assume the following conditions.

(i) The Tension 'T' of the string is same throughout.

(ii) The effect of gravity is ignored due to large tension

(iii) The motion of string is in small transverse vibration i.e. in perpendicular direction, no vibration horizontally.



Now, we consider a small element AB of length δx.

Let T_1 and T_2 be the tensions at points A and B. α and β are angles made by T_1 and T_2 with horizontal.

$\because$ There is no motion in horizontal direction, the horizontal components will cancel each other.

$\therefore$ By figure, $T = T_1 \cos\alpha = T_2 \cos\beta$ ——— (1)

Hence, the resultant force acting vertically upwards is,

$T_2 \sin\beta - T_1 \sin\alpha$ ——— (2)

By Newton's second law of motion,

force = mass × acceleration.

$T_2 \sin\beta - T_1 \sin\alpha = (\rho\,\delta x)\left(\frac{\partial^2 u}{\partial t^2}\right)$ [where u is the displacement along 'x']

Dividing throughout by T.

$$\frac{T_2}{T}\sin\beta - \frac{T_1}{T}\sin\alpha = \frac{\rho}{T}\delta x\frac{\partial^2 u}{\partial t^2}.$$

By (1), $\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha} = \frac{\rho}{T}\delta x\frac{\partial^2 u}{\partial t^2}$

$\Rightarrow \tan\beta - \tan\alpha = \frac{\rho}{T}\delta x\frac{\partial^2 u}{\partial t^2}$.

∵ $\tan\beta$ and $\tan\alpha$, represents slopes at $B(x+\delta x)$ and $A\,x$, respectively,

$\tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$, $\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x \Rightarrow \left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{\rho}{T}\delta x\frac{\partial^2 u}{\partial t^2}$

$\Rightarrow \lim_{\delta x\to 0}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$

$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial t^2}}$ is one dimensional wave eqn. with $c^2 = \frac{T}{\rho}$.

—X—

Q6. (E) The one dimension heat equation is given by

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad -(1)$$

Let $u(x,t) = X(x)\,T(t)$ be the solution of (1)

$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XT) = c^2\frac{\partial^2}{\partial x^2}(XT) \Rightarrow X\frac{\partial T}{\partial t} = c^2 T\frac{\partial^2 X}{\partial x^2}$

$\Rightarrow \frac{1}{c^2 T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k$ (say)

If we separate the variables, we get

$\frac{dT}{dt} = c^2 kT$; $\frac{d^2X}{dx^2} = kX$

$(D - c^2k)T = 0$; $(D^2 - k)X = 0$,

Case-i) when $k=0$ $\Rightarrow$ $\frac{dT}{dt}=0$; $\frac{d^2X}{dx^2}=0$

$T=c_1$. $\frac{dX}{dx}=c_2$

$\Rightarrow X=c_2x+c_3$

$\therefore$ The solution is given by, $u(x,t)=c_1(c_2x+c_3)=Ax+B$.

Case-ii) when k is positive i.e. $k=p^2$ (say)

$\Rightarrow (D-p^2c^2)T=0$; $(D^2-p^2)X=0$

$m-p^2c^2=0 \Rightarrow m=p^2c^2$; $m^2-p^2=0 \Rightarrow m=\pm p$

$T=c_1e^{p^2c^2t}$; $X=c_2e^{px}+c_3e^{-px}$

$u(x,t)=c_1e^{p^2c^2t}\{c_2e^{px}+c_3e^{-px}\}=e^{p^2c^2t}\{A\cdot e^{px}+Be^{-px}\}$

Case-iii) when k is negative, i.e. $k=-p^2$ (say)

$(D+p^2c^2)T=0$; $(D^2+p^2)X=0$

A.E. $\Rightarrow m+p^2c^2=0$; A.E. $\Rightarrow m^2+p^2=0$

$m=-p^2c^2$; $m=\pm ip$

$T=c_1e^{-p^2c^2t}$; $X=c_2\cos px+c_3\sin px$

$u(x,t)=c_1e^{-p^2c^2t}\{c_2\cos px+c_3\sin px\}$

$u(x,t)=e^{-p^2c^2t}\{A\cos px+B\sin px\}$.

