USN :

CMR Institute of Technology, Bangalore
DEPARTMENT OF MECHANICAL ENGINEERING
I - INTERNAL ASSESSMENT

Semester: 6-CBCS 2018
Subject: FINITE ELEMENT METHODS (18ME61)
Faculty: Mr Prashanth Hatti

Date: 19 May 2021
Time: 09:00 AM - 10:00 AM
Max Marks: 50

Answer All Questions

Q.No			Marks	CO	PO	BT/CL
1	a	Define simplex, complex and multiplex elements	6	CO1	PO1,PO2,PO3,PO5,PO11,PO12	L1
	b	Explain, with sketch, plane stress and plane strain condition.	4	CO1	PO1,PO2,PO3,PO5,PO11,PO12	L2
2		Explain different steps in FEM	10	CO1	PO1,PO2,PO3,PO5,PO11,PO12	L2
3		Derive equilibrium equation in elasticity of 3D elastic body subjected to body force.	10	CO1	PO1,PO2,PO3,PO5,PO11,PO12	L3
4		Using Rayleigh-Ritz method,compute the value of central deflection for simply supported beam with centrally applied load, considering trigonometric functions.	10	CO2	PO1,PO3,PO5,PO6,PO7,PO8,PO11,PO12	L3
5		Derive the equation maximum deflection in cantilever beam of length ‘l’, Young’s Modulus ‘E’ subjected to concentrated ‘P’ at free end by Rayleigh-Ritz method.	10	CO2	PO1,PO3,PO5,PO6,PO7,PO11,PO12	L3

Solution for 1st IAT

FINITE ELEMENT METHODS

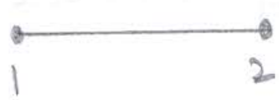
1. a **SIMPLEX, COMPLEX AND MULTIPLEX ~~Comp~~ ELEMENTS**

SIMPLEX

Simplex elements are those for which approximating polynomial consists of constant & linear terms.

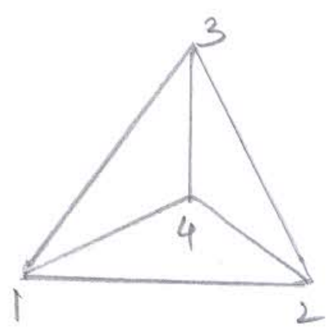
Example:

1D line element



2D triangular element

3D



Polynomial Eqns.

1D $\quad u(x) = a_1 + a_2 x$

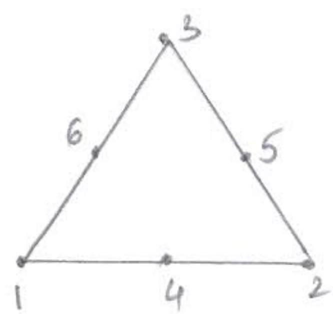
2D $\quad u(x,y) = a_1 + a_2 x + a_3 y$

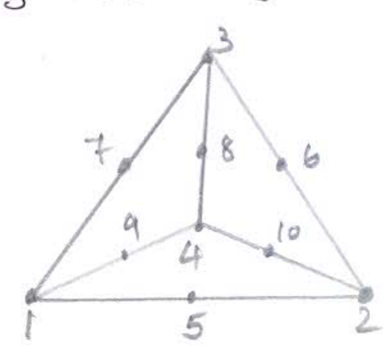
3D $\quad u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z$

COMPLEX

Complex elements are those for which approximating polynomial consists of quadratic, cubic & higher order terms in addition to constant & linear terms

Examples:





Polynomial eqns

For Quadratic

1D $\quad u(x) = a_1 + a_2 x + a_3 x^2$

2D $\quad u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$

3D $\quad u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2$
$\quad + a_8 xy + a_9 yz + a_{10} zx$

