IAT-3: Solution

DMS - 18CS36

1. (i) ~~Method of~~ Proof by Contradiction

(ii) Direct Method

(iii) Indirect Method

2. S = {62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 79}

62 = 31+31

63 = 61 + 2

64 = 59+5

65 =

67 =

68 =

69 =

70 =

71 =

65 can not be written as a sum of two primes.

So, the answer is <u>No.</u>

3. (i) False

(ii) False

4)

Example 10 *In how many ways can one distribute eight identical balls into four distinct containers so that* (i) *no container is left empty?* (ii) *the fourth container gets an odd number of balls?*

▶ (i) First we distribute one ball into each container. Then we distribute the remaining 4 balls into 4 containers. The number of ways of doing this is the required number. This number is

$$C(4+4-1,4) = C(7,4) = \frac{7!}{4!\,3!} = 35.$$

(ii) If the fourth container has to get an odd number of balls, we have to put one or three or five or seven balls into it.

Suppose we put one ball into it (the fourth container). Then the remaining 7 balls can be distributed into the remaining three containers in

$$C(3+7-1,7) = C(9,7) \quad \text{ways.}$$

Similarly, putting 3 balls into the fourth container and the remaining 5 into the remaining 3 containers can be done in

$$C(3+5-1,5) = C(7,5) \quad \text{ways}$$

Next, putting 5 balls into the fourth container and the remaining 3 into the remaining 3 containers can be done in

$$C(3+3-1,3) = C(5,3) \quad \text{ways}$$

Lastly, putting 7 balls into the fourth container and the remaining 1 into the remaining 3 container can be done in

$$C(3+1-1,1) = C(3,1) = 3 \quad \text{ways.}$$

Thus, the total number ways of distributing the given balls so that the fourth container gets an odd number of balls is

$$C(9,7) + C(7,5) + C(5,3) + 3 = \frac{9!}{7!\,2!} + \frac{7!}{5!\,2!} + \frac{5!}{3!\,2!} + 3$$
$$= 36 + 21 + 10 + 3 = 70$$ ∎

5

Example 11 *How many positive integers n can we form using the digits* 3, 4, 4, 5, 5, 6, 7 *if we want n to exceed* 5, 000, 000*?*

► Here n must be of the form

$$n = x_1x_2x_3x_4x_5x_6x_7$$

where $x_1, x_2, \ldots, x_7$ are the given digits with $x_1 = 5, 6$ or 7. Suppose we take $x_1 = 5$. Then $x_2x_3x_4x_5x_6x_7$ is an arrangement of the remaining 6 digits which contains two 4's and one each of 3, 5, 6, 7. The number of such arrangements is

$$\frac{6!}{2!\,1!\,1!\,1!\,1!} = 360.$$

Next, suppose we take $x_1 = 6$. Then, $x_2x_3x_4x_5x_6x_7$ is an arrangement of 6 digits which contains two each of 4 and 5 and one each of 3 and 7. The number of such arrangements is

$$\frac{6!}{1!\,2!\,2!\,1!} = 180.$$

Similarly, if we take $x_1 = 7$, the number of arrangements is

$$\frac{6!}{1!\,2!\,2!\,1!} = 180.$$

Accordingly, by the Sum Rule, the number of n's of the desired type is

$$360 + 180 + 180 = 720.$$ ■

6)

(iii) The general term in the expansion of $(2x^3 - 3xy^2 + z^2)^6$ is

$$\binom{6}{n_1,\ n_2,\ n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}$$

For $n_3 = 0$, $n_2 = 2$, $n_1 = 3$ this becomes

$$\binom{6}{3,\ 2,\ 0}(2^3x^9)(3^2x^2y^4) = 72 \times \frac{6!}{3!\,2!\,0!}x^{11}y^4.$$

Thus, the required coefficient is

$$72 \times \frac{6 \times 5 \times 4}{2} = 4320.$$

Example *A sequence* $\{a_n\}$ *is defined recursively by*

$$a_1 = 4, \; a_n = a_{n-1} + n \quad \textit{for} \quad n \geq 2.$$

Find a_n *in explicit form.*

Using the given recursive formula repeatedly, we find that

$$\begin{aligned}
a_n &= a_{n-1} + n \\
&= [a_{n-2} + (n-1)] + n \\
&= a_{n-3} + (n-2) + (n-1) + n \\
&= a_{n-4} + (n-3) + (n-2) + (n-1) + n \\
&\quad \cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots \\
&= a_1 + 2 + 3 + 4 + \cdots + n \\
&= (a_1 - 1) + (1 + 2 + 3 + \cdots + n)
\end{aligned}$$