6(a) $f(x^2+y^2, z-xy)=0 \Rightarrow f(u,v)=0$ —— (1)

where $u = x^2+y^2$, $v = z-xy$.

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = p-y.$$

$$\frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial y} = q-x$$

diff eq^n (1), w.r.t. 'x' and 'y', —— (2)

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \Rightarrow 2x\frac{\partial f}{\partial u} + (p-y)\frac{\partial f}{\partial v} = 0$$ —— (3)

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \Rightarrow 2y\frac{\partial f}{\partial u} + (q-x)\frac{\partial f}{\partial v} = 0$$ —— (4)

from eq^n (2), $2x\frac{\partial f}{\partial u} = -(p-y)\frac{\partial f}{\partial v}$ —— (5)

from eq^n (3), $2y\frac{\partial f}{\partial u} = -(q-x)\frac{\partial f}{\partial v}$

dividing (4) by (5)

$$\frac{x}{y} = \frac{p-y}{q-x} \Rightarrow \boxed{x(q-x) = y(p-y)}$$ is the required PDE.

6(b) Solve: $(x^2-yz)p + (y^2-zx)q = z^2-xy$.

Given equation is of the form, $Pp + Qq = R$,

$\Rightarrow P = x^2-yz, \quad Q = y^2-zx, \quad R = z^2-xy$

the Auxiliary Equation are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx-dy}{(x^2-y^2)+z(x-y)} = \frac{dy-dz}{(y^2-z^2)+x(y-z)} = \frac{dz-dx}{(z^2-x^2)+y(z-x)}$$

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$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

$$\Rightarrow \frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

from ① & ② $\Rightarrow \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$

$\Rightarrow \log(x-y) = \log(y-z) + \log c_1$, on Integration

$\Rightarrow c_1 = \frac{x-y}{y-z}$

from ② & ③ $\Rightarrow \frac{d(y-z)}{y-z} = \frac{d(z-x)}{z-x}$

$\Rightarrow \log(y-z) = \log(z-x) + \log c_2 \Rightarrow c_2 = \frac{y-z}{z-x}$

$\therefore$ the general solution is given by

$$\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$$

7. Model Question Paper - 1 - 18MAT21

a. $\sum_{n=1}^{\infty} \frac{n!}{(n^n)^2}$

Here $u_n = \frac{n!}{(n^n)^2} = \frac{n!}{n^{2n}}$, $u_{n+1} = \frac{(n+1)!}{(n+1)^{2(n+1)}} = \frac{(n+1)!}{(n+1)^{2(n+1)}}$

$u_{n+1} = \frac{(n+1)!}{(n+1)^{2n}(n+1)^2} = \frac{(n+1)\cdot n!}{(n+1)^{2n}(n+1)(n+1)}$

$\therefore u_{n+1} = \frac{n!}{(n+1)^{2n}(n+1)}$

$\frac{u_{n+1}}{u_n} = \frac{n^{2n}}{(n+1)^{2n}(n+1)}$

$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{n^{2n}}{n^{2n}[1+1/n]^{2n}(n+1)}$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^{2n}} \lim_{n\to\infty} \frac{1}{(n+1)}$

$= \lim_{n\to\infty} \frac{1}{[(1+1/n)^n]^2} \times 0.$

$= \lim_{n\to\infty} \frac{1}{e^2} \times 0.$

$= 0 < 1.$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ is convergent

(b) The Bessel differential eq of order n is in the form:

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \quad - (1)$$

Where n is a non-ve real constant.