For Cubic

1D $\quad u(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$

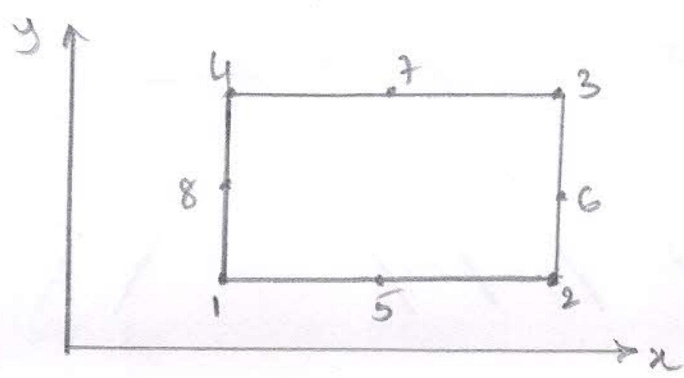
2D $\quad u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy + a_7 x^3$
$\quad + a_8 y^3 + a_9 x^2 y + a_{10} x y^2$

3D $\quad u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2$
$\quad + a_8 xy + a_9 yz + a_{10} zx + a_{11} x^3 + a_{12} y^3$
$\quad + a_{13} z^3 + a_{14} x^2 y + a_{15} y^2 x + a_{16} y^2 z + a_{17} y z^2$
$\quad + a_{18} z^2 x + a_{19} z x^2 + a_{20} xyz.$

MULTIPLEX

Multiplex elements are those whose boundaries are parallel to coordinate axes to achieve inter element continuity

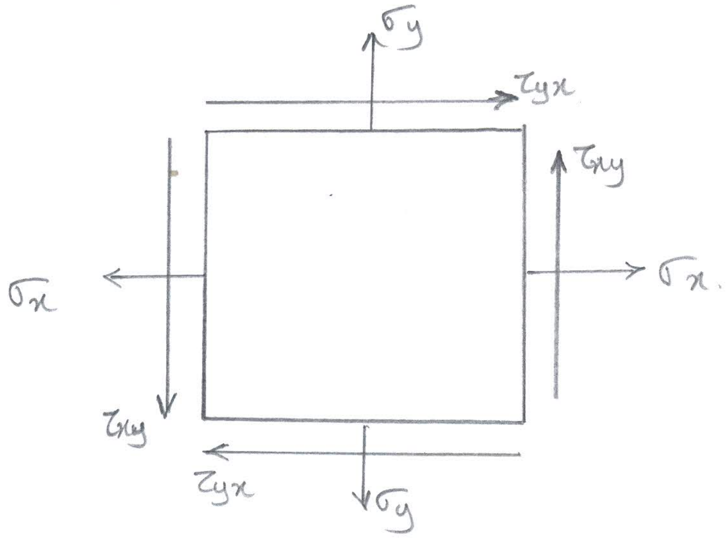
Ex:- Rectangular element.



1.b PLANE STRESS

Plane stress is defined to be a state of stress in which the normal stress, σ_z & shear stresses, τ_{xz} & τ_{yz}, directed perpendicular to the x-y plane are assumed to be zero.

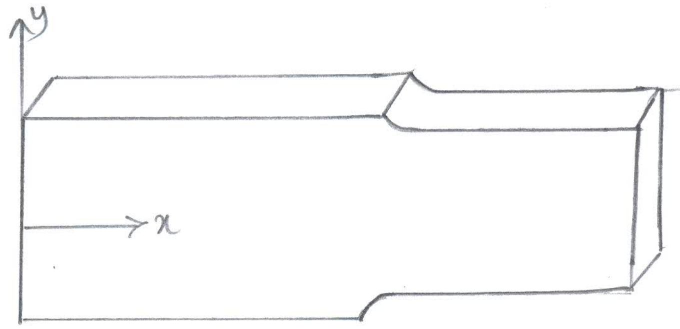
The two dimensional state of stress is illustrated below where σ_x, σ_y are normal stresses. τ_{xy} & τ_{yx} are shear stresses.



Stress vector

$$[\sigma] = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

The geometry of the body is essentially that of a plate with one dimension much smaller than the others.



