USN :

CMR Institute of Technology, Bangalore
DEPARTMENT OF ELECTRICAL & ELECTRONICS ENGINEERING
I - INTERNAL ASSESSMENT

Semester: 4-CBCS 2018
Subject: ELECTROMAGNETIC FIELD THEORY (18EE45)
Faculty: Mr Hemachandra G

Date: 21 May 2021
Time: 09:00 AM - 10:30 AM
Max Marks: 50

Instructions to Students :

Answer Any FIVE FULL Questions. Read the questions carefully and write the answer based on the marks allotted. All the Best

Answer any 5 question(s)

Q.No			Marks	CO	PO	BT/CL
1	a	Given vectors, $\vec{A} = 2\,\hat{a}_x + 3\,\hat{a}_y + 4\,\hat{a}_z$ and $\vec{B} = 5\,\hat{a}_x - 3\,\hat{a}_y - 2\,\hat{a}_z$ (a) Find the angle between $\vec{A}$ and $\vec{B}$ using Scalar product (b) $\vec{A} \times \vec{B}$.	6	CO1	PO1	L2
	b	Define the following (i) Scalar quantity (ii) Vector quantity (iii) Dot product (iv) Cross Product	4	CO1	PO1	L1
2	a	Show that the following vector fields are perpenducular each other. $\vec{A} = \rho\,\sin\phi\,\hat{a}_\rho + \rho\,\cos\phi\,\hat{a}_\phi + \rho\,\hat{a}_z$ $\vec{B} = \rho\,\sin\phi\,\hat{a}_\rho + \rho\,\cos\phi\,\hat{a}_\phi - \rho\,\hat{a}_z$	4	CO1	PO1,PO2	L3
	b	Write the differential lengths and Diffrential normal surface area for cartesian, cylindrical and spherical coordinate systems. **(Diagrams are not required)**	6	CO1	PO1	L1
3	a	Derive the relation between cartesian and cylindrical coordinate system. Convert the point $P(2, 6, 10)$ cartesian to cylindrical coordinate system.	5	CO1	PO1,PO2	L2
	b	Derive the relation between cartesian and spherical coordinate system. Convert the point $P(2, 60^o, 30^o)$ spherical to cartesian coordinate system.	5	CO1	PO1,PO2	L2
4	a	Write the Expression for Gradient of a scalar field in different coordinate systems.	3	CO1	PO1	L1
	b	Given $\vec{B} = y\,\hat{a}_x - x\,\hat{a}_y$, Express $\vec{B}$ in Cylindrical coordinates using phasor diagram	7	CO1	PO1,PO2	L3
5	a	Write the Expression for Divergence of a vector field in different coordinate systems.	3	CO1	PO1	L1
	b	Express the following vector in cartesian coordinate systems$\vec{H} = \rho^2\,\cos\phi\,\hat{a}_\rho - \rho\,\sin\phi\,\hat{a}_\phi$	7	CO1	PO1,PO2	L3
6		Express the following vector in cartesian coordinates using phasor diagram $\vec{D} = r\,\sin\phi\,\hat{a}_r - \frac{1}{r}\sin\theta\,\cos\phi\,\hat{a}_\theta + r^2\,\hat{a}_\phi$	10	CO1	PO1,PO2	L3
7		Express the following vector in spherical coordinate system using phasor diagram $\vec{G} = z\,\hat{a}_x + x\,\hat{a}_y - y\,\hat{a}_z$	10	CO1	PO1,PO2	L3
8	a	Find the gradient of the following scalar field $(i)\, U = \frac{4}{r}\sin\theta\cos\phi$	3	CO1	PO1,PO2	L3
	b	Evaluate the divergence of the following vector fields $(i)\, \vec{A} = 3\rho\sin\phi\,\hat{a}_\rho - 5\rho^2 z\,\hat{a}_\phi + 8z\cos^2\phi\,\hat{a}_z$ $(ii)\, \vec{B} = r^2\,\cos\phi\,\hat{a}_r + 2r\,\hat{a}_\phi$	7	CO1	PO1,PO2	L3

Electromagnetic field theory

IAT - 1

(1) (a) Given vectors $\vec{A} = 2\hat{a}_x + 3\hat{a}_y + 4\hat{a}_z$

$\vec{B} = 5\hat{a}_x - 3\hat{a}_y - 2\hat{a}_z$

(a) Determine the angle between $\vec{A}$ and $\vec{B}$ using Scalar product.