We employ frobenius method to solve this eq as we have.

$$y'' = x^2 = P_0(x) \text{ \& } P_0(x) = 0 \text{ at } x = 0$$

We assume the series sol of (1) in the form, $y = \sum_{r=0}^{\infty} a_r x^{k+r}$

$$y' = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$$

$$y'' = \sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

(1) becomes

$$\sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{0}^{\infty} a_r (k+r) x^{k+r} +$$

$$\sum_{0}^{\infty} a_r x^{k+r+2} - n^2 \sum_{0}^{\infty} a_r x^{k+r} = 0$$

collecting, first, second & fourth terms

$$\sum_{0}^{\infty} a_r x^{k+r} [(k+r)(k+r-1) + (k+r) - n^2] +$$

$$\sum_{0}^{\infty} a_r x^{k+r+2} = 0$$

$$\sum_{0}^{\infty} a_r x^{k+r} [(k+r)^2 - n^2] + \sum_{0}^{\infty} a_r x^{k+r+2} = 0$$

ie $a_0 [k^2 - n^2] = 0$ (equate coeff of x^k to zero)

$\because a_0 \neq 0 \; k = \pm n.$

equate coeff of x^{k+1} to zero.

$$a_1\left[(k+1)^2 - n^2\right] = 0$$

$\Rightarrow a_1 = 0 \quad \because (k+1)^2 - n^2 = 0$ would mean $(k+1)^2 = n^2$ or $k+1 = \pm n$ which cannot be accepted as we have already $k = \pm n$.

equate coeff of x^{k+r} $(r \geq 2)$ to zero.

$$a_r\left[(k+r)^2 - n^2\right] + a_{r-2} = 0$$

$$a_r = \frac{-a_{r-2}}{\left[(k+r)^2 - n^2\right]} \quad (r \geq 2) \quad -(3)$$

$k = n$, (3) becomes

$$a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} = -\frac{a_{r-2}}{2nr + r^2}.$$

$r = 2, 3, 4 \ldots$

$$a_2 = -\frac{a_0}{4n+4} = -\frac{a_0}{4(n+1)}, \quad a_3 = -\frac{a_1}{6n+9} = 0$$

$\because a_1 = 0, \; a_3 = a_5 = a_7 = 0 \ldots$

$$a_4 = \frac{-a_2}{8n+16} = -\frac{a_2}{8(n+2)} = \frac{a_0}{32(n+1)(n+2)} \text{ \& so on.}$$

we substitute these values in the exp form of (2)

$$y = x^k\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right]$$

Also let the sol $k = +n$ be denoted by y_1

$$y_1 = x^n\left[a_0 - \frac{a_0}{4(n+1)} x^2 + \frac{a_0}{32(n+1)(n+2)} x^4 - \cdots\right]$$

$$y_1 = a_0 x^n\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} - \cdots\right] \quad (4)$$

$\because$ $K = -n$, let the sol for $k = -n$ be denoted by y_2.

Replacing n by $-n$ in (4) we have

$$y_2 = a_0 x^{-n}\left[1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5(-n+1)(-n+2)} - \cdots\right]$$

The complete sol of (1) is given by (5)

$y = Ay_1 + By_2$, A, B are arbitrary constants

we shall now standardize the sol as in (4) by choosing $a_0 = \frac{1}{2^n\Gamma(n+1)}$ & the same be denoted by Y_1.

$$Y_1 = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{n+1} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\cdot 2} - \cdots\right]$$

$$Y_1 = \left(\frac{x}{2}\right)^n\left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\Gamma(n+1)\cdot 2} - \cdots\right]$$

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$\Gamma n = (n-1)\Gamma n-1$ $\quad$ $\Gamma n+3 = (n+2)\Gamma n+2$.

$\Gamma n+2 = (n+1)\Gamma n+1$

As a consequence of these results we now have

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma n+1} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma n+2} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma n+3 \cdot 2} - \cdots\right]$$

This can be put in the form

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\Gamma n+1 \cdot 0!}\left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{\Gamma n+2 \cdot 1!}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{\Gamma n+3 \cdot 2!}\left(\frac{x}{2}\right)^4 + \cdots\right]$$

$$= \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma n+r+1 \cdot r!}\left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma n+r+1 \cdot r!}$$

This func is called the Bessel func of the first kind of order n denoted by $J_n(x)$.