Typical loading & boundary conditions for plane stress problems in two dimensional elasticity

i) Loadings may be point forces or distributed forces applied over thickness of the plate.

ii) Supports may be fixed points or fixed edges or roller supports.

For isotropic material; $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

& $\gamma_{xz} = \gamma_{yz} = 0$

$$[\sigma] = [D][\varepsilon]$$

where $$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

in which $[D]$ is the stress/strain matrix,

E is modulus of elasticity,

ν is Poisson's ratio.

The Strains in plane stress then are

$$[\varepsilon] = [C][\sigma]$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

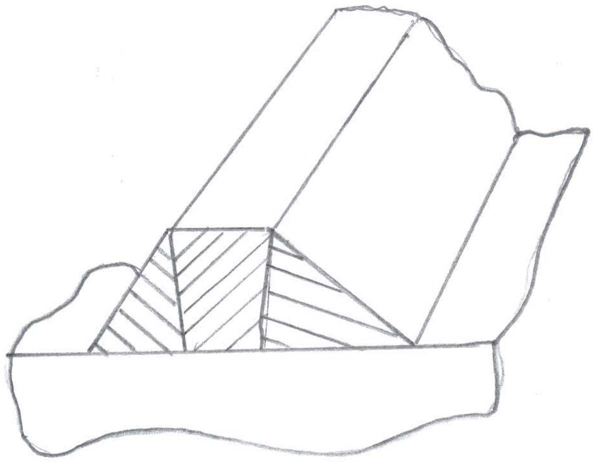
$$[C]^{-1} = D$$

PLANE STRAIN

Plane strain is defined to be a state of strain in which the strain normal to the x-y plane, ϵ_z & the shear strain γ_{xz} & γ_{yz} are assumed to be zero.

In plane strain, one deals with a situation in which the dimension of structure in one direction, say the z coordinate direction, is very large in comparison with dimensions of the structure in other two directions.

Some important practical applications of this representation occur in the analysis of dams, tunnels, & other geotechnical works.



For isotropic material

$$\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

$$\tau_{xz} = \tau_{yz} = 0$$

yields

$$\{\sigma\} = [D][\varepsilon]$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

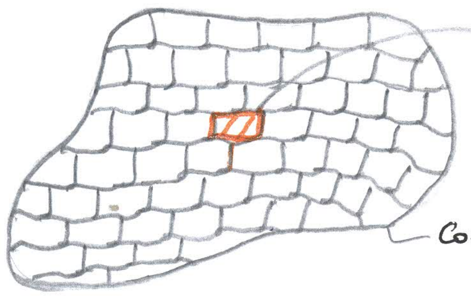
with

$$\sigma_z = \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}(\epsilon_x + \epsilon_y)\right]$$

2 **STEPS IN FEM**

1. **Discritization of a given Continum**

In this step, the Continum is Subdivided into no of parts Called finite elements. The type of element Selected depends on the Kind of analysis namely 1D, 2D or 3D.



Types of Elements

1-d Element

Element Shape - Line

Element type - rod, bar, beam, pipe, axisymmetric shell etc

Bar element

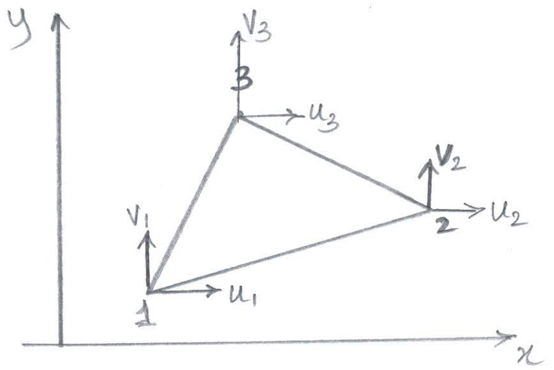
Common one dimensional element is bar element having 2 nodes.

Practical applications - Long shafts, beams, pin joint, Connection elements etc.