Sol:-

$$\vec{A} = 2\hat{a}_x + 3\hat{a}_y + 4\hat{a}_z$$

$$\vec{B} = 5\hat{a}_x - 3\hat{a}_y - 2\hat{a}_z$$

$$\boxed{\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1}$$

$$\vec{A} \cdot \vec{B} = (2\hat{a}_x + 3\hat{a}_y + 4\hat{a}_z) \cdot (5\hat{a}_x - 3\hat{a}_y - 2\hat{a}_z)$$

$$= (2)(5) + (3)(-3) + (4)(-2)$$

$$= 10 - 9 - 8 = -7$$ - 1 Mark

$$|\vec{A}| = \sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{4+9+16} = \sqrt{29}$$

$$|\vec{B}| = \sqrt{(5)^2 + (-3)^2 + (-2)^2} = \sqrt{25+9+4} = \sqrt{38}$$ - 1 Mark

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{-7}{\sqrt{29}\sqrt{38}}$$

$$\theta = \cos^{-1}\left(\frac{-7}{\sqrt{29}\sqrt{38}}\right) = 102.1^\circ$$ 2 Marks

(b) $\vec{A} \times \vec{B}$

$$\vec{A} = 2\hat{a}_x + 3\hat{a}_y + 4\hat{a}_z$$

$$\vec{B} = 5\hat{a}_x - 3\hat{a}_y - 2\hat{a}_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & 3 & 4 \\ 5 & -3 & -2 \end{vmatrix}$$ – 1 Mark.

$$= \hat{a}_x(-6+12) + \hat{a}_y(20+4) + \hat{a}_z(-6-15)$$

$$= 6\hat{a}_x + 24\hat{a}_y - 21\hat{a}_z.$$ – 1 Mark

6 Marks

① ⓑ Define the following

(i) Scalar quantity :-

A physical quantity that can be completely described by its magnitude is called 'scalar quantity'.

Example: mass, time, temperature, work .... etc.

– 1 Mark

(ii) Vector quantity :-

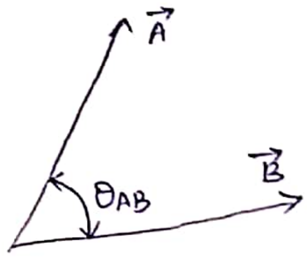
A physical quantity that can be described by its magnitude and as well as direction is called vector quantity.

Example:- Force, velocity, ... etc.

– 1 Mark

1 (b) (iii) Dot Product :-

The Scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A} \cdot \vec{B}$



$$\vec{A} \cdot \vec{B} = AB \cos\theta_{AB}$$

where, $A = |\vec{A}|$, $B = |\vec{B}|$.

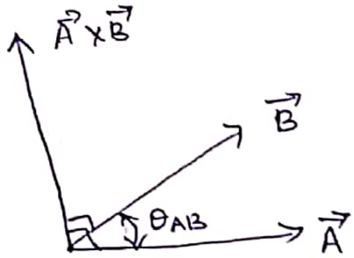
θ_{AB} = Smallest angle between $\vec{A}$ and $\vec{B}$.

– 1 Mark

(iv) Cross Product :-

The cross product between two vectors $\vec{A}$ and $\vec{B}$ written as, $\vec{A} \times \vec{B}$.

$$\vec{A} \times \vec{B} = |\vec{A}| \, |\vec{B}| \sin\theta_{AB} \, \hat{a}_n$$



where

$\hat{a}_n$ is the unit normal to the plane containing $\vec{A}$ and $\vec{B}$.

– 1 Mark

4 Marks

(2) Show that the following vector fields are perpendicular to each other

$$\vec{A} = \rho \sin\phi \, \hat{a}_\rho + \rho \cos\phi \, \hat{a}_\phi + \rho \, \hat{a}_z$$

$$\vec{B} = \rho \sin\phi \, \hat{a}_\rho + \rho \cos\phi \, \hat{a}_\phi - \rho \, \hat{a}_z$$

$$\vec{A} \cdot \vec{B} = |\vec{A}| \, |\vec{B}| \cos\theta$$

$$\vec{A} \cdot \vec{B} = (\rho \sin\phi)(\rho \sin\phi) + (\rho \cos\phi)(\rho \cos\phi) + (\rho)(-\rho)$$

$$= \rho^2 \sin^2\phi + \rho^2 \cos^2\phi - \rho^2$$

$$\vec{A}\cdot\vec{B} = \rho^2(\sin^2\phi + \cos^2\phi) - \rho^2$$

$$= \rho^2 - \rho^2$$

$$= 0.$$

– 2 Marks

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|A|\,|B|}$$

$$\cos\theta = 0.$$

$$\theta = 90^\circ$$

– 1 Mark

$\vec{A}\cdot\vec{B} = 0.$

If the dot product is zero, then
The above two vectors fields are $\perp^r$ to each other.