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r+n} \frac{1}{\Gamma n+r+1 \cdot r!}$$

The sol $K = -n$ (in respect of y_2) be denoted by $J_{-n}(x)$

Hence the general sol of Bessel's eq is given by

$y = a J_n(x) + b J_{-n}(x)$

where a & b are arbitrary constants & n not an integer.

(c) $f(x) = x^4 + 3x^3 + 5x - x^2 - 2$.

$P_0(x) = 1, \; P_1(x) = x, \; P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}[5x^3 - 3x], \; P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$x^2 = \frac{1}{3}[2P_2(x) + P_0(x)], \; x^3 = \frac{1}{5}[2P_3(x) + 3P_1(x)]$

$x^4 = \frac{1}{35}[8P_4(x) + 30x^2 - 3] = \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$

$f(x) = \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)] + \frac{3}{5}[2P_3(x) + 3P_1(x)] - \frac{1}{3}[2P_2(x) + P_0(x)] + 5P_1(x) - 2P_0(x)$.

$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1(x) - \frac{224}{105}P_0(x)$

([redacted])

8.

a. $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

$u_n = \frac{n^2}{3^n}$.

$u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$

$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2} = \frac{n^2\left[1+\frac{1}{n}\right]^2}{3^n \cdot 3} \cdot \frac{3^n}{n^2}$

$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{\left[1+1/n\right]^2}{3}$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{3}\left[1+\frac{1}{n}\right]^2$.

$= \frac{1}{3}[1+0]$

$= \frac{1}{3} < 1$

Thus it is convergent.

(b) Orthogonal property of Bessel Functions.

Statement: If α & β are 2 distinct roots of $J_n(x) = 0$ then

$$\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = 0 \quad \text{if } \alpha \neq \beta$$

Proof: W.K.T $J_n(\lambda x)$ is a solution of the eq

$$x^2 y'' + x y' + (\lambda^2 x^2 - n^2) y = 0$$

If $u = J_n(\alpha x)$, $v = J_n(\beta x)$ the associated diff eqs are

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0 \quad -①$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0 \quad -②$$

×ly ① by $\frac{v}{x}$ & ② by $\frac{u}{x}$ we obtain

$$x v u'' + v u' + \alpha^2 u v x - \frac{n^2 u v}{x} = 0$$

$$x u v'' + u v' + \beta^2 u v x - \frac{n^2 u v}{x} = 0$$

on subtracting we obtain

$$x[v u'' - u v''] + [v u' - u v'] + (\alpha^2 - \beta^2) u v x = 0$$

$$\frac{d}{dx}\left[x(u'v - uv')\right] = (\beta^2 - \alpha^2) u v x.$$

Integrating b.s w.r.to x b/w 0 & 1

$$\left[x(u'v - uv')\right]_0^1 = (\beta^2 - \alpha^2)\int_0^1 x u v\, dx$$

$$(u'v - uv')_{x=1} - 0 = (\beta^2 - \alpha^2)\int_0^1 x u v\, dx$$

7

$\therefore u = J_n(\alpha x),\ v = J_n(\beta x)$ we have
$u' = \alpha J_n'(\alpha x),\ v' = \beta J_n'(\beta x)$ & as a consequence of these (3) becomes

$$\left[J_n(\beta x)\,\alpha J_n'(\alpha x) - J_n(\alpha x)\,\beta J_n'(\beta x)\right]_{x=1}$$
$$= (\beta^2-\alpha^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$$

Hence $\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{\beta^2-\alpha^2}\left(\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\right]$ — (4)

$\therefore \alpha$ & β are distinct roots of
$J_n(x) = 0$ we have $J_n(\alpha) = 0$ & $J_n(\beta) = 0$ with the result the R.H.S of (4) becomes zero provided

$\beta^2 - \alpha^2 \neq 0$ or $\beta \neq \alpha$

$\therefore \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$ if $\alpha \neq \beta$.