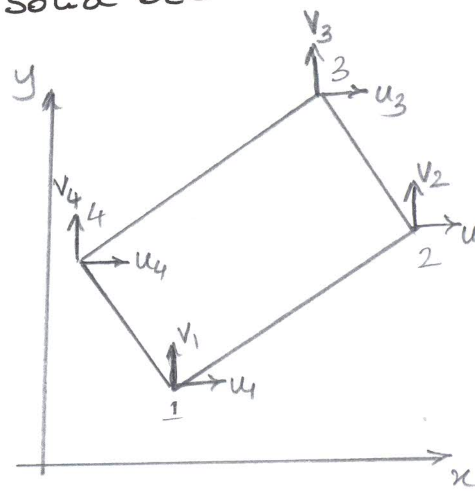
2-d Element

Element shape - quad, tria

Element type - thin shell, plate, membrane, plane stress, plane strain, axi-symmetric solid etc



TRIA3 OR CST



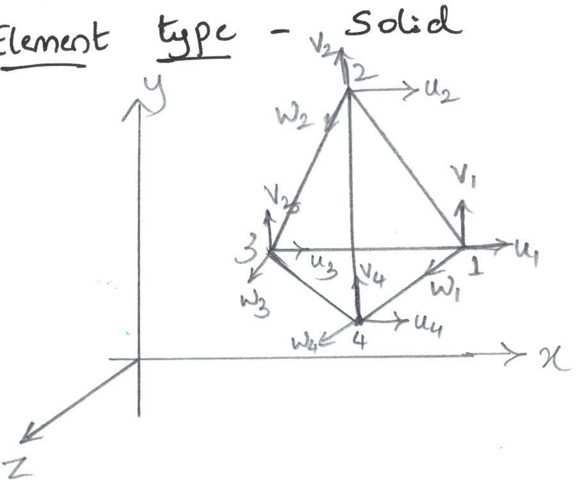
QUAD 4

Practical applications - Sheet metal parts, plastic components like instrumental panels etc.

3-d Element

Element shape - tetra, penta, hex, Pyramid

Element type - Solid



Practical application - Transmission casing, engine block, Crankshaft etc

2. Selection of displacement model for each finite element

The displacement variation for each element is unknown. Hence a mathematical model is represented for each finite element. They can be either a Polynomial or trignometric function.
A polynomial function is adopted due to Simplicity of mathematical calculations.

3. Generation of stiffness matrix for each finite element discritized for the Continum

The stiffness matrix [K] is derived by using Principle of minimum potential energy.

4. Generation of global Stiffness matrix

Overall stiffness matrix is formed by global addition of elemental stiffness matrix

$$[K] = K_1 + K_2$$

5. Imposition of equilibrium equation

$$[K][q] = [F]$$

where [K] - stiffness matrix (global)
[q] - Nodal displacements
[F] - Nodal forces

6. Enforcing the boundary Condition of given problem

There are two methods to enforce the boundary Conditions.

i) Elimination Method.

ii) Penalty Method.

Elimination Method

In this method after imposing the boundary Conditions, depending on the Constraints the Corresponding row & Column of the equilibrium is eliminated.

Penalty Method

In this method, large value of the global stiffness matrix 'c' is added to the 1^{st} & last element of diagonal of matrix. On applying the boundary Condition, the overall equilibrium eqn gets modified.

$$[C] = \max|K_{ij}| \times 10^4$$

7. Determination of unknowns

Computation of Stress & Strains

$$[\sigma] = [E][B][q]$$

where $[B]$ - Strain displacement matrix

$$- \frac{1}{l_e}\begin{bmatrix}-1 & 1\end{bmatrix}$$

$$[q] = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \text{ - Nodal displacements.}$$

