Model Question Paper-2 with effect from 2018-19
(CBCS Scheme)

USN | | | | | | | | | | |

18MAT11

First Semester B.E. Degree Examination
Calculus and Linear Algebra
(Common to all Branches)

Time: 3 Hrs Max.Marks: 100

Note: Answer any FIVE full questions, choosing at least ONE question from each module.

Module-1

1. (a) With usual notation, prove that $\tan\phi = r[d\theta/dr]$. **(06 Marks)**

   (b) Find the radius of curvature at the point $(3a,3a)$ on the curve $x^3 + y^3 = 3axy$. **(06 Marks)**

   (c) Show that the evolute of the ellipse: $x^2/a^2 + y^2/b^2 = 1$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$ **(08 Marks)**

OR

2. (a) Find the pedal equation of the curve : $l/r = 1 + e\cos\theta$. **(06 Marks)**

   (b) Find the radius of curvature for the curve $\theta = \left[\sqrt{r^2 - a^2}/a\right] + \cos^{-1}[a/r]$ at any point on it. **(06 Marks)**

   (c) Show that the angle between the pair of curves: $r^2\sin 2\theta = 4$ & $r^2 = 16\sin 2\theta$ is $\pi/3$. **(08 Marks)**

Module-2

3. (a) Using Maclaurin's series, prove that $\log(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \ldots$ **(06 Marks)**

   (b) Evaluate (i) $\lim_{x\to a}[2-(x/a)]^{\tan\left(\frac{\pi x}{2a}\right)}$ (ii) $\lim_{x\to 0}[(1/x)]^{2\sin x}$ **(07 Marks)**

   (c) Examine the function $f(x,y) = 2(x^2 - y^2) - x^4 + y^4$ for its extreme values. **(07 Marks)**

OR

4. (a) If $u = f(2x-3y, 3y-4z, 4z-2x)$, show that $(1/2)u_x + (1/3)u_y + (1/4)u_z = 0$ **(06 Marks)**

   (b) If $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$, show that $J[(x,y,z)/(r,\theta,\phi)] = r^2\sin\theta$. **(07 Marks)**

   (c) A rectangular box, open at the top, is to have a volume of 32 cubic ft. Find the dimension of the box requiring least material for its construction. **(07 Marks)**

Module-3

5. (a) Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$ by changing into polar coordinates. **(06 Marks)**

   (b) Find the volume the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes, using double integration. **(07 Marks)**

   (c) Show that $\int_0^\infty \frac{e^{-x^2}dx}{\sqrt{x}} \times \int_0^\infty \sqrt{x}\,e^{-x^2}dx = \frac{\pi}{2\sqrt{2}}$ **(07 Marks)**

OR

6. (a) Evaluate: $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$ **(06 Marks)**

(b) Find by double integration, the centre of gravity of the area of the cardioid: $r = a(1+\cos\theta)$. **(07 Marks)**

(c) With usual notations, show that $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx = \beta(m,n)$. **(07 Marks)**

Module-4

7. (a) A copper ball originally at $80^0 C$ cools down to $60^0 C$ in 20 minutes, if the temperature of the air being $40^0 C$. What will be the temperature of the ball after 40 minutes from the original? **(06 Marks)**

(b) Find the orthogonal trajectories of the family of curves $r^n \cos n\theta = a^n$,where a is a parameter. **(07 Marks)**

(c) Solve : $\left[3x^2y^4 + 2xy\right]dx + \left[2x^3y^3 - x^2\right]dy = 0$. **(07 Marks)**

OR

8. (a) Solve the differential equation $L[di/dt] + Ri = 200\sin 300t$, when $L = 0.05$ & $R = 100$ and find the value of the current i at any time t, if initially there is no current in the circuit. What value does i approach after a long time. **(06 Marks)**

(b) Solve : $\left[r\sin\theta - r^2\right]d\theta - \left[\cos\theta\right]dr = 0$. **(07 Marks)**

(c) Solve: $p^4 - [x+2y+1]p^3 + [x+2y+2xy]p^2 - 2xyp = 0$, where $p = dy/dx$. **(07 Marks)**

Module-5

9. (a) For what values λ and μ the system of equations $x+y+z=6;\ x+2y+3z=10;\ x+2y+\lambda z=\mu$, has (a) no solution (b) a unique solution and (iii) infinite number of solutions. **(06 Marks)**

(b) Using Rayleigh's power method, find largest eigen value and eigen vector of the matrix:

$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ by taking $X^{(0)} = [1,0,0]^T$ as initial eigen vector.(Perform 7 iterations) **(07 Marks)**

(c) Use Gauss-Jordan method solve the system of equations:

$83x + 11y - 4z = 95;\ 7x + 52y + 13z = 104; 3x + 8y + 29z = 71$ **(07 Marks)**

OR

10. (a) Find the rank of the matrix $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ by applying elementary row operations. **(06 Marks)**

(b) Reduce the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ into the diagonal form. **(06 Marks)**

(c) Solve the system of equations $2x - 3y + 20z = 25;\ 20x + y - 2z = 17;\ 3x + 20y - z = -18$, using Gauss-Seidel method. (Carry out 4 iterations). **(07 Marks)**

*****

Calculus and Linear Algebra (18MAT11) (1)

Solution for Model Question Paper – 2 (2018-2019)

1.a

Consider a polar curve $r=f(\theta)$ and a point $P(r,\theta)$ on the curve.

Let PT be the tangent to the curve at P and ψ be the angle made by the tangent PT with the +ve X-axis, ϕ be the angle between the radius vector OP and the tangent PT.

