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Internal Assessment Test II – October 2019

Sub:	Calculus and Linear Algebra					Sub Code:	18MAT11	
Date:	24/10/2019	Duration:	90 mins	Max Marks:	50	Sem / Sec:	I / A,B,C,D,E,F & G	OBE

Question 1 is compulsory and answer any SIX questions from the rest.

		MARKS	CO	RBT
1.	Solve: $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$.	[08]	CO4	L3
2.	Find the evolute of the parabola $y^2 = 4ax$.	[07]	CO1	L3
3.	Obtain the Maclaurin's expansion of $\log(secx + tanx)$ upto the first three non-vanishing terms.	[07]	CO2	L3
4.	Evaluate (i) $\lim_{x\to 0}\left(\frac{1}{x}\right)^{2\sin x}$ (ii) $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x}{3}\right)^{\left(\frac{1}{x}\right)}$.	[07]	CO2	L3

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5.	If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ then prove that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$.	[07]	CO2	L3
6.	Find the extreme values of $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.	[07]	CO2	L3
7.	Show that the rectangular box of maximum volume and a given surface area is a cube.	[07]	CO2	L3
8.	If $u = \frac{yz}{x},\ v = \frac{zx}{y},\ w = \frac{xy}{z}$, then show that $J\left(\frac{u,v,w}{x,y,z}\right) = 4$.	[07]	CO2	L3

Calculus And Linear Algebra – 18MAT11
Solutions to IAT-2 October 2019.

1. Let $M(x,y) = xy^3 + y = y(xy^2+1)$.
$N(x,y) = 2(x^2y^2 + x + y^4)$.

$$\frac{\partial M}{\partial y} = 3xy^2+1 \quad , \quad \frac{\partial N}{\partial x} = 2(2xy^2+1)$$
$$= 4xy^2+2$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. ∴ The given equation is not exact.

Now, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3xy^2+1-4xy^2-2$
$= -xy^2-1$.
$= -(xy^2+1)$ .... close to M.

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y(xy^2+1)} \times -(xy^2+1)$$
$$= -\frac{1}{y} = g(y).$$

$$\therefore I.F = e^{-\int g(y)dy} = e^{-\int\left(-\frac{1}{y}\right)dy} = e^{\log y} = y.$$

Multiplying the given eq^n with y, we get.
$$(xy^4+y^2)\,dx + 2(x^2y^3+xy+y^5)\,dy = 0. \quad (1)$$

Now $M = xy^4+y^2 \Rightarrow \frac{\partial M}{\partial y} = 4xy^3+2y$

$N = 2(x^2y^3+xy+y^5) \Rightarrow \frac{\partial N}{\partial x} = 2(2xy^3+y)$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$\therefore$ Eqn (1) is exact.

Thus, the solution is

$$\int M\,dx + \int N(y)\,dy = c.$$

$$\Rightarrow \int (xy^4 + y^2)\,dx + 2\int y^5\,dy = c.$$

$$\Rightarrow \frac{1}{2}x^2y^4 + xy^2 + \frac{2y^6}{6} = c$$

$$\Rightarrow \frac{1}{2}x^2y^4 + xy^2 + \frac{y^6}{3} = c$$ is the general Solution.

2. The parametric equations of the parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$.

$$\therefore \frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a.$$

$$\therefore y_1 = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}.$$

$$y_2 = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{1}{t}\right)\frac{1}{2at}$$

$$= -\frac{1}{t^2}\cdot\frac{1}{2at} = \frac{-1}{2at^3}.$$

Then the co-ordinates of centre of curvature $(\bar{x}, \bar{y})$ at 't' are

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{\frac{-1}{2at^3}}$$

$$= at^2 + 2at^3\left(\frac{1}{t} + \frac{1}{t^3}\right)$$

$$= at^2 + 2at^2 + 2a.$$

$$= 3at^2 + 2a. \quad \text{—— (1)}$$

and $\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 2at + \frac{\left(1+\frac{1}{t^2}\right)}{\frac{-1}{2at^3}}$