3 EQUILIBRIUM EQUATION FOR 3D STATE OF STRESS

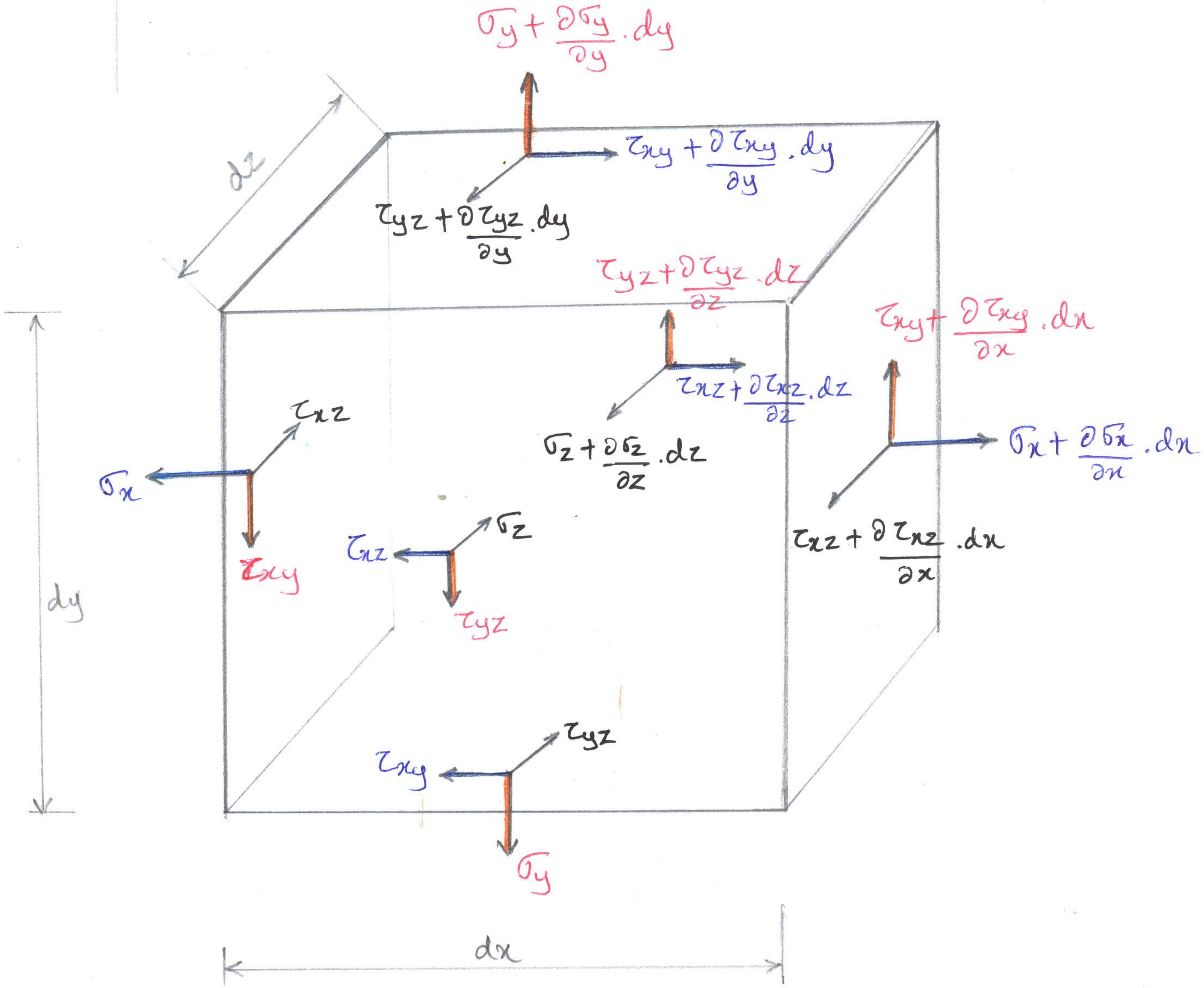


Figure shows a rectangular bar subjected to normal stress σ_x, σ_y, σ_z & shear stress τ_{xy}, τ_{yz}, τ_{xz}.

Let X, Y & Z be the body forces.