Diagram in Page.39

From the diagram we have, $\psi=\phi+\theta$

We know that

$$\frac{dy}{dx} = \text{slope of the tangent PT}$$

$$= \tan\psi$$

$$= \tan(\phi+\theta)$$

$$\frac{dy}{dx} = \frac{\tan\phi+\tan\theta}{1-\tan\phi\tan\theta} \longrightarrow (1)$$

Let (x,y) be the cartesian coordinate of P, then we have $x=r\cos\theta$, $y=r\sin\theta$

Now $$\frac{dx}{d\theta} = r(-\sin\theta)+\cos\theta\frac{dr}{d\theta}$$

$$\frac{dy}{d\theta} = r\cos\theta+\sin\theta\frac{dr}{d\theta}$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{r\cos\theta + \sin\theta \frac{dr}{d\theta}}{-r\sin\theta + \cos\theta \frac{dr}{d\theta}}$$

Dividing both Nr and Dr of RHS by $\cos\theta \frac{dr}{d\theta}$

we get, $$\frac{dy}{dx} = \frac{r\, d\theta/dr + \tan\theta}{1 - r\frac{d\theta}{dr}\tan\theta} \longrightarrow (2)$$

Comparing (1) & (2) we get

$$\underline{\tan\phi = r\frac{d\theta}{dr}}$$

$\boxed{1.b}$ $x^3 + y^3 = 3axy$

Diff. w.r.to x

$$3x^2 + 3y^2 y_1 = 3a[xy_1 + y]$$

$$\cancel{3}(x^2 + y^2 y_1) = \cancel{3}(axy_1 + ay)$$

$$y^2 y_1 = axy_1 + ay - x^2$$

$$y_1(y^2 - ax) = ay - x^2$$

$$y_1 = \frac{ay - x^2}{y^2 - ax}$$

at $(3a, 3a)$

$$y_1 = \frac{3a^2 - 9a^2}{9a^2 - 3a^2} = -1$$

$$y_1 = -1$$

$$y_2 = \frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2}$$ (3)

At $(3a, 3a)$

$$y_2 = \frac{(9a^2-3a^2)(-a-6a)-(3a^2-9a^2)(-6a-a)}{(9a^2-3a^2)^2}$$

$$= \frac{\cancel{(9a^2-3a^2)}[-7a-7a]}{(9a^2-3a^2)^{\cancel{2}}}$$

$$= \frac{-14a}{6a^2}$$

$$= -\frac{\cancel{14}\,7}{\cancel{6}_3 a}$$

$$= \frac{-7}{3a}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{-7/3a}$$

$$= \frac{\sqrt{8}}{-7}(3a)$$

$$= \frac{2\sqrt{2}}{-7}(3a)$$

$$\rho = \frac{6a\sqrt{2}}{-7}$$

1.c

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

4

The parametric form of the given curve is

$x = a\cos t, \quad y = b\sin t$

$\cancel{y_1 = \frac{dy}{dx} =}$ $\frac{dx}{dt} = -a\sin t, \quad \frac{dy}{dt} = b\cos t$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$y_1 = \frac{b\cos t}{-a\sin t} = \frac{-b}{a}(\cot t)$$

$$y_2 = \frac{-b}{a}(-\mathrm{cosec}^2 t)\frac{dt}{dx}$$

$$= \frac{b}{a}\mathrm{cosec}^2 t\left(\frac{1}{-a\sin t}\right)$$

$$y_2 = \frac{-b}{a^2}\mathrm{cosec}^3 t$$

We have $\quad \bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$\bar{x} = a\cos t - \frac{(-b/a)\cot t\left(1+\frac{b^2}{a^2}\cot^2 t\right)}{\left(-\frac{b}{a^2}\right)\mathrm{cosec}^3 t}$$

$$= a\cos t - a\sin^3 t\,\frac{\cos t}{\sin t}\left(1 + \frac{b^2}{a^2}\frac{\cos^2 t}{\sin^2 t}\right)$$

$$= a\cos t - a\sin^2 t\cos t - \frac{b^2}{a}\cos^3 t$$

$$\bar{x} = a\cos t\left(1 - \sin^2 t\right) - \frac{b^2}{a}\cos^3 t$$

$$= a\cos^3 t - \frac{b^2}{a}\cos^3 t$$

$$\bar{x} = \cos^3 t\left(\frac{a^2 - b^2}{a}\right)$$

$$\bar{y} = b\sin t + \frac{\left(1 + \frac{b^2}{a^2}\cot^2 t\right)}{\left(-b/a^2\right)\operatorname{cosec}^3 t}$$

$$\bar{y} = b\sin t - \frac{a^2}{b}\sin^3 t\left(1 + \frac{b^2}{a^2}\cot^2 t\right)$$

$$\bar{y} = b\sin t - \frac{a^2}{b}\sin^3 t - b\cos^2 t\sin t$$

$$= b\sin t\left(1 - \cos^2 t\right) - \frac{a^2}{b}\sin^3 t$$

$$= b\sin^3 t - \frac{a^2}{b}\sin^3 t$$

$$= \sin^3 t\left(\frac{b^2 - a^2}{b}\right)$$

$$\bar{y} = -\sin^3 t\left(\frac{a^2 - b^2}{b}\right)$$

Now $\bar{x} = \cos^3 t\left(\frac{a^2 - b^2}{a}\right)$

$$a\bar{x} = \left(a^2 - b^2\right)\cos^3 t$$

$$\left(a\bar{x}\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}\cos^2 t \longrightarrow (1)$$

$$\bar{y} = -\sin^3 t\left(\frac{a^2 - b^2}{b}\right)$$

$$b\bar{y} = -(a^2-b^2)\sin^3 t$$

$$(b\bar{y})^{2/3} = (a^2-b^2)^{2/3}\sin^2 t \rightarrow (2)$$

Eqs (1) + (2)

$$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}\left[\cos^2 t + \sin^2 t\right]$$

$$= (a^2-b^2)^{2/3}$$

$\therefore$ The evolute is $(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$

$\boxed{2.a}$

$$\frac{l}{r} = 1+e\cos\theta$$

$$\log l - \log r = \log(1+e\cos\theta)$$

Diff w.r.to θ

$$\cancel{\frac{1}{r}\frac{dr}{d\theta}} \quad 0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+e\cos\theta}(e)(-\sin\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{e\sin\theta}{1+e\cos\theta}$$

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

$$\cot\phi = \frac{e\sin\theta}{1+e\cos\theta}$$

we have $p = r\sin\phi$

~~p=~~

$$\frac{1}{p} = \frac{1}{r\sin\phi} = \frac{1}{r}\operatorname{cosec}\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1+\cot^2\phi\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right] \longrightarrow (1)$$