– 1 Mark

4 Marks

② ⓑ Cartesian Coordinate Systems

Differential length.

$$\vec{dl} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$ – 1 Mark

Differential normal Surface Area,

$$\vec{ds} = dx\,dy\,\hat{a}_z$$
$$= dy\,dz\,\hat{a}_x$$
$$= dz\,dx\,\hat{a}_y$$ – 1 Mark

Cylindrical Coordinate Systems

Differential length

$$\vec{dl} = d\rho\,\hat{a}_\rho + \rho\,d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$$ – 1 Mark

Differential normal Surface Area

$$\vec{ds} = \rho\,d\rho\,d\phi\,\hat{a}_z$$
$$= d\rho\,dz\,\hat{a}_\phi$$
$$= \rho\,d\phi\,dz\,\hat{a}_\rho$$ – 1 Mark

Spherical Coordinate Systems

Differential length

$$\vec{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$$ — 1 Mark

Differential normal Surface Area

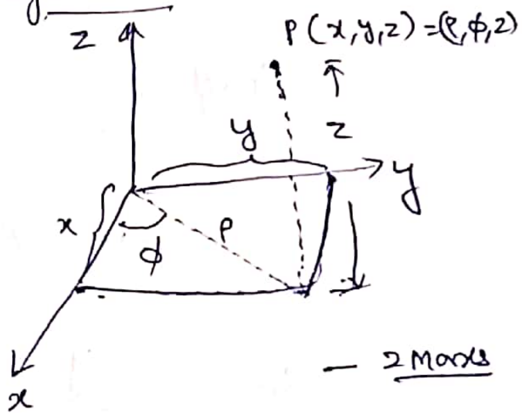
$$\vec{ds} = r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r$$

$$= r\sin\theta\,dr\,d\phi\,\hat{a}_\theta$$

$$= r\,dr\,d\theta\,\hat{a}_\phi$$ — 1 Mark

6 Marks

3 a) Relation between cartesian and cylindrical coordinate systems



$\cos\phi = \dfrac{x}{\rho}$ | $\sin\phi = \dfrac{y}{\rho}$ | $z = z$

$x = \rho\cos\phi$ | $y = \rho\sin\phi$

— 2 Marks

$$x^2+y^2 = \rho^2\cos^2\phi + \rho^2\sin^2\phi$$

$$= \rho^2(\sin^2\phi + \cos^2\phi)$$

$$x^2+y^2 = \rho^2$$

$$\rho = \sqrt{x^2+y^2}$$

$$\frac{y}{x} = \frac{\rho\sin\phi}{\rho\cos\phi}$$

$$\tan\phi = \frac{y}{x}$$

$$\phi = \tan^{-1}(y/x)$$

$$z = z.$$

— 2 Marks

Convert (2,6,10) to Cylindrical Coordinate System.

$x = 2, \quad y = 6, \quad z = 10$

$\rho = \sqrt{x^2+y^2} = \sqrt{(2)^2+(6)^2} = \sqrt{4+36} = \sqrt{40}$

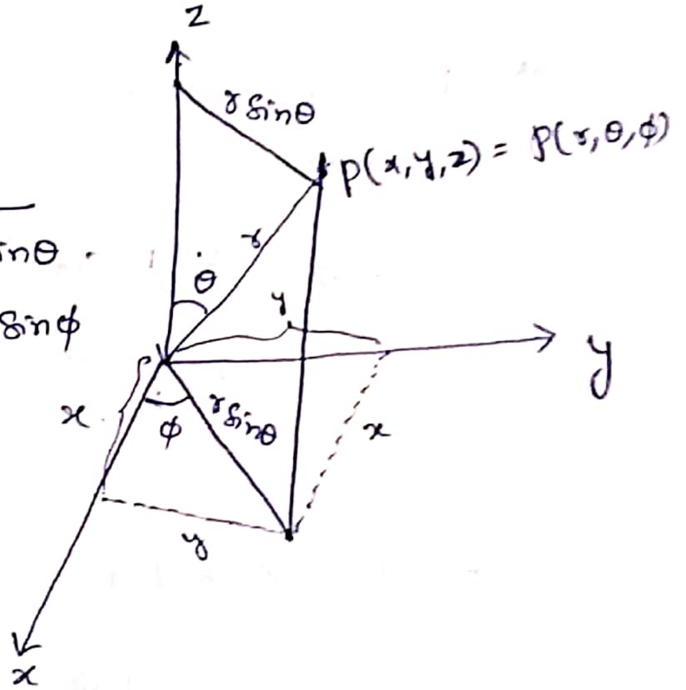
$\phi = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{6}{2}\right) = \tan^{-1}(3) = 71.56°.$

$z = 10.$

— mark

5 marks

③ ⓑ Relation between Cartesian and spherical Coordinate Systems.