For bar to be in equilibrium.

$$\Sigma F_x = 0 \; ; \; \Sigma F_y = 0 \; ; \; \Sigma F_z = 0$$

$$\sum F_x = 0 \qquad \xrightarrow{\ } \ +ve \qquad \xleftarrow{\ } \ -ve$$

$$\rightarrow \left[\sigma_x + \frac{\partial \sigma_x}{\partial x}.dx\right] dy.dz - \sigma_x.dy.dz$$

$$+ \left[\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}.dy\right] dx.dz - \tau_{xy}.dx.dz$$

$$+ \left[\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z}.dz\right] dx.dy - \tau_{xz}.dx.dy$$

$$+ X\ dx.dy.dz = 0$$

$$\rightarrow \cancel{\sigma_x.dy.dz} + \frac{\partial \sigma_x}{\partial x}.dx.dy.dz - \cancel{\sigma_x.dy.dz}$$

$$+ \cancel{\tau_{xy}\ dx.dz} + \frac{\partial \tau_{xy}}{\partial y}.dy.dx.dz - \cancel{\tau_{xy}.dx.dz}$$

$$+ \cancel{\tau_{xz}.dx.dy} + \frac{\partial \tau_{xz}}{\partial z}.dx.dy.dz - \cancel{\tau_{xz}.dx.dy}$$

$$+ X.dx.dy.dz = 0$$

$$\rightarrow \frac{\partial \sigma_x}{\partial x}.dx.dy.dz + \frac{\partial \tau_{xy}}{\partial y} dx.dy.dz + \frac{\partial \tau_{xz}}{\partial z} dx.dy.dz$$

$$+ X\ dx.dy.dz = 0$$

$$\rightarrow dx.dy.dz \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X\right] = 0$$

$$dx.dy.dz \neq 0$$

$$\boxed{\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0} \rightarrow (1)$$

$\Sigma F_y = 0$ $\uparrow +ve$ $\downarrow -ve$

$$\rightarrow \left[\sigma_y + \frac{\partial \sigma_y}{\partial y}.dy\right] dx.dz - \sigma_y.dx.dz$$

$$+ \left[\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}.dx\right] dy.dz - \tau_{xy}.dy.dz$$

$$+ \left[\tau_{yz} + \frac{\partial \tau_{yz}}{\partial z}.dz\right] dy.dx - \tau_{yz}.dy.dx$$

$$+ Y\, dx.dy.dz = 0$$

$$\rightarrow \cancel{\sigma_y.dx.dz} + \frac{\partial \sigma_y}{\partial y}.dx.dy.dz - \cancel{\sigma_y.dx.dz}$$

$$+ \cancel{\tau_{xy}\, dy.dz} + \frac{\partial \tau_{xy}}{\partial x}.dx.dy.dz - \cancel{\tau_{xy}\, dy.dz}$$

$$+ \cancel{\tau_{yz}.dy.dx} + \frac{\partial \tau_{yz}}{\partial z}.dx.dy.dz - \cancel{\tau_{yz}.dy.dx}$$

$$+ Y.dx.dy.dz = 0$$

$$\rightarrow dx.dy.dz \left[\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z}\right] = 0$$

$$dx.dy.dz \neq 0$$

$$\boxed{\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = 0} \rightarrow (2)$$

$\Sigma F_z = 0$ $\quad \swarrow$ +ve $\quad \nearrow$ -ve

$$\rightarrow \left[\sigma_z + \frac{\partial \sigma_z}{\partial z} . dz\right] dy . dx - \sigma_z . dy . dx$$

$$+ \left[\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} . dy\right] dx . dz - \tau_{yz} . dx . dz$$

$$+ \left[\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} . dx\right] dy . dz - \tau_{xz} . dy . dz$$

$$+ Z \; dx . dy . dz = 0$$

$$\rightarrow \cancel{\sigma_z . dy . dx} + \frac{\partial \sigma_z}{\partial z} . dx . dy . dz - \cancel{\sigma_z . dy . dx}$$

$$+ \cancel{\tau_{yz} \; dx . dz} + \frac{\partial \tau_{yz}}{\partial y} . dy . dx \; dz - \cancel{\tau_{yz} . dx . dz}$$