Also we have $\frac{l}{r} = 1+e\cos\theta \longrightarrow (2)$

$$\frac{l}{r} - 1 = e\cos\theta$$

Also, $e^2\sin^2\theta = e^2(1-\cos^2\theta)$

$$= e^2 - e^2\cos^2\theta$$

$$e^2\sin^2\theta = e^2 - \left(\frac{l}{r}-1\right)^2 \longrightarrow (3)$$

Using (3) & (2) in (1)

$$\therefore \frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{e^2-\left(\frac{l}{r}-1\right)^2}{(l/r)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{r^2}{l^2}\left(e^2-\left(\frac{l}{r}-1\right)^2\right)\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{l^2}\left[e^2-\left(\frac{l}{r}-1\right)^2\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{l^2}\left[e^2-\frac{l^2}{r^2}+2\frac{l}{r}-1\right]$$

$$\frac{1}{p^2} = \cancel{\frac{1}{r^2}}+\frac{e^2}{l^2}-\cancel{\frac{1}{r^2}}+\frac{2}{lr}-\frac{1}{l^2}$$

$$\frac{1}{p^2} = \frac{e^2-1}{l^2}+\frac{2}{lr}$$ is the required pedal equation.

$\boxed{2 \cdot b}$ $\theta = \frac{\sqrt{r^2-a^2}}{a} + \cos^{-1}\left(\frac{a}{r}\right)$

Diff w.r.to θ

$$1 = \frac{1}{a}\frac{1}{\cancel{2}\sqrt{r^2-a^2}}\cancel{2}rr_1 + \frac{1}{\sqrt{1-\frac{a^2}{r^2}}}a\left(\frac{-1}{r^2}\right)r_1$$

$$= \frac{rr_1}{a\sqrt{r^2-a^2}} - \frac{1}{\sqrt{r^2-a^2}}ar_1\left(\frac{1}{r}\right)$$

$$= \frac{r_1}{\sqrt{r^2-a^2}}\left[\frac{r}{a}-\frac{a}{r}\right]$$

$$1 = \frac{r_1}{\sqrt{r^2-a^2}}\left[\frac{r^2-a^2}{ar}\right]$$

$$r_1 = \frac{ar\sqrt{r^2-a^2}}{r^2-a^2}$$

$$r_1 = \frac{ar}{\sqrt{r^2-a^2}}$$

Diff w.r.to θ

$$r_2 = a\left[\frac{\sqrt{r^2-a^2}\,r_1 - r\frac{1}{\cancel{2}\sqrt{r^2-a^2}}\cancel{2}rr_1}{r^2-a^2}\right]$$

$$r_2 = a\left[\frac{(r^2-a^2)r_1 - r^2r_1}{(r^2-a^2)^{3/2}}\right]$$

$$r_2 = a\left[\frac{\cancel{r^2r_1} - a^2r_1 - \cancel{r^2r_1}}{(r^2-a^2)^{3/2}}\right]$$

$$r_2 = \frac{-a^3r_1}{(r^2-a^2)^{3/2}}$$

(9) (8)

$$r_2 = \frac{-a^3}{(r^2-a^2)^{3/2}} \frac{ar}{\sqrt{r^2-a^2}}$$

$$r_2 = \frac{-a^4 r}{(r^2-a^2)^2}$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{\left(r^2 + \frac{a^2r^2}{(\sqrt{r^2-a^2})^2}\right)^{3/2}}{r^2+2\left(\frac{a^2r^2}{r^2-a^2}\right) - r\left[\frac{-a^4r}{(r^2-a^2)^2}\right]}$$

$$\cancel{\rho = \frac{\left(r^2(r^2-a^2) - a^2r^2\right)^{3/2}}{}}$$

$$\rho = \frac{\left(\frac{r^2(r^2-a^2)+a^2r^2}{r^2-a^2}\right)^{3/2}}{\left(\frac{r^2(r^2-a^2)^2+2a^2r^2(r^2-a^2)+a^4r^2}{(r^2-a^2)^2}\right)}$$

$$= \frac{\left(\frac{r^4}{r^2-a^2}\right)^{3/2}}{\left(\frac{r^2}{(r^2-a^2)^2}\right)\left(r^4+\cancel{a^4}-\cancel{2r^2a^2}+\cancel{2a^2r^2}-\cancel{2a^4}+\cancel{a^4}\right)}$$

$$= \left[\frac{r^6}{(r^2-a^2)^{3/2}}\right] \frac{(r^2-a^2)^2}{r^6}\left(\cancel{r^4}\right)$$

$$\boxed{\rho = \gamma^8 \sqrt{r^2 - a^2}}$$

$$\rho = \sqrt{r^2 - a^2}$$

$\boxed{2.c}$ $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

$2 \log r + \log(\sin 2\theta) = \log 4$

Diff w.r.to θ

$2 \frac{1}{r} \frac{dr}{d\theta} + \frac{1}{\sin 2\theta}(2\cos 2\theta) = 0$

$\frac{1}{r} \frac{dr}{d\theta} = -\cot 2\theta = \cot(-2\theta)$

$\cot\phi_1 = \cot(-2\theta)$

$\Rightarrow \boxed{\phi_1 = -2\theta}$

$r^2 = 16 \sin 2\theta$

$2 \log r = \log 16 + \log(\sin 2\theta)$

Diff w.r.to θ

$2 \frac{1}{r} \frac{dr}{d\theta} = 0 + \frac{1}{\sin 2\theta}(2\cos 2\theta)$

$\frac{1}{r} \frac{dr}{d\theta} = \cot 2\theta$

$\cot\phi_2 = \cot 2\theta$

$\phi_2 = 2\theta$

$|\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta$

we have $r^2 \sin 2\theta = 4$ & $r^2 = 16 \sin 2\theta \rightarrow (1)$

$\cancel{r^2 = \sin 2\theta}$

$$r^2 = \frac{4}{\sin 2\theta} \longrightarrow (2)$$

From (1) & (2)

$$\frac{4}{\sin 2\theta} = 16 \sin 2\theta$$

$$(\sin 2\theta)^2 = \frac{4}{16}$$

$$(\sin 2\theta)^2 = \frac{1}{4}$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = \sin^{-1}\left(\frac{1}{2}\right)$$

$$2\theta = \frac{\pi}{6}$$

$$\theta = \frac{\pi}{12}$$

$$\therefore |\phi_1 - \phi_2| = 4\left(\frac{\pi}{12}\right) = \frac{\pi}{3}$$

Module-02 [Differential calculus - ii]