$\cos\phi = \dfrac{x}{r\sin\theta}$ | $\sin\phi = \dfrac{y}{r\sin\theta}.$

$x = r\sin\theta\cos\phi$ | $y = r\sin\theta\sin\phi$

$\cancel{\sin\phi} = \cancel{\dfrac{y}{r\sin\theta}}$

$\cos\theta = \dfrac{z}{r}$

$z = r\cos\theta.$

— 2 marks

$$x^2+y^2+z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$$

$$= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta$$

$$= r^2\sin^2\theta + r^2\cos^2\theta = r^2$$

$r = \sqrt{x^2+y^2+z^2}$,

$\frac{y}{x} = \frac{r \sin\theta \sin\phi}{r \sin\theta \cos\phi}$

$\tan\phi = y/x$

$\phi = \tan^{-1}(y/x)$

$z = r\cos\theta$

$\cos\theta = \frac{z}{r}$

$\cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

— 2 Marks

Given point $(2, 60^\circ, 30^\circ)$

$r = 2, \ \theta = 60^\circ, \ \phi = 30^\circ$

$x = r\sin\theta\cos\phi = 2\sin 60^\circ \cos 30^\circ = 1.5$

$y = r\sin\theta\sin\phi = 2\sin 60^\circ \sin 30^\circ = 0.866$

$z = r\cos\theta = 2\cos 60^\circ = 1.$

$P(2, 60^\circ, 30^\circ) = P(1.5, 0.866, 1).$

— 1 Mark

5 Marks

4 (a) Gradient of Scalar field

Cartesian Coordinate System

$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$ — 1 mark

cylindrical Coordinate System,

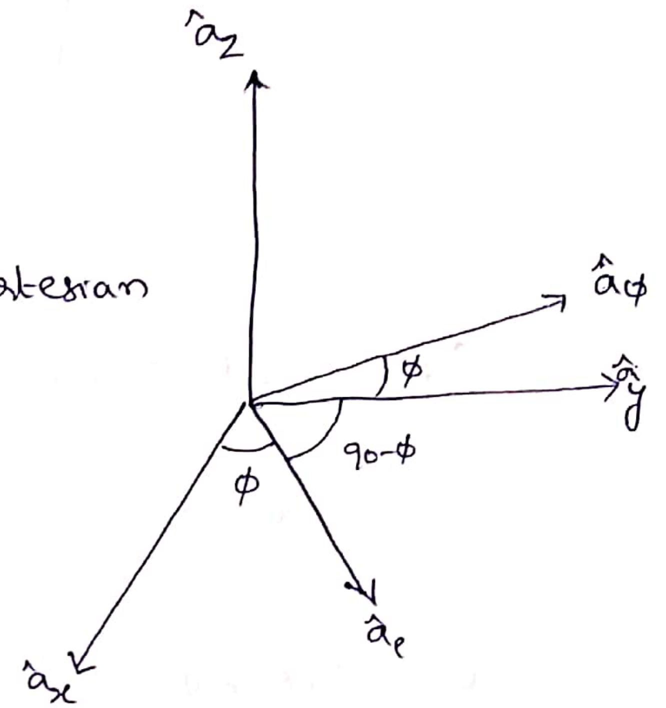
$\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$ — 1 Mark

In Spherical Coordinate System,

$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$ — 1 Mark

3 Marks

(4) (b) $\vec{B} = y\,\hat{a}_x - x\,\hat{a}_y$



The given vector is in Cartesian Coordinate System,

$\vec{B} = y\,\hat{a}_x - x\,\hat{a}_y$

$\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

$B_x = y,\ B_y = -x,\ B_z = 0.$ — 1 Mark

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$ — 2 Mark

$\cancel{B_\rho} = \cancel{B_x}\ \cancel{\cos\phi}$ $\quad x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z. \rightarrow$ 1 Mark

$$B_\rho = B_x\cos\phi + B_y\sin\phi$$
$$= y\cos\phi + (-x)\sin\phi$$
$$= \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$