$$+ \cancel{\tau_{xz} . dy . dz} + \frac{\partial \tau_{xz}}{\partial x} . dx . dy . dz - \cancel{\tau_{xz} . dy . dz}$$

$$+ Z \; dx . dy . dz = 0$$

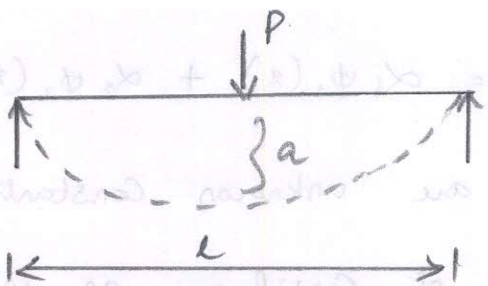
$$\rightarrow dx . dy . dz \left[\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x}\right] = 0$$

$$dx . dy . dz \neq 0$$

$$\boxed{\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x} = 0} \rightarrow (3)$$

Eqns 1, 2 & 3 are equilibrium equation for 3D state of stress.

4



Sol 1) Potential Energy functional

$$\pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$$

$$S.E \text{ for beams} = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \rightarrow (1)$$

$$W.P = P.y$$

2) Assume the displacement function

$$y = a \sin \frac{\pi x}{l}$$

Boundary condition: At $x = 0$, $y = 0$

$x = l$, $y = 0$

$\frac{dy}{dx} \neq 0$ at $x=0$, $y=0$

$$y = a \sin \frac{\pi x}{l}$$

$$\frac{dy}{dx} = a \cdot \cos \frac{\pi x}{l} \cdot \frac{\pi}{l}$$

$$\frac{d^2y}{dx^2} = -a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}$$

Sub. this in eqn (1)

$$S.E = \frac{EI}{2}\int_0^l \left[-\frac{a\pi^2}{l^2}\sin\frac{\pi x}{l}\right]^2 dx$$

$$= \frac{EI}{2} \cdot -a^2\frac{\pi^4}{l^4}\int_0^l \sin^2\frac{\pi x}{l}\, dx$$

$$= \frac{EI}{2} \cdot a^2\frac{\pi^4}{l^4}\int_0^l \frac{1-\cos(2\pi x/l)}{2} \cdot dx$$

$$= \frac{EI}{4} \cdot a^2\frac{\pi^4}{l^4}\left[1 - \sin\frac{2\pi x}{l} \cdot \frac{2\pi}{l}\right]_0^l$$

$$= \frac{EI}{4} \cdot \frac{a^2\pi^4}{l^4} \cdot l$$

$$S.E = \frac{EI}{4} \cdot \frac{a^2\pi^4}{l^3}$$

$$W.P = P \cdot y_{max}$$

$y = y_{max}$ at $x = l/2$

$$y_{max} = a\sin\left[\frac{\pi}{\cancel{l}} \cdot \frac{l}{2}\right] = a\sin\left(\frac{\pi}{2}\right) = a$$

$$\therefore W.P = Pa$$

3) P.E functional $\pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$

$$\pi = \frac{EI}{4} \cdot \frac{a^2\pi^4}{l^3} - Pa$$

4) Using principle of minimum P.E

$$\frac{\partial \pi}{\partial a} = 0$$

$$\frac{\partial \pi}{\partial a} = \frac{2EI\,a\pi^4}{4l^3} - P = 0$$

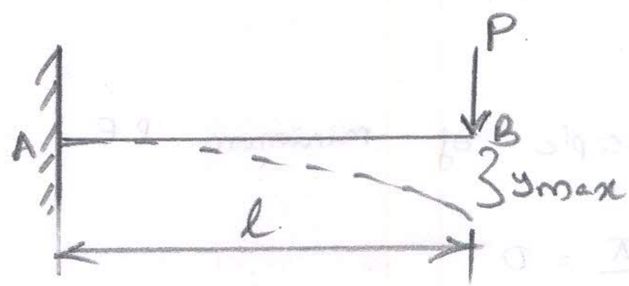
$$\frac{2EI\,a\pi^4}{4l^3} = P$$

$$a = \frac{4Pl^3}{2EI\,\pi^4}$$

$$= \frac{Pl^3}{\frac{EI}{2}\pi^4}$$

$$\boxed{a = \frac{Pl^3}{48.7\,EI}}$$

5



Sol 1) Potential Energy functional

$$\pi = \underset{(+ve)}{S.E} + \underset{(-ve)}{W.P}$$

$$S.E = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$W.P = P.\, y_{max}$$