3a) The Maclaurin's series of $\log(\sec x + \tan x)$

To prove: $y(x) = x + \frac{x^3}{6} + \frac{x^5}{24} + ---$

W.K.T The Maclaurin's series of

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + ----$$

given $y = \log(\sec x + \tan x)$ $\quad \therefore y(0) = 0$

$$y_1 = \frac{1}{(\sec x + \tan x)} (\sec x \tan x + \sec^2 x)$$

$$y_1 = \sec x \qquad \therefore y_1(0) = 1$$

$$y_2 = \sec x \tan x$$

$$y_2 = y_1 \cdot \tan x \qquad \therefore y_2(0) = 0$$

$$y_3 = y_1 \cdot \sec^2 x + \tan x \, y_2$$

$$y_3 = (y_1)^3 + y_2 \tan x \qquad \therefore y_3(0) = 1$$

$$y_4 = 3y_1^2 y_2 + y_2 \sec^2 x + \tan x \cdot y_3$$

$$y_4 = 3y_1^2 y_2 + y_2 \cdot y_1^2 + y_3 \cdot \tan x \qquad \therefore y_4(0) = 0$$

$$y_5 = 3y_1^2 y_3 + \underbrace{6y_2 y_1 y_2}_{0} + \underbrace{2y_2^2 y_1}_{0} + \underbrace{y_1^2 y_3 + y_3 y_1^2}_{2} + \underbrace{\tan x \, y_4}_{0}$$

$$y_5(0) = 3 + 2 = 5$$

$$\therefore \log(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + -----$$

b) $K = \lim_{x \to a} \left[2 - \left(\frac{x}{a}\right)\right]^{\tan\left(\frac{\pi x}{2a}\right)}$ $\quad$ 1^{∞} form

$$\log K = \lim_{x \to a} \tan\left(\frac{\pi x}{2a}\right) \log\left[2 - \left(\frac{x}{a}\right)\right] \qquad \infty \times 0$$

$$\log K = \lim_{x \to a} \frac{\log\left[2 - \left(\frac{x}{a}\right)\right]}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \frac{0}{0}$$

$$\log K = \lim_{x \to a} \frac{\frac{1}{\left(2 - \left(\frac{x}{a}\right)\right)} \times -\frac{1}{a}}{-\operatorname{cosec}^2 \frac{\pi x}{2a} \times \frac{\pi}{2a}}$$

$$\log K = \frac{2}{\pi}$$

$$\boxed{K = e^{2/\pi}}$$

b) $k = \lim_{x \to 0} \left(\frac{1}{x}\right)^{2\sin x}$ $\quad \underline{\infty^0 \text{ form}}$

$\log k = \lim_{x\to 0} 2\sin x \log\left(\frac{1}{x}\right)$ $\quad \underline{0 \times \infty}$

$\log k = \lim_{x\to 0} 2 \cdot \frac{\log\left(\frac{1}{x}\right)}{\operatorname{cosec} x}$ $\quad \frac{\infty}{\infty}$

$\log k = \lim_{x\to 0} 2 \cdot \frac{x \times -\frac{1}{x^2}}{-\operatorname{cosec} x \cot x} = \lim_{x\to 0} \frac{+2\sin^2 x}{x\cos x}$ $\quad \frac{0}{0}$

$\log k = \lim_{x\to 0} \frac{4\sin x \cdot \cos x}{-x\sin x + \cos x}$

$\boxed{k = 1}$

3. c) $f(x,y) = 2(x^2 - y^2) - x^4 + y^4$

To find: Extreme values of a function.

$f_x = 4x - 4x^3, \quad f_y = -4y + 4y^3$

To find the stationary pts (x,y) such that $f_x = 0$ and $f_y = 0$

i.e $x(1-x^2) = 0$ and $y(-1+y^2) = 0$

$x = 0, x = \pm 1 \qquad y = 0, y = \pm 1$

$\therefore$ the stationary pts are $(0,0), (0,1), (0,-1), (1,0)$ $(-1,0)$ $(1,1)$ $(1,-1)$ $(-1,1)$ $(-1,-1)$

	$(0,0)$	$(0,1)$	$(0,-1)$	$(1,0)$	$(-1,0)$	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$r = f_{xx} = 4 - 12x^2$	4	$4>0$	$4>0$	$-8<0$	$-8<0$	-8	-8	-8	-8
$t = f_{yy} = -4 + 12y^2$	-4	$+8$	8	-4	-4	8	8	8	8
$s = f_{xy} = 0$	0	0	0	0	0	0	0	0	0
$rt - s^2$	$-16<0$	$32>0$	$32>0$	$32>0$	$32>0$	$-64<0$	$-64<0$	$-64<0$	$-64<0$
	Saddupt	Minm		Maxm		Saddupt			

$\therefore f_{max(1,0)} = 1$ and $f_{min(0,1)} = -1$

$f_{max}(-1,0) = 1$

$f_{min}(0,-1) = -1$

$\therefore$ the max^m value of the given function is 1.

the min^m value of the given function is -1.