$$= 2at - 2at^3\left(1+\frac{1}{t^2}\right)$$

$$= 2at - 2at^3 - 2at$$

$$= -2at^3 \quad \text{—— (2)}$$

$$\Rightarrow t = \left(-\frac{\bar{y}}{2a}\right)^{1/3}$$

Now (1) gives $\bar{x} = 3a\left(-\frac{\bar{y}}{2a}\right)^{2/3} + 2a$

$$\Rightarrow (\bar{x} - 2a) = 3a\left(\frac{\bar{y}}{2a}\right)^{2/3}$$

$$\Rightarrow (\bar{x} - 2a)^3 = 27a^3\left(\frac{\bar{y}}{2a}\right)^2$$

$$\Rightarrow 4a^2(\bar{x} - 2a)^3 = 27a^3\bar{y}^2.$$

$\therefore$ The evolute is $27a^3y^2 = 4a^2(x-2a)^3$.

$$\Rightarrow 27ay^2 = 4(x-2a)^3.$$

3. $y = \log(\sec x + \tan x)$ $\qquad y(0) = \log(\sec 0 + \tan 0)$

$= 0.$

$$y_1 = \frac{1}{\sec x + \tan x} \times (\sec x . \tan x + \sec^2 x)$$

$$= \frac{\sec x(\tan x + \sec x)}{(\sec x + \tan x)}$$

$= \sec x.$ $\qquad y_1(0) = \sec 0 = 1.$

$y_2 = \sec x . \tan x.$

$= y_1 \tan x$ $\qquad y_2(0) = (1)(0)$

$= 0.$

$y_3 = y_1 \sec^2 x + y_2 \tan x$ $\qquad y_3(0) = 1(1) + 0 = 1$

$= y_1^3 + y_2 \tan x.$

$y_4 = 3y_1^2 y_2 + y_2 \sec^2 x + y_3 \tan x$

$y_4(0) = 0+0+0 = 0$

$= 3y_1^2 y_2 + y_2 y_1^2 + y_3 \tan x.$

$= 4y_1^2 y_2 + y_3 \tan x.$

$y_5 = 8y_1 y_2^2 + 4y_1^2 y_3 + y_3 \sec^2 x + y_4 \tan x$

$y_5(0) = 0+4+1+0 = 5$

Maclaurin's expansion is given by,

$$y = y(0) + x\, y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots$$

$$= 0 + x(1) + 0 + \frac{x^3}{6}(1) + 0 + \frac{x^5}{120}(5)$$

$$= x + \frac{x^3}{6} + \frac{x^5}{24}$$

4. (i) Let $k = \lim_{x \to 0} \left(\frac{1}{x}\right)^{2\sin x}$. (∞^0 form)

Applying log both sides,

$$\log k = \lim_{x \to 0} (2\sin x) \log\left(\frac{1}{x}\right).$$

($0 \times \infty$ form)

$$= 2\lim_{x \to 0} \frac{\log\left(\frac{1}{x}\right)}{\operatorname{cosec} x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

Applying L'Hospital's rule,

$$\log k = 2 \lim_{x \to 0} \frac{x \times \left(-\frac{1}{x^2}\right)}{-\operatorname{cosec} x \times \cot x}$$

$$= 2 \lim_{x \to 0} \frac{\sin x}{x} \times \frac{\tan x}{x} \times x$$

$$= 2 \times 1 \times 1 \times 0 = 0.$$

$$\therefore k = e^0 = 1.$$

(ii) Let $k = \lim_{x \to 0} \left(\frac{1^x + 2^x + 3^x}{3} \right)^{1/x}$ (1^∞ form) ⑥