2) Assume displacement function

$$y = a_2x^2 + a_3x^3$$

B.C At A, $x=0$, $y=0$ At A, $x=0$, $\frac{dy}{dx}=0$

$$y = a_2x^2 + a_3x^3$$

$$\frac{dy}{dx} = 2a_2 x + 3a_3 x^2$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3 x$$

$y = y_{max}$ at $x = l$

$$y = a_2 x^2 + a_3 x^3$$

$$y_{max} = a_2 l^2 + a_3 l^3$$

3) P.E functional

$$\pi = \frac{EI}{2}\int_0^l (2a_2 + 6a_3 x)^2\, dx - P\left(a_2 l^2 + a_3 l^3\right)$$

$$= \frac{EI}{2}\int_0^l \left[4a_2^2 + 36a_3^2 x^2 + 24 a_2 a_3 x\right] dx - P\left(a_2 l^2 + a_3 l^3\right)$$

$$= \frac{EI}{2}\left[4a_2^2 x + 36a_3^2 \frac{x^3}{3} + 24 a_2 a_3 \frac{x^2}{2}\right]_0^l - P\left(a_2 l^2 + a_3 l^3\right)$$

$$\pi = \frac{EI}{2}\left[4a_2^2 l + 36 a_3^2 \frac{l^3}{3} + 24 a_2 a_3 \frac{l^2}{2}\right] - P\left(a_2 l^2 + a_3 l^3\right)$$

4) Using Principle of minimum P.E

$$\frac{\partial \pi}{\partial a_2} = 0 \quad ; \quad \frac{EI}{2}\left[8a_2 l + 12 a_3 l^2\right] - Pl^2 = 0 \rightarrow (1)$$

$$\frac{\partial \pi}{\partial a_3} = 0 \quad ; \quad \frac{EI}{2}\left[24 a_3 l^3 + 12 a_2 l^2\right] - Pl^3 = 0 \rightarrow (2)$$

Multiply eqn (1) by $2l$

$$\frac{EI}{2}\left[16 a_2 l^2 + 24 a_3 l^3\right] - 2Pl^3 = 0$$

$$\frac{EI}{2}\left[12 a_2 l^2 + 24 a_3 l^3\right] - Pl^3 = 0$$

$$- \qquad\qquad - \qquad\qquad +$$

$$\frac{EI}{2}\left[4 a_2 l^2\right] - Pl^3 = 0$$

$$a_2 = \frac{2Pl^3}{4l^2 \cdot EI}$$

$$\boxed{a_2 = \frac{Pl}{2EI}}$$

Sub. a_2 in eqn (2)

$$\frac{EI}{2}\left[12 \cdot \frac{Pl}{2EI} \cdot l^2 + 24 a_3 l^3\right] - Pl^3 = 0$$

$$\frac{6Pl^3}{EI} + 24 a_3 l^3 = \frac{2Pl^3}{EI}$$

$$24 a_3 l^3 = \frac{2Pl^3}{EI} - \frac{6Pl^3}{EI} = \frac{-4Pl^3}{EI}$$

$$a_3 = \frac{-4Pl^3}{24 l^3 \cdot EI} = \frac{-P}{6EI}$$

$$\boxed{a_3 = \frac{-P}{6EI}}$$

Max' deflection

$$y_{max} = a_2 l^2 + a_3 l^3$$

$$= \frac{P.\, l^3}{2EI} - \frac{Pl^3}{6EI}$$

$$\boxed{y_{max} = \frac{Pl^3}{3EI}}$$