Q.4 a) If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$

To prove: $\frac{1}{2}\left(\frac{\partial u}{\partial x}\right) + \frac{1}{3}\left(\frac{\partial u}{\partial y}\right) + \frac{1}{4}\left(\frac{\partial u}{\partial z}\right) = 0$

$u \to f(p, q, r) \to (x, y, z)$

$\therefore u \to (x, y, z)$

$\therefore u_x = u_p \cdot p_x + u_q \cdot q_x + u_r \cdot r_x$

$u_x = u_p \cdot (2) + u_q \cdot (0) + u_r \cdot (-2)$

$u_y = u_p \cdot p_y + u_q \cdot q_y + u_r \cdot r_y$

$u_y = u_p \cdot (-3) + u_q (3) + u_r \cdot (0)$

$u_z = u_p \cdot p_z + u_q \cdot q_z + u_r \cdot r_z$

$u_z = u_p (0) + u_q \cdot (-4) + u_r \cdot (4)$

$\frac{1}{2}(u_x) + \frac{1}{3}(u_y) + \frac{1}{4}(u_z) = \cancel{u_p} - \cancel{u_r} + (\cancel{-u_p}) + \cancel{u_q} - \cancel{u_q} + \cancel{u_r} = 0$ //

b) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

To prove:- $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta.$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^3\theta(\cos^2\phi + \sin^2\phi)$$

$$= r^2\sin\theta(\cos^2\theta + \sin^2\theta)$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta \ //.$$

4.c

Let x, y, z represents the length, breadth, height of the rectangular box. let V be its volume and S be the surface Area which has to be minimum so that least material will be used for its construction.

$$V = xyz = 32$$

$$S = xy + 2yz + 2zx \quad \Big|\because S = 2(xy + yz + zx) - xy$$

To find: x, y, z such that S is minimum, subject to the condition that $xyz = 32$.

$$F = (xy + 2yz + 2zx) + \lambda(xyz)$$

$$F_x = 0, \quad F_y = 0, \quad F_z = 0$$

$$(y + 2z) + \lambda yz = 0, \qquad \lambda = -(y + 2z)/yz$$

$$(x + 2z) + \lambda xz = 0, \qquad \lambda = -(x + 2z)/xz$$

$$(2y + 2x) + \lambda xy = 0, \qquad \lambda = -(2y + 2x)/xy$$

$$\frac{(y+2z)}{yz} = \frac{(x+2z)}{xz} = \frac{(2y+2x)}{xy}$$

$$xy + 2xz = xy + 2yz \quad ; \quad xy + 2\cancel{z}y = 2\cancel{z}y + 2zx$$

$$\boxed{x = y} \qquad\qquad y = 2z$$

$$z = \frac{x}{2} \quad \Big|\because y = x$$

$$\therefore xyz = 32$$

$$x \times x \times \frac{x}{2} = 32$$

$$x^3 = 64$$

$$x = 4$$

$$y = 4$$

$$z = 2$$

$\therefore$ the dimensions are 3, 4, 2 so that least material is used for construction.

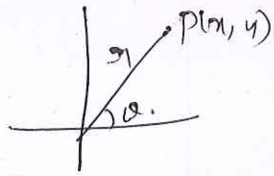
5)

a) $I = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$

Put $x = r\cos\theta \quad y = r\sin\theta \Rightarrow x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

$\therefore I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$



$I = \int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty} e^{-t}\,.\frac{dt}{2}\,d\theta$ Put $r^2 = t$

$\Rightarrow 2r\,dr = dt$

$= \frac{1}{2}\int_0^{\pi/2} \left(-e^{-t}\right)_{t=0}^{\infty} d\theta = -\frac{1}{2}\int_0^{\pi/2} (0-1)\,d\theta$

$= \frac{1}{2}\,(\theta)_0^{\pi/2} = \pi/4$

$\therefore \boxed{I = \pi/4}$

b) $V = \iiint dx\,dy\,dz = \iint z\, dx\,dy$

$x/a + y/b + z/c = 1 \Rightarrow z = c(1 - x/a - y/b)$

x varies from 0 to a

y varies from 0 to $b(1 - x/a)$

$$\therefore V = \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} c\,(1 - x/a - y/b)\,dy\,dx$$

$$= c \int_{x=0}^{a} \left(y - \frac{x}{a}y - \frac{y^2}{2b}\right)_0^{b(1-x/a)} dx.$$

$$= c \int_{x=0}^{a} \left\{ b(1-x/a) - \frac{x}{a} b(1-x/a) - \frac{b^2}{2b}(1-x/a)^2 \right\} dx$$

$$= c \int_{x=0}^{a} b\,(1-x/a)\left\{1 - x/a - \tfrac{1}{2}(1-x/a)\right\} dx.$$

$$= c \int_{x=0}^{a} b\,(1-x/a)\, \tfrac{1}{2}(1-x/a)\, dx$$

$$= \frac{bc}{2}\int_{x=0}^{a} (1-x/a)^2 dx = \frac{bc}{2}\left(\frac{-a}{3}(1-x/a)^3\right)_{x=0}^{a}$$

$$V = \frac{-abc}{6}(0-1) = \frac{abc}{6}.$$

$$\therefore V = \frac{abc}{6} \text{ cubic unit.}$$

c) $\int_0^\infty \frac{e^{-x^2}dx}{\sqrt{x}} \int_0^\infty \sqrt{x}\, e^{-x^2} dx = \frac{\pi}{2\sqrt{2}}$

Let $I_1 = \int_0^\infty \frac{e^{-x^2} dx}{\sqrt{x}.} = \int_0^\infty e^{-x^2} . x^{-1/2} dx$

We know that $\therefore \Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx.$

$\therefore I_1 = 1/2 \, \Gamma 1/4$ (Since $2n-1 = -1/2 \Rightarrow 2n = 1/2 \Rightarrow n = 1/4$

Let $I_2 = \int_0^\infty e^{-x^2} . x^{1/2} dx$ here $2n-1 = 1/2$

$2n = 3/2$

$n = 3/4$

$I_2 = 1/2 \, \Gamma 3/4$

$I_1 . I_2 = 1/4 \, \Gamma 1/4 \, \Gamma 3/4 = \frac{1}{4} . \pi\sqrt{2} = \frac{\pi}{2\sqrt{2}}.$