Applying log both sides,

$$\log k = \lim_{x \to 0} \frac{1}{x} \log\left(\frac{1^x + 2^x + 3^x}{3} \right)$$ ($\infty \times 0$ form)

$$= \lim_{x \to 0} \frac{\log\left(\frac{1^x + 2^x + 3^x}{3} \right)}{x}$$ ($\frac{0}{0}$ form)

Applying L' Hospital's rule,

$$\log k = \lim_{x \to 0} \frac{\frac{3}{1^x + 2^x + 3^x} \times \frac{1}{3} \times (1^x \log 1 + 2^x \log 2 + 3^x \log 3)}{1}$$

$$= \frac{3}{3} \times \frac{1}{3} (\log 1 + \log 2 + \log 3)$$

$$= \frac{1}{3} \log 6 = \log 6^{1/3}$$

$\therefore k = 6^{1/3}$

5. Let $2x - 3y = P$

$3y - 4z = Q$

$4z - 2x = R$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial P} \frac{\partial P}{\partial x} + \frac{\partial u}{\partial R} \cdot \frac{\partial R}{\partial x}$$

$$= \frac{\partial u}{\partial P} (2) + \frac{\partial u}{\partial R} (-2)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial P}\frac{\partial P}{\partial y} + \frac{\partial u}{\partial Q}\frac{\partial Q}{\partial y}$$

$$= \frac{\partial u}{\partial P}(-3) + \frac{\partial u}{\partial Q}(3)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial Q}\frac{\partial Q}{\partial z} + \frac{\partial u}{\partial R}\frac{\partial R}{\partial z}$$

$$= \frac{\partial u}{\partial Q}(-4) + \frac{\partial u}{\partial R}(4)$$

Now, $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z}$

$$= 6\left[2\frac{\partial u}{\partial P} - 2\frac{\partial u}{\partial R}\right] + 4\left[-3\frac{\partial u}{\partial P} + 3\frac{\partial u}{\partial Q}\right] + 3\left[-4\frac{\partial u}{\partial Q} + 4\frac{\partial u}{\partial R}\right]$$

$$= 12\frac{\partial u}{\partial P} - 12\frac{\partial u}{\partial R} - 12\frac{\partial u}{\partial P} + 12\frac{\partial u}{\partial Q} - 12\frac{\partial u}{\partial Q} + 12\frac{\partial u}{\partial R}$$

$$= 0.$$

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6. $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$

$f_x = 3x^2 + 3y^2 - 6x$

$f_y = 6xy - 6y$

To find critical points, $3x^2 + 3y^2 - 6x = 0$

and $6xy - 6y = 0$

$\Rightarrow 6y(x-1) = 0.$

$\Rightarrow y = 0$ (or) $x = 1$.

When $y = 0$,

$3x^2 + 3(0) - 6x = 0$

$\Rightarrow 3x(x-2) = 0$

$\Rightarrow x = 0$ (or) $x = 2$.

When $x = 1$.

$3(1) + 3y^2 - 6(1) = 0$

$\Rightarrow 3y^2 - 3 = 0$

$\Rightarrow 3(y+1)(y-1) = 0$

$\Rightarrow y = -1$ (or) $y = 1$.

$\therefore$ The critical points are

$(0,0), (2,0), (1,-1), (1,1)$.

Now, $r = f_{xx} = 6x - 6$

$s = f_{xy} = 6y$

$t = f_{yy} = 6x - 6$.

Case (i):- $(0,0)$

$r = -6, \ s = 0, \ t = -6$

$\Rightarrow rt - s^2 = (-6)(-6) - 0 = 36$.

$r < 0$ and $rt - s^2 > 0$.

$\therefore (0,0)$ is a point of maximum.

The maximum value is $f(0,0) = 4$.

Case (ii):- $(2,0)$

$r = 6, \ s = 0, \ t = 6$.

$\Rightarrow rt - s^2 = (6)(6) - 0 = 36$.

$r > 0$ and $rt - s^2 > 0$.

$\therefore (2,0)$ is a point of minimum.