$B_\phi = -B_x \sin\phi + B_y \cos\phi$

$= -y \sin\phi - x \cos\phi$

$= -\rho \sin\phi (\sin\phi) - (\rho \cos\phi)(\cos\phi)$

$= -\rho \sin^2\phi - \rho \cos^2\phi$

$= -\rho (\sin^2\phi + \cos^2\phi)$

$B_\phi = -\rho$

$B_z = 0.$ — 2 Marks

$\vec{B} = B_\rho \hat{a}_\rho + B_\phi \hat{a}_\phi + B_z \hat{a}_z$

$\vec{B} = -\rho \hat{a}_\phi$ — 1 Mark

7 Marks

5 (a) **Divergence of vector field**

In Cartesian Coordinate Systems

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ — 1 Mark

In cylindrical coordinate systems

$$\nabla \cdot \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$ — 1 Mark

In spherical coordinate systems

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

– 1 mark

3 Marks

(5) (b) Express the following vector in Cartesian coordinate system.

The given vector field

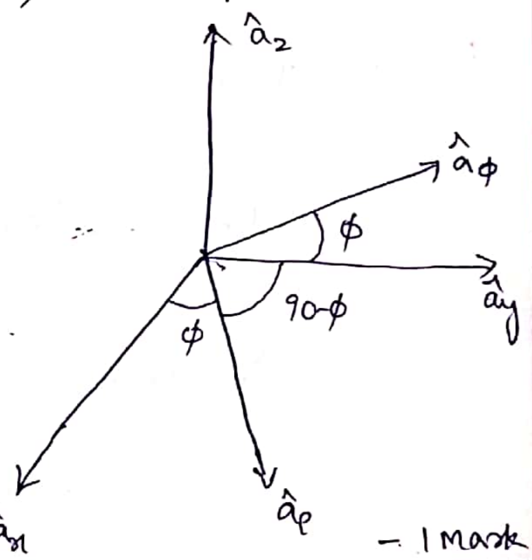
$$\vec{H} = \rho^2 \cos\phi\, \hat{a}_\rho - \rho \sin\phi\, \hat{a}_\phi \quad -(1)$$

$$\vec{H} = H_\rho \hat{a}_\rho + H_\phi \hat{a}_\phi + H_z \hat{a}_z \quad -(2)$$

from eq (1) & (2),

$$H_\rho = \rho^2 \cos\phi, \quad H_\phi = -\rho\sin\phi, \quad H_z = 0.$$ – 1 Mark

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} H_\rho \\ H_\phi \\ H_z \end{bmatrix}$$



– 1 Mark

2 Marks

$\rho = \sqrt{x^2+y^2}$,

$x = \rho\cos\phi, \quad y = \rho\sin\phi$

$$H_x = H_\rho \cos\phi - H_\phi \sin\phi$$

$$= (\rho^2 \cos\phi)(\cos\phi) - (-\rho \sin\phi) \sin\phi$$

$$= \rho^2 \cos^2\phi + \rho \sin^2\phi$$

$$= x^2 + y \cdot \frac{y}{\sqrt{x^2+y^2}}$$

$$H_y = H_\rho \sin\phi + H_\phi \cos\phi$$

$$= (\rho^2 \cos\phi) \sin\phi + (-\rho \sin\phi) \cos\phi$$

$$= (\rho \cos\phi)(\rho \sin\phi) - (\rho \sin\phi) \cos\phi$$

$$= xy - y \cdot \frac{x}{\sqrt{x^2+y^2}}$$

$H_z = H_z = 0.$ — 2 Marks

$$\vec{H} = H_x \hat{a}_x + H_y \hat{a}_y + H_z \hat{a}_z$$

$$= \left(x^2 + \frac{y^2}{\sqrt{x^2+y^2}}\right) \hat{a}_x + \left(xy - \frac{xy}{\sqrt{x^2+y^2}}\right) \hat{a}_y$$ — 1 Mark

7 Marks.

(6) Express the following vector in cartesian coordinate system.