6)
a) $$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx.}{\sqrt{1-x^2-y^2-z^2}}$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{z=0}^{K} \frac{dz}{\sqrt{K^2-z^2}}\,dy\,dx \quad \left(\text{let } K = \sqrt{1-x^2-y^2}\right.$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\sin^{-1} z/K\right)_{z=0}^{K} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\sin^{-1}1 - \sin^{-1}0\right) dy\,dx = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \pi/2\,dy\,dx$$

$$= \pi/2 \int_{x=0}^{1} (y)_0^{\sqrt{1-x^2}}\,dx = \pi/2 \int_{x=0}^{1} \sqrt{1-x^2}\,dx.$$

$$= \pi/2 \left(\frac{x\sqrt{1-x^2}}{2} + 1/2 \sin^{-1}x\right)_0^1$$

$$= \pi/2 \cdot \left(1/2 \left(\sin^{-1} - \sin^{-1}0\right)\right)$$

$$= \pi/2 \cdot 1/2 \cdot \pi/2 = \pi^2/8.$$

$$\left(\text{using } \int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + a^2/2 \sin^{-1}(x/a)\right)$$

6) b) $r = a(1+\cos\theta)$

Since cardioid is symmetrical about initial line $\bar{y} = 0$

$$\bar{x} = \frac{\iint \rho\, r\cos\theta\, r\, d\theta\, dr}{\iint \rho\, r\, d\theta\, dr}$$

$$\bar{x} = \frac{\int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} \cos\theta \cdot r^2\, dr\, d\theta}{\int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} r\, dr\, d\theta} = \frac{\int_{-\pi}^{\pi} \cos\theta \left(r^3/3\right)_0^{a(1+\cos\theta)} d\theta}{\int_{-\pi}^{\pi} \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta}$$

$$= \frac{2a}{3} \frac{\int_{-\pi}^{\pi} \cos\theta\,(1+\cos\theta)^3\, d\theta}{\int_{-\pi}^{\pi} (1+\cos\theta)^2\, d\theta}$$

$$= \frac{2a}{3} \cdot \frac{2\int_{0}^{\pi} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2\int_{0}^{\pi} (1+\cos^2\theta)\, d\theta}$$

$$= \frac{2a}{3} \cdot \frac{2\int_{0}^{\pi/2} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2\int_{0}^{\pi/2} (1+\cos^2\theta)\, d\theta} = \frac{2a}{3} \frac{\left(3\times\frac{1}{2}\times\frac{\pi}{2} + \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)}{\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2}}$$

$\therefore \bar{x} = 5a/6$

$$\boxed{\bar{x} = 5a/6 \ \& \ \bar{y} = 0}$$

6. c)

We know that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Consider the second term & substitute $x = 1/t$

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^0 \frac{(1/t)^{m-1}}{(1+1/t)^{m+n}} \frac{dt}{-t^2}$$

$$RHS = \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

$= \underline{\text{LHS}}$

Module- 4

Q7. (a) According to Newton's Law of Cooling

$$T = t_2 + (t_1 - t_2)e^{-kt} \quad \text{——— (1)}$$

we have, by data t_1 = Initial temperature = $80^\circ C$

t_2 = Surrounding temperature = $40^\circ C$

$T = 60$, when $t = 20$

Now, $60 = 40 + 40e^{-20k} \Rightarrow 20 = 40e^{-20k}$

$e^{-20k} = 1/2$ ——— (2)

Now, $T = ?$ when $t = 40$

$T = 40 + 40e^{-40k} = 40 + 40\left(\frac{1}{2}\right)^2$

$= 40 + 40\left(\frac{1}{4}\right) = 40 + 10 = 50^\circ C$

(b) Given ~~$r^n = a^n$~~ $r^n \cos n\theta = a^n$

taking 'log' both sides

$n \log r + \log \cos n\theta = n \log a$

diff w.r.t. 'θ'

$\frac{n}{r}\frac{dr}{d\theta} + \frac{(-n \sin n\theta)}{\cos n\theta} = 0$

$\frac{1}{r}\frac{dr}{d\theta} = \tan n\theta$

Replacing $\frac{dr}{d\theta} \rightarrow -r^2\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = \tan n\theta \Rightarrow -\cot n\theta \, d\theta = \frac{dr}{r}$

On integration, we get $\log b - \frac{1}{n}\log \sin n\theta = \log r$ ~~$\log b$~~

$n \log b = n \log r + \log \sin n\theta \Rightarrow \boxed{b^n = r^n \sin n\theta}$

c) Given: $[3x^2y^4 + 2xy]\,dx + [2x^3y^3 - x^2]\,dy = 0$

$M = 3x^2y^4 + 2xy \quad ; \quad N = 2x^3y^3 - x^2$

$\dfrac{\partial M}{\partial y} = 12x^2y^3 + 2x \qquad \dfrac{\partial N}{\partial x} = 6x^2y^3 - 2x$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{6x^2y^3 + 4x}{y(3x^2y^3 + 2x)} = \frac{2}{y} = g(y)$$

$$\text{I.f.} = e^{-\int g(y)\,dy} = e^{-\int \frac{2}{y}\,dy} = e^{-2\log y} = 1/y^2$$

multiplying given eqn by I.f)

$$\left[3x^2y^2 + \frac{2x}{y}\right]dx + \left[2x^3y - \frac{x^2}{y^2}\right]dy = 0$$

Now, the eqn is Exact therefore the solution is given by

$$\int M\,dx + \int (\text{terms of } N \text{ without } x)\,dy = c$$

$$\int \left(3x^2y^2 + \frac{2x}{y}\right)dx + \int 0\,dy = c$$

$$\Rightarrow \frac{3x^3y^2}{3} + \frac{2x^2}{2y} = c \Rightarrow \boxed{x^3y^2 + \frac{x^2}{y} = c}$$

Q8. (a) $L\dfrac{di}{dt} + Ri = 200\sin 300t$, $\quad L = 0.05$, $R = 100$

$$\frac{di}{dt} + \frac{100}{0.05}\,i = \frac{200}{0.05}\sin 300t$$

$$\text{I.f.} = e^{R\int dt} = e^{Rt}$$

Q8.(a) Given $L\frac{di}{dt} + Ri = 200\sin 300t$, $L = 0.05$, $R = 100$

Considering $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\sin\omega t$, $E = 200$, $\omega = 300$