(c) $P_4(\cos\theta) = \frac{1}{8}[35\cos 4\theta + 20\cos 2\theta + 9]$

$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$

$P_4(\cos\theta) = \frac{1}{8}[35(\cos^4\theta) - 30\cos^2\theta + 3]$

$= \frac{1}{8}\left[35\{[\cos 2\theta]\}^2 - 30\left(\frac{1+\cos 2\theta}{2}\right) + 3\right]$

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$$= \frac{1}{8}\left[35\left(\frac{1+\cos 2\theta}{2}\right)^2 - 30\left(\frac{1+\cos 2\theta}{2}\right) + 3\right]$$

$$= \frac{1}{8}\left[\frac{35}{4}\left(1+\cos^2 2\theta + 2\cos 2\theta\right) - 15\left(1+\cos 2\theta\right) + 3\right]$$

$$= \frac{1}{8}\left[\frac{35}{4} + \frac{35}{4}\cos^2 2\theta + \frac{35}{2}\cos 2\theta - 15 - 15\cos 2\theta + 3\right]$$

$$= \frac{1}{8}\left[\frac{35}{4} + \frac{35}{4}\left(\frac{1+\cos 4\theta}{2}\right) + \frac{35}{2}\cos 2\theta - 15 - 15\cos 2\theta + 3\right]$$

$$= \frac{1}{8}\left[\frac{35}{4} + 3 - 15 + \frac{35}{8} + \frac{35}{8}\cos 4\theta + \frac{35}{2}\cos 2\theta - 15\cos 2\theta\right]$$

$$= \frac{1}{8}\left[\frac{35}{4}\left(\frac{\cos 4\theta + 4\cos 2\theta + 3}{2}\right) - 15\cos 2\theta - 12\right]$$

$$= \frac{1}{64}\left[35\cos 4\theta + 140\cos 2\theta + 105 - 120\cos 2\theta - 96\right]$$

$$= \frac{1}{64}\left[35\cos 4\theta + 20\cos 2\theta + 9\right]$$

Module-5

QP-1

9(a)

$$f(x) = x\sin x + \cos x$$

$$f'(x) = x\cos x + \sin x - \sin x = x\cos x$$

Also $x_0 = \pi$ (in radians)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{(\pi\sin\pi + \cos\pi)}{\pi\cos\pi}$$

$$x_1 = \pi - \frac{1}{\pi} = 2.8233$$

Now $x_2 = 2.8233 - \dfrac{f(2.8233)}{f'(2.8233)}$

$$x_2 = 2.7986$$

Next, $x_3 = 2.7986 - \dfrac{f(2.7986)}{f'(2.7986)}$

$$x_3 = 2.7984$$

Also $x_4 = 2.7984 - \dfrac{f(2.7984)}{f'(2.7984)}$

$$x_4 = 2.7984$$

$\therefore$ The real root is 2.7984

9(b)

x	$y = f(x)$	I difference	II difference	III difference	IV difference
1.7	5.474	0.			
		0.576			
1.8	6.050		0.06		
		0.636		0.007	
1.9	6.686		0.067		0
		0.703		0.007	
2.0	7.389		0.074		0.001
		0.777		0.008	
2.1	8.166		0.082		
		0.859			
2.2	9.025				

To find f(2.18) we use Newton's backward interpolation formula.

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

where $r = \frac{x - x_n}{h} = \frac{2.18 - 2.2}{0.1} = -0.2$

$\nabla y_n = 0.859$, $\nabla^2 y_n = 0.082$, $\nabla^3 y_n = 0.008$ $\nabla^4 y_n = 0.001$

$$y(2.18) = 9.025 + (-0.2)(0.859) + \frac{(-0.2)(0.8)}{2}(0.082) + \frac{(-0.2)(0.8)(1.8)}{6}(0.008) + \frac{(0.2)(0.8)(1.8)(2.8)}{24}(0.001)$$

$$= 9.025 - 0.1718 - 0.00656 - 0.00038 - 0.0000336$$

$y(2.18) = 8.8462$

9(c) $h = \frac{\pi/2 + \pi/2}{6} = \frac{\pi}{6} = 30^\circ$ | $n = 6$

x	-90°	-60°	-30°	0	30	60	90
$y = \cos x$	0	0.5	0.8660	1	0.8660	0.5	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