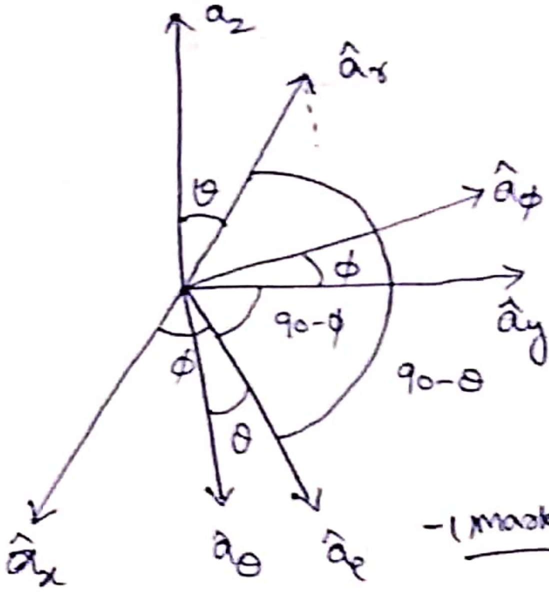
$$\vec{D} = r\sin\phi\, \hat{a}_r - \frac{1}{r}\sin\theta\cos\phi\, \hat{a}_\theta + r^2\, \hat{a}_\phi \quad -(1)$$

Sol:- The Given vector is in Spherical coordinate system,

$$\vec{D} = D_r\hat{a}_r + D_\theta\hat{a}_\theta + D_\phi\hat{a}_\phi \quad -(2)$$

By comparing (1) & (2),

$$D_r = r\sin\phi, \quad D_\theta = \frac{-1}{r}\sin\theta\cos\phi, \quad D_\phi = r^2$$



-1 Mark

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} D_r \\ D_\theta \\ D_\phi \end{bmatrix}$$

— 2 Mark

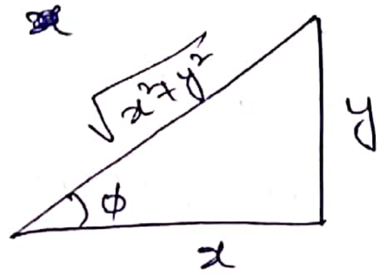
$r = \sqrt{x^2+y^2+z^2}$ | $y = r\sin\theta\sin\phi$

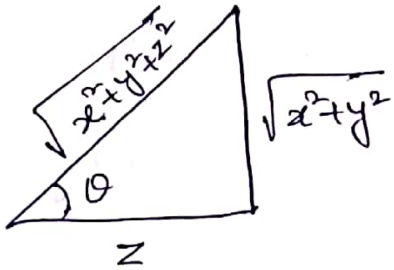
$x = r\sin\theta\cos\phi$ | $z = r\cos\theta$.

$$D_x = D_r \sin\theta \cos\phi + D_\theta \cos\theta \cos\phi - D_\phi \sin\phi$$

$$= r \sin\phi \sin\theta \cos\phi + \frac{1}{r} \sin\theta \cos\phi \cos\theta \cos\phi - r^2 \sin\phi$$

$$= (r \sin\theta \cos\phi) \sin\phi - \frac{1}{r} (\sin\theta \cos\phi) \cos\theta \cos\phi - r^2 \sin\phi$$





$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}} \qquad \sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}} \qquad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

— 1 mark

$$D_x = x \cdot \frac{y}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{x}{\sqrt{x^2+y^2+z^2}} \cdot \frac{z}{\sqrt{x^2+y^2+z^2}} \cdot \frac{x}{\sqrt{x^2+y^2}}$$

$$- (x^2+y^2+z^2) \cdot \frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{xy}{\sqrt{x^2+y^2}} - \frac{x^2 z}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)^{3/2}} - \frac{y(x^2+y^2+z^2)}{\sqrt{x^2+y^2}}$$

— 2 marks

$$D_y = D_r \sin\theta \sin\phi + D_\theta \cos\theta \sin\phi + D_\phi \cos\phi$$

$$= r\sin\phi \sin\theta \sin\phi - \frac{1}{r} \sin\theta\cos\phi \cos\theta \sin\phi + r^2 \cos\phi$$

$$= (r \sin\theta \sin\phi) \sin\phi - \frac{1}{r} (\sin\theta \cos\phi) \cos\theta \cdot \sin\phi + r^2 \cos\phi$$

$$= x \cdot \frac{y}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{x}{\sqrt{x^2+y^2+z^2}} \cdot \frac{z}{\sqrt{x^2+y^2+z^2}} \cdot \frac{y}{\sqrt{x^2+y^2}}$$

$$+ (x^2+y^2+z^2) \cdot \frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{xy}{\sqrt{x^2+y^2}} - \frac{xyz}{(x^2+y^2+z^2)^{3/2}\sqrt{x^2+y^2}} + \frac{(x^2+y^2+z^2)\, x}{\sqrt{x^2+y^2}}$$

-2 Marks

$$D_z = D_r \cos\theta - D_\theta \sin\theta$$

$$= r \sin\phi \cos\theta - \left(\frac{-1}{r}\right) \sin\theta \cos\phi \sin\theta$$

$$= \sqrt{x^2+y^2+z^2} \cdot \frac{y}{\sqrt{x^2+y^2}} \cdot \frac{z}{\sqrt{x^2+y^2+z^2}} + \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{x}{\sqrt{x^2+y^2+z^2}} \cdot \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