The current at any time 't' is given by

$$i = \frac{EL}{R^2+\omega^2}\left[\frac{R}{L}\sin\omega t - \omega\cos\omega t + \omega e^{-R/L\, t}\right]$$

$$i = \frac{\sin 300t}{5} - \frac{3}{100}\cos 300t + \frac{3}{100}e^{-2000t}$$

When $t \to \infty$. $\boxed{i = \frac{\sin 300t}{5} - \frac{3}{100}\cos 300t}$

Q8. (b) Solve: $[r\sin\theta - r^2]\,d\theta - [\cos\theta]\,dr = 0$

$\cos\theta\frac{dr}{d\theta} - r\sin\theta = -r^2$

$\frac{dr}{d\theta} - r\tan\theta = -r^2\sec\theta$ [Bernoulli's Equation] —— (1)

$r^{-2}\frac{dr}{d\theta} - r^{-1}\tan\theta = -\sec\theta$

Let $r^{-1} = t \Rightarrow -r^{-2}\frac{dr}{d\theta} = \frac{dt}{d\theta}$

$-\frac{dt}{d\theta} - \tan\theta \cdot t = -\sec\theta \Rightarrow \frac{dt}{d\theta} + t\tan\theta = \sec\theta$ [Linear eq^n] —— (2)

I.F. $= e^{+\int\tan\theta\, d\theta} = e^{-\log\cos\theta} = (\cos\theta)^{-1} = \sec\theta$

∴ Sol^n of (2) is

$t\cos^{-1}\theta = \int\sec\theta.\cos^{-1}\theta\, d\theta + C \Rightarrow t\cos^{-1}\theta = \theta + C$

∴ Sol^n of (1) is $\frac{\cos\theta}{r} = \theta + C$

Sol^n of (2), is given by

$$t\sec\theta = \int \sec^2\theta\, d\theta + c \Rightarrow t\sec\theta = \tan\theta + c$$

$\therefore$ Solution of (1), is

$$\frac{\sec\theta}{x} = \tan\theta + c$$

Q8(c) Wrong Question

Module-5.

9. a) $x+y+z=6$

$x+2y+3z=10$

$x+2y+\lambda z=\mu$.

The corresponding augmented matrix is,

$$(A:B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right)$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right)$$

$R_3 \to R_3 - R_2$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right)$$

To have,

(i) no solution., $r \neq r'$.

$\lambda - 3 = 0$ and $\mu - 10 \neq 0$.

$\Rightarrow$ $\lambda = 3$ and $\mu \neq 10$.

28

(ii) a unique solution, $r = r' = n$.

$r = r' = 3$.

$\lambda - 3 \neq 0$

$\Rightarrow \lambda \neq 3$.

(iii) infinite number of solutions, $r = r' < n$.

$r = r' < 3$.

$\lambda - 3 = 0$ and $\mu - 10 = 0$

$\Rightarrow \lambda = 3$ and $\mu = 10$.

b) $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}, \quad X^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$AX^{(0)} = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0.5 \\ -0.5 \end{pmatrix}$$
$$= \lambda^{(1)} X^{(1)}.$$

$$AX^{(1)} = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0.5 \\ -0.5 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -4 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0.8 \\ -0.8 \end{pmatrix} = \lambda^{(2)} X^{(2)}.$$

$$AX^{(2)} = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0.8 \\ -0.8 \end{pmatrix} = \begin{pmatrix} 5.6 \\ 5.21 \\ -5.21 \end{pmatrix} = 5.6\begin{pmatrix} 1 \\ 0.93 \\ -0.93 \end{pmatrix}$$
$$= \lambda^{(3)} X^{(3)}.$$

$$AX^{(3)} = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0.93 \\ -0.93 \end{pmatrix} = \begin{pmatrix} 5.86 \\ 5.74 \\ -5.74 \end{pmatrix} = 5.86\begin{pmatrix} 1 \\ 0.98 \\ -0.98 \end{pmatrix}$$
$$= \lambda^{(4)} X^{(4)}.$$

3

29

$$Ax^{(4)} = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0.98 \\ -0.98 \end{pmatrix} = \begin{pmatrix} 5.96 \\ 5.9 \\ -5.9 \end{pmatrix}$$

$$= 5.96 \begin{pmatrix} 1 \\ 0.99 \\ -0.99 \end{pmatrix}$$

$$= \lambda^{(5)} x^{(5)}.$$

$$Ax^{(5)} = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0.99 \\ -0.99 \end{pmatrix} = \begin{pmatrix} 5.98 \\ 5.98 \\ -5.98 \end{pmatrix} = 5.98 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \lambda^{(6)} x^{(6)}.$$

$$Ax^{(6)} = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ +1 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \lambda^{(7)} x^{(7)}$$

Largest eigenvalue = 6

Corresponding eigenvector = $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

c) $83x + 11y - 4z = 95$

$7x + 52y + 13z = 104$

$3x + 8y + 29z = 71$.

The augmented matrix is,

$$(A:B) = \left(\begin{array}{ccc|c} 83 & 11 & -4 & 95 \\ 7 & 52 & 13 & 104 \\ 3 & 8 & 29 & 71 \end{array}\right).$$

A

$$\sim \left(\begin{array}{ccc|c} 83 & 11 & -4 & 95 \\ 7 & 52 & 13 & 104 \\ 3 & 8 & 29 & 71 \end{array}\right)$$

$$R_1 \to \frac{R_1}{83} \quad \sim \left(\begin{array}{ccc|c} 1 & 11/83 & -4/83 & 95/83 \\ 7 & 52 & 13 & 104 \\ 3 & 8 & 29 & 71 \end{array}\right)$$