The minimum value is $f(2,0) = 0$.

Case(iii) :- $(1,-1)$

$r = 0,\ s = -6,\ t = 0.$

$rt - s^2 = 0 - (-6)^2 = -36$

$r = 0$ and $rt - s^2 < 0.$

$\therefore$ $(1,-1)$ is a point of neither maximum nor minimum.

Thus $(1,-1)$ is a saddle point.

Case(iv) :- $(1,1)$

$r = 0,\ s = 6,\ t = 0$

$rt - s^2 = -36 < 0$

$\therefore f(x,y)$ attains neither maximum nor minimum at $(1,1)$. $(1,1)$ is a saddle point.

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7. Let x, y, z be the length, breadth and height of the rectangular box.

Volume, $V = xyz.$

Surface area $= 2xy + 2yz + 2zx = k$ (Say).

Here, $f(x,y,z) = xyz$

$\phi(x,y,z) = 2xy + 2yz + 2zx - k.$

Auxiliary equation is

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$= xyz + \lambda(2xy + 2yz + 2zx - k)$$

$F_x = yz + \lambda(2y + 2z)$

$F_y = xz + \lambda(2x + 2z)$, $F_z = xy + \lambda(2y + 2x)$

Now, solving $F_x = 0,\ F_y = 0,\ F_z = 0$.

$$yz + 2\lambda(y+z) = 0 \Rightarrow \lambda = -\frac{yz}{2(y+z)}$$

$$xz + 2\lambda(x+z) = 0 \Rightarrow \lambda = \frac{-xz}{2(x+z)}$$

$$xy + 2\lambda(x+y) = 0 \Rightarrow \lambda = -\frac{xy}{2(x+y)}.$$

$$\Rightarrow -\frac{yz}{2(y+z)} = -\frac{xz}{2(x+z)} = -\frac{xy}{2(x+y)}.$$

Consider

$$\frac{\cancel{+}y\cancel{z}}{\cancel{2}(y+z)} = \frac{\cancel{+}x\cancel{z}}{\cancel{2}(x+z)} \qquad \frac{\cancel{+}\cancel{y}z}{\cancel{2}(y+z)} = \frac{\cancel{+}x\cancel{y}}{\cancel{2}(x+y)}$$

$$\Rightarrow \cancel{xy} + zy = \cancel{xy} + xz \qquad \Rightarrow \cancel{xz} + yz = xy + \cancel{xz}$$

$$\Rightarrow y = x \qquad \Rightarrow z = x.$$

$$x = y = z.$$

$\therefore$ The rectangular box is a cube.

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8. $u = \frac{yz}{x}, \quad v = \frac{zx}{y}, \quad w = \frac{xy}{z}.$

$$\frac{\partial u}{\partial x} = -\frac{yz}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{z}{x}, \quad \frac{\partial u}{\partial z} = \frac{y}{x}$$

$$\frac{\partial v}{\partial x} = \frac{z}{y}, \quad \frac{\partial v}{\partial y} = -\frac{zx}{y^2}, \quad \frac{\partial v}{\partial z} = \frac{x}{y}.$$

$$\frac{\partial w}{\partial x} = \frac{y}{z}, \quad \frac{\partial w}{\partial y} = \frac{x}{z}, \quad \frac{\partial w}{\partial z} = -\frac{xy}{z^2}.$$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left[\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right]$$

$$- \frac{z}{x}\left[-\frac{xyz}{yz^2} - \frac{xy}{yz}\right]$$

$$+ \frac{y}{x}\left[\frac{xz}{yz} + \frac{xyz}{y^2z}\right]$$

$$= 0 - \frac{z}{x}\left[-\frac{x}{z} - \frac{x}{z}\right]$$

$$+ \frac{y}{x}\left[\frac{x}{y} + \frac{x}{y}\right]$$

$$= 1 + 1 + 1 + 1$$

$$= 4.$$