Using $a_1 = 4$ and the standard result

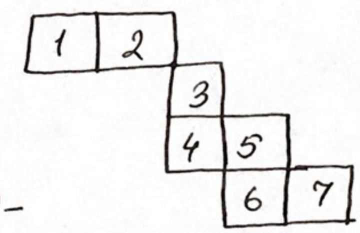
$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1),$$

this becomes

$$a_n = 3 + \frac{1}{2}n(n+1).$$

This is the explicit formula for a_n.* ■

8. $r_1 = n = 7$



$r_2 = 16$

{Non-capturing positions when 2 rooks are placed on board-

(1,3), (1,4), (1,5), (1,6), (1,7), (2,3), (2,4), (2,5), (2,6), (2,7)
(3,5), (3,6), (3,7), (4,6), (4,7), (5,7)}

$r_3 = 13$ (Non-capturing positions when 3 rooks are placed on board)

(1, 3, 5), (1, 3, 6), (1, 3, 7), (2, 3, 5), (2, 3, 6) (2, 3, 7)

(3, 5, 7) (1, 4, 6), (1, 4, 7), (1, 5, 7) (2, 4, 6), (2, 4, 7), (2, 5, 7)

$r_4 = 2$ (when 4 rooks are placed -

(1, 3, 5, 7), (2, 3, 5, 7)

$\therefore$ The rook polynomial is -

$$r(C, x) = 1 + 7x + 16x^2 + 13x^3 + 2x^4$$

9) Replace n by 15 in the following solution

4)

Example *There are n pairs of children's gloves in a box. Each pair is of a different colour. Suppose the right gloves are distributed at random to n children, and thereafter the left gloves are also distributed to them at random. Find the probability that* (i) *no child gets a matching pair,* (ii) *every child gets a matching pair,* (iii) *exactly one child gets a matching pair, and* (iv) *at least 2 children get matching pairs.*

▶ Any one distribution of n right gloves to n children determines a set of n places for the n pairs of gloves. Let us take these as the natural places for the pairs of gloves. The left gloves can be distributed to n children in $n!$ ways.

(i) The event of no child getting a matching pair occurs if the distribution of the left gloves is a derangement. The number of derangements is d_n. Therefore, the required probability, in this case, is

$$p_1 = \frac{d_n}{n!} = \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

(ii) The event of every child getting a matching pair occurs in only one distribution of the left gloves. Therefore, the required probability, in this case, is $p_2 = \dfrac{1}{n!}$.

(iii) The event of exactly one child getting a matching pair occurs when only one left glove is in the natural place, and all others are in wrong places. The number of such distributions is d_{n-1}. The required probability, in this case, is

$$p_3 = \frac{d_{n-1}}{n!} = \frac{1}{n}\left\{1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right\}.$$

10) (i) Number of ways of selecting 7 questions out of 10 $= {}^{10}C_7$

$= 120$

(ii) Number of ways of selecting three from first five questions $=$

${}^{5}C_3$

Number of ways of selecting 4 questions from last five $=$

${}^{5}C_4$

$\therefore$ Answer in this particular case $= {}^{5}C_3 \times {}^{5}C_4 = 50$

11) WKT $(x+y)^n = \sum_{r=0}^{n} {}^{n}C_r x^r y^{n-r}$

$$(2x-3y)^7 = \sum_{r=0}^{7} {}^{7}C_r (2x)^r (-3y)^{7-r}$$

$$= \sum_{r=0}^{7} {}^{7}C_r 2^r (-3)^{7-r} x^r y^{7-r}$$

$x^5 y^2$ corresponds to $r=5$ in the above term.

$\therefore$ the coefficient of x^5y^2 is ${}^{7}C_5 2^5 (-3)^2 = 6048$

12)

(vi) Here, $a_1 = 3$, $a_2 = 1$, $a_3 = 3, \ldots\ a_n = 2 - (-1)^n$ and $a_{n+1} = 2 - (-1)^{n+1}$. These give

$$a_{n+1} - a_n = -(-1)^{n+1} + (-1)^n = (-1)^n \left\{1 + (-1)^2\right\} = 2(-1)^n$$

or

$$a_{n+1} = a_n + 2(-1)^n \quad \text{for} \quad n \geq 1.$$

Thus, $a_1 = 3$ and $a_{n+1} = a_n + 2(-1)^n$ for $n \geq 1$ ■

is a recursive definition of the given sequence.

(13.) No. of rows = 4

No. of columns = 5

The min no. is 4. $\therefore$ The degree of the rook poly. is 4.

(14.) $d_4 = 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$

(15.) $x_1 + x_2 + x_3 + x_4 = 7$

Here $n = 4$, $r = 7$

The no. of non negative integers solution is

$$C(n+r-1, r) = C(10, 7) = 120$$