$R_2 \to R_2 - 7R_1$

$R_3 \to R_3 - 3R_1$

$$\sim \left(\begin{array}{ccc|c} 1 & 11/83 & -4/83 & 95/83 \\ 0 & 4239/83 & 1107/83 & 7967/83 \\ 0 & 631/83 & 2419/83 & 5608/83 \end{array}\right)$$

$$R_2 \to \frac{83}{4239} R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 11/83 & -4/83 & 95/83 \\ 0 & 1 & 41/157 & 7967/4239 \\ 0 & 631/83 & 2419/83 & 5608/83 \end{array}\right)$$

$$R_3 \to R_3 - \frac{631}{83} R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 11/83 & -4/83 & 95/83 \\ 0 & 1 & 41/157 & 7967/4239 \\ 0 & 0 & 4264/157 & 225845/4239 \end{array}\right).$$

$$R_3 \to \frac{157}{4264} R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 11/83 & -4/83 & 95/83 \\ 0 & 1 & 41/157 & 7967/4239 \\ 0 & 0 & 1 & 225845/115128 \end{array}\right)$$

$$R_2 \to R_2\left(\frac{157}{41}\right) - R_3$$

$$R_1 \to \frac{4}{83} R_3 + R_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 11/83 & 0 & 1.23912 \\ 0 & 157/41 & 0 & 5.23524 \\ 0 & 0 & 1 & 1.9617 \end{array}\right)$$

$$R_2 \to \frac{41}{157} R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 11/83 & 0 & 1.23912 \\ 0 & 1 & 0 & 1.36717 \\ 0 & 0 & 1 & 1.9617 \end{array}\right)$$

$$R_1 \to R_1 - \frac{11}{83} R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1.05793 \\ 0 & 1 & 0 & 1.36717 \\ 0 & 0 & 1 & 1.9617 \end{array}\right)$$

(6) (32)

$\therefore x = 1.06$

$y = 1.367$

$z = 1.962.$

10) a) $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}.$

$R_1 \to \frac{1}{4} R_1 \quad \sim \begin{pmatrix} 1 & 1/4 & -1/4 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}$

$R_2 \to 2R_1 - R_2$
$R_3 \to 2R_1 + R_3 \quad \sim \begin{pmatrix} 1 & 1/4 & -1/4 \\ 0 & -5/2 & 1/2 \\ 0 & 3/2 & 9/2 \end{pmatrix}$

$R_2 \to -\frac{2}{5} R_2 \quad \sim \begin{pmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & -1/5 \\ 0 & 3/2 & 9/2 \end{pmatrix}$

$R_3 \to \frac{3}{2} R_2 - R_3$

$\sim \begin{pmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & -1/5 \\ 0 & 0 & -24/5 \end{pmatrix}$

$\therefore \rho(A) = 3.$

b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 - 4 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$

$\Rightarrow \lambda^2 - 3\lambda + \lambda - 3 = 0$

$\Rightarrow \lambda(\lambda - 3) + 1(\lambda - 3) = 0$

$\Rightarrow \lambda = -1, 3 \rightarrow$ eigenvalues of A.

The matrix equation is, $(A - \lambda I)X = 0$.

$$\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow (1-\lambda)x + 2y = 0$

$2x + (1-\lambda)y = 0$.

Let $\lambda = -1$.

$$\left.\begin{aligned} 2x + 2y = 0 \\ 2x + 2y = 0 \end{aligned}\right\} \quad x + y = 0.$$

Let $y = k_1 \neq 0$

$\Rightarrow x = -k_1$

$\therefore X_1 = \begin{pmatrix} -k_1 \\ k_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, if $k_1 = 1$.

Let $\lambda = 3$,

$$\left.\begin{matrix} -2x + 2y = 0 \\ 2x - 2y = 0 \end{matrix}\right\} \quad x - y = 0$$

$$x = y.$$

Let $y = k_2 \neq 0$

$\Rightarrow x = k_2$.

Pick $k_2 = 1$.

$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\therefore$ The modal matrix is $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.

Thus, $P^{-1} = \dfrac{1}{-1-1}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$.

$$= -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

$$\therefore P^{-1}AP = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 1-2 & 2-1 \\ -1-2 & -2-1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -1 & 1 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 1+1 & -1+1 \\ 3-3 & -3-3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} = \mathrm{Diag}(-1, 3) = D$$

9

35

c) $$\left.\begin{aligned} 2x-3y+20z &= 25 \\ 20x+y-2z &= 17 \\ 3x+20y-z &= -18 \end{aligned}\right\} \text{— (1)}$$

System (1) is not diagonally dominant.

$$\left.\begin{aligned} 20x+y-2z &= 17 \\ 3x+20y-z &= -18 \\ 2x-3y+20z &= 25 \end{aligned}\right\} \text{— (2)}$$

System (2) is diagonally dominant.

$$x = \frac{1}{20}(17-y+2z)$$

$$y = \frac{1}{20}(-18-3x+z)$$

$$z = \frac{1}{20}(25-2x+3y)$$

Let the initial values of (x, y, z) be $(0,0,0)$.

Iteration 1 :-

$$x_1 = \frac{1}{20}(17-0+0) = \frac{17}{20} = 0.85$$

$$y_1 = \frac{1}{20}(-18-3(0.85)) = -1.0275$$

$$z_1 = \frac{1}{20}(25-2(0.85)+3(-1.0275))$$

$$= 1.0109.$$

10

37

Iteration 2 :-

$x_2 = \frac{1}{20}\left[17 + 1.0275 + 2(1.0109)\right] = 1.002465$

$y_2 = \frac{1}{20}\left[-18 - 3(1.002465) + 1.0109\right] = -0.9998$

$z_2 = \frac{1}{20}\left[25 - 2(1.002465) + 3(-0.9998)\right]$

$= 0.9997835.$

Iteration 3 :-

$x_3 = \frac{1}{20}\left[17 + 0.9998 + 2(0.9997835)\right] = 0.99997$

$y_3 = \frac{1}{20}\left[-18 - 3(0.99997) + 0.9998\right] = -1.0000055$

$z_3 = \frac{1}{20}\left[25 - 2(0.99997) + 3(-1)\right] = 1.000003.$

Iteration 4 :-

$x_4 = \frac{1}{20}\left[17 + 1 + 2(1)\right] = 1$

$y_4 = \frac{1}{20}\left[-18 - 3 + 1\right] = -1$

$z_4 = \frac{1}{20}\left[25 - 2 - 3\right] = 1.$

$\therefore x = 1, y = -1, z = 1.$

Q. 1. a)