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$$D_z = \frac{yz}{\sqrt{x^2+y^2}} + \frac{x\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^{3/2}}.$$

— 1 mark

$$\vec{D} = D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z$$

$$= \left[\frac{xy}{\sqrt{x^2+y^2}} - \frac{x^2 z}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)^{3/2}} - \frac{y(x^2+y^2+z^2)}{\sqrt{x^2+y^2}}\right]\hat{a}_x$$

$$+ \left[\frac{xy}{\sqrt{x^2+y^2}} - \frac{xyz}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)^{3/2}} + \frac{x(x^2+y^2+z^2)}{\sqrt{x^2+y^2}}\right]\hat{a}_y$$

$$+ \left[\frac{yz}{\sqrt{x^2+y^2}} + \frac{x\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^{3/2}}\right]\hat{a}_z$$

— 1 Mark

4

10 Marks

(7) Express the following vector in spherical Coordinate System.

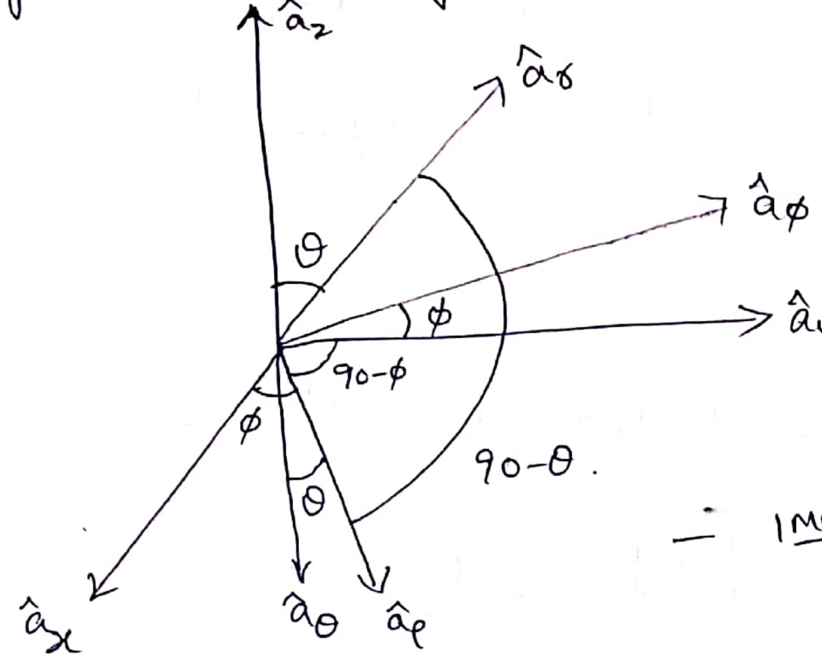
$$\vec{G} = z\,\hat{a}_x + x\,\hat{a}_y - y\,\hat{a}_z$$

The Given vector field is in Cartesian Coordinates

$$\vec{G} = G_x \hat{a}_x + G_y \hat{a}_y + G_z \hat{a}_z$$

$G_x = z, \quad G_y = x, \quad G_z = -y.$

$\begin{bmatrix} \cancel{G_r} \\ \cancel{G_\phi} \\ \cancel{G_\phi} \end{bmatrix}$



— 1 Mark

$$\begin{bmatrix} G_r \\ G_\theta \\ G_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix}$$

— 2 Marks

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

— 1 Mark

$G_r = G_x \sin\theta\cos\phi + G_y \sin\theta\sin\phi + G_z \cos\theta$

$= z\sin\theta\cos\phi + x\sin\theta\sin\phi - y\cos\theta$

$= r\cos\theta\sin\theta\cos\phi + r\sin\theta\sin\theta\sin\phi - r\sin\theta\sin\phi\cos\theta$