weddle's rule for $n = 6$ is given by

$$\int_a^b y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

Substituting the values & summing up, we get

$$= \frac{3\,\pi/6}{10}\left[12.732\right]$$

$$= \frac{3\pi}{60}\left[12.732\right]$$

$$\int_{-\pi/2}^{\pi/2} \cos x\,dx = 1.9999$$

10(a) Given $a = 2$, $b = 3$ $\quad x \log_{10} x - 1.2 = 0$

$f(a) = -0.6 \quad f(b) = 0.23$

I iteration

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$x_1 = 2.7229$

$f(x_1) = f(2.7229) = -0.0154$

~~Since~~ Root lies between 2.7229 and 3

$\therefore a = 2.7229 \quad b = 3$

$f(a) = -0.0154 \quad f(b) = 0.23$

II iteration

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = 2.7403$$

$f(x_2) = f(2.7403) = -0.000301$

$\therefore$ Root lies between 2.7403 and 3.

$a = 2.7403 \quad b = 3$

$f(a) = -0.0003 \quad f(b) = 0.23$

III iteration

$$x_3 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = 2.7406$$

$\therefore x_2 \approx x_3 \approx 2.740$ (upto 3 decimal places)

10 (b) $x_0 = 2$, $x_1 = 10$, $x_2 = 17$; $y_0 = 1$, $y_1 = 3$, $y_2 = 4$ } $y = f(x)$? polynomial.

Lagrange's interpolation formula

$$y = \frac{(x - x_1)(x - x_2)\cancel{(x - x_3)}}{(x_0 - x_1)(x_0 - x_2)\cancel{(x_0 - x_3)}} y_0 + \frac{(x - x_0)(x - x_2)\cancel{(x - x_3)}}{(x_1 - x_0)(x_1 - x_2)\cancel{(x_1 - x_3)}} y_1 + \frac{(x - x_0)(x - x_1)\cancel{(x - x_3)}}{(x_2 - x_0)(x_2 - x_1)\cancel{(x_2 - x_3)}} y_2$$

$$y = \frac{(x-10)(x-17)}{(2-10)(2-17)}\cdot 1 + \frac{(x-2)(x-17)}{(10-2)(10-17)}\times 3 + \frac{(x-2)(x-10)}{(17-2)(17-10)}\, 4$$

$$= \frac{(x^2 - 10x - 17x + 170)}{120} + \frac{(x^2 - 17x - 2x + 34)3}{-56} + \frac{(x^2 - 10x - 2x + 20)(4)}{105}$$

$$= \frac{1}{120}\left[x^2 - 17x + 170\right] - \frac{1}{56}\left[x^2 - 19x + 34\right]3 + \frac{\left[x^2 - 12x + 20\right]4}{105}$$

$$= x^2\left[\frac{1}{120} - \frac{3}{56} + \frac{4}{105}\right] + x\left[-\frac{17}{120} + \frac{57}{56} - \frac{48}{105}\right] + \left[\frac{170}{120} - \frac{102}{56} - \frac{80}{105}\right]$$

$$y = -\frac{x^2}{140} - \frac{44}{105}x - \frac{7}{6}$$

10 (c) Given $n = 3$ [4 equidistant ordinates]

Length of each strip $(h) = \frac{3-0}{3} = 1$

x	0	1	2	3
$y = \frac{1}{(1+x)^2}$	1	$1/4$	$1/9$	$1/16$
	y_0	y_1	y_2	y_3

Simpson's $3/8^{th}$ rule for $n=3$ is given by

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0 + y_3) + 3(y_1 + y_2)\right]$$

$$= \frac{3}{8}\left[\left(1 + \frac{1}{16}\right) + 3\left(\frac{1}{4} + \frac{1}{9}\right)\right] = 0.8047$$

$$\therefore \int_0^3 \frac{dx}{(1+x)^2} = 0.8047$$