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$$G_r = r \sin\theta \cos\theta (\cos\phi - \sin\phi) + r \sin^2\theta \sin\phi$$ — 2 Mark

$$G_\theta = G_x \cos\theta \cos\phi + G_y \cos\theta \sin\phi - G_z \sin\theta$$

$$= z \cos\theta \cos\phi + x \cos\theta \sin\phi - (-y) \sin\theta$$

$$= z \cos\theta \cos\phi + x \cos\theta \sin\phi + y \sin\theta$$

$$= r \cos\theta \cos\theta \cos\phi + r \sin\theta \cos\phi \cos\theta \sin\phi + r \sin\theta \sin\phi \sin\theta$$

$$= r \cos^2\theta \cos\phi + r \sin\theta \cos\theta \sin\phi \cos\phi + r \sin^2\theta \sin\phi$$ — 2 Mark

$$G_\phi = -G_x \sin\phi + G_y \cos\phi$$

$$= -z \sin\phi + x \cos\phi$$

$$= -r \cos\theta \sin\phi + r \sin\theta \cos\phi \cos\phi$$

$$= -r \cos\theta \sin\phi + r \sin\theta \cos^2\phi.$$ — 1 Mark

$$\vec{G} = G_r \hat{a}_r + G_\theta \hat{a}_\theta + G_\phi \hat{a}_\phi$$

$$= [r \sin\theta \cos\theta (\cos\phi - \sin\phi) + r \sin^2\theta \sin\phi] \hat{a}_r$$

$$+ [r \cos^2\theta \cos\phi + r \sin\theta \cos\theta \sin\phi \cos\phi + r \sin^2\theta \sin\phi] \hat{a}_\theta$$

$$+ (-r \cos\theta \sin\phi + r \sin\theta \cos^2\phi) \hat{a}_\phi$$ — 1 Mark

10 Marks.

(8) (a) Find the Gradient of the following Scalar field

$$U = \frac{4}{r} \sin\theta \cos\phi$$

$$\nabla U = \frac{\partial U}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial U}{\partial \theta} \hat{a}_\theta + \frac{1}{r\sin\theta} \frac{\partial U}{\partial \phi} \hat{a}_\phi$$

— 1Mark

$$\frac{\partial U}{\partial r} = \frac{\partial}{\partial r}\left(\frac{4}{r} \sin\theta \cos\phi\right) = -\frac{4}{r^2} \sin\theta \cos\phi$$

$$\frac{\partial U}{\partial \theta} = \frac{\partial}{\partial \theta} \frac{4}{r}(\sin\theta \cos\phi) = \frac{4}{r} \cos\theta \cos\phi$$

$$\frac{\partial U}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\frac{4}{r} \sin\theta \cos\phi\right) = \frac{4}{r} \sin\theta (-\sin\phi)$$

$$= -\frac{4}{r} \sin\theta \sin\phi$$

— 1Mark

$$\nabla U = \left(-\frac{4}{r^2} \sin\theta \cos\phi\right) \hat{a}_r + \frac{1}{r} \cdot \left(\frac{4}{r} \cos\theta \cos\phi\right) \hat{a}_\theta + \frac{1}{r\sin\theta} \cdot \left(-\frac{4}{r} \sin\theta \sin\phi\right) \hat{a}_\phi$$

$$= -\frac{4}{r^2} \sin\theta \cos\phi \, \hat{a}_r + \frac{4}{r^2} \cos\theta \cos\phi \, \hat{a}_\theta - \frac{4}{r^2} \sin\phi \, \hat{a}_\phi$$

— 1Mark

3 Marks.

8 (b) Divergence of the following vector fields.

(i) $\vec{A} = 3\rho \sin\phi \; \hat{a}_\rho - 5\rho^2 z \; \hat{a}_\phi + 8z \cos^2\phi \; \hat{a}_z$

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \quad \text{— 1 Mark}$$

$A_\rho = 3\rho \sin\phi$, $A_\phi = -5\rho^2 z$, $A_z = 8z\cos^2\phi$

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial[\rho(3\rho\sin\phi)]}{\partial\rho} + \frac{1}{\rho}\frac{\partial(-5\rho^2 z)}{\partial\phi} + \frac{\partial(8z\cos^2\phi)}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial(3\rho^2\sin\phi)}{\partial\rho} + \frac{1}{\rho}(0) + 8\cos^2\phi$$

$$= \frac{6\rho}{\rho}\sin\phi + 8\cos^2\phi = 6\sin\phi + 8\cos^2\phi$$

— 3 Mark

4 Marks.

(ii) $\vec{B} = r^2\cos\phi \; \hat{a}_r + 2r \; \hat{a}_\phi$

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 B_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}$$

$B_r = r^2\cos\phi$, $B_\theta = 0$, $B_\phi = 2r$ — 1 Mark

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, r^2\cos\phi) + 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(2r)$$

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^4 \cos\phi)$$

$$= \frac{1}{r^2} (4 r^3 \cos\phi)$$

$$= 4r \cos\phi$$

— 2 Mark

3 Marks.

7 Marks