FLUID MECHANICS MAY 2021

IAT-1 SCHEME AND SOLUTION

SOLUTION

1		Derive CONTINUITY EQUATION IN CARTESIAN CO-ORDINATES for a fluid flow in 3-Dimenions	10	CO4	PO1	L2

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z directions respectively. Mass of fluid entering the face $ABCD$ per second

$$= \rho \times \text{Velocity in } x\text{-direction} \times \text{Area of } ABCD$$
$$= \rho \times u \times (dy \times dz)$$

Then mass of fluid leaving the face $EFGH$ per second $= \rho u\, dydz + \frac{\partial}{\partial x}(\rho u\, dydz)\, dx$

∴ Gain of mass in x-direction

$$= \text{Mass through } ABCD - \text{Mass through } EFGH \text{ per second}$$
$$= \rho u\, dydz - \rho u\, dydz - \frac{\partial}{\partial x}(\rho u\, dydz)dx$$
$$= -\frac{\partial}{\partial x}(\rho u\, dydz)\, dx$$
$$= -\frac{\partial}{\partial x}(\rho u)\, dx\, dydz \qquad \{\because dydz \text{ is constant}\}$$

Similarly, the net gain of mass in y-direction

$$= -\frac{\partial}{\partial y}(\rho v)\, dxdydz$$

and in z-direction

$$= -\frac{\partial}{\partial z}(\rho w)\, dxdydz$$

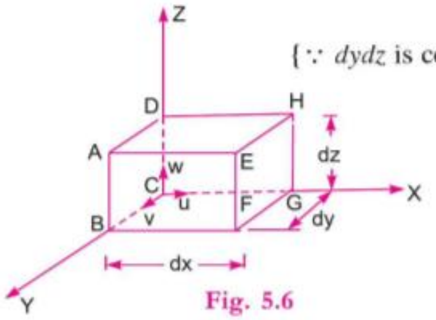

Fig. 5.6

∴ Net gain of masses $= -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass of fluid in the element is $\rho.\, dx.\, dy.\, dz$ and its rate of increase with time is $\frac{\partial}{\partial t}(\rho\, dx.\, dy.\, dz)$ or $\frac{\partial \rho}{\partial t}.\, dx\, dy\, dz.$

Equating the two expressions,

or $$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t}.\, dxdydz$$

or $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \text{ [Cancelling } dx.dy.dz \text{ from both sides]} \quad ...(5.3A)$$

Equation (5.3A) is the continuity equation in cartesian co-ordinates in its most general form. This equation is applicable to :

(*i*) Steady and unsteady flow,

(*ii*) Uniform and non-uniform flow, and

(*iii*) Compressible and incompressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.3*A*) becomes as

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad ...(5.3B)$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad ...(5.4)$$

Equation (5.4) is the continuity equation in three-dimensions. For a two-dimensional flow, the component $w = 0$ and hence continuity equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad ...(5.5)$$

2	a	Explain two different fluid flow analysis with suitable example	5	CO4	PO1	L1

The fluid motion is described by two methods. They are —(*i*) Lagrangian Method, and (*ii*) Eulerian Method. In the Lagrangian method, a **single fluid particle** is followed during its motion and its velocity, acceleration, density, etc., are described. In case of Eulerian method, the velocity, acceleration, pressure, density etc., are described **at a point** in flow field. The Eulerian method is commonly used in fluid mechanics.

	b	Write the expression for velocity and acceleration of a fluid in x,y and z direction. Differentiate between local and convective acceleration	5	CO4	PO1	L1

Let V is the resultant velocity at any point in a fluid flow. Let u, v and w are its component in x, y and z directions. The velocity components are functions of space-co-ordinates and time. Mathematically, the velocity components are given as

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

and Resultant velocity, $V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x, a_y and a_z are the **total acceleration** in x, y and z directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But $\frac{dx}{dt} = u, \frac{dy}{dt} = v$ and $\frac{dz}{dt} = w$

∴ $$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Similarly, $$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \qquad ...(5.6)$$

$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

For steady flow, $\frac{\partial V}{\partial t} = 0$, where V is resultant velocity

3		Define Viscosity and explain how viscosity varies in fluids with respect to temperature	10	CO1	PO1	L1

Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. When two layers of a fluid, a distance '*dy*' apart, move one over the other at different velocities, say *u* and *u* + *du* as shown in Fig. 1.1, the viscosity together with relative velocity causes a shear stress acting between the fluid layers.

The top layer causes a shear stress on the adjacent lower layer while the lower layer causes a shear stress on the adjacent top layer. This shear stress is proportional to the rate of change of velocity with respect to *y*. It is denoted by symbol τ (Tau).

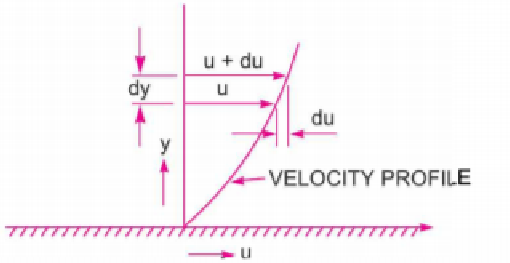

Fig. 1.1 *Velocity variation near a solid boundary.*

Mathematically, $\tau \propto \frac{du}{dy}$

or $$\tau = \mu \frac{du}{dy} \quad ...(1.2)$$

where μ (called *mu*) is the constant of proportionality and is known as the co-efficient of dynamic viscosity or only viscosity. $\frac{du}{dy}$ represents the rate of shear strain or rate of shear deformation or velocity gradient.

From equation (1.2), we have $$\mu = \frac{\tau}{\left(\frac{du}{dy}\right)} \quad ...(1.3)$$

Thus viscosity is also defined as the shear stress required to produce unit rate of shear strain.

1.3.4 Variation of Viscosity with Temperature. Temperature affects the viscosity. The viscosity of liquids decreases with the increase of temperature while the viscosity of gases increases with the increase of temperature. This is due to reason that the viscous forces in a fluid are due to cohesive forces and molecular momentum transfer. In liquids, the cohesive forces predominates the molecular momentum transfer, due to closely packed molecules and with the increase in temperature, the cohesive forces decreases with the result of decreasing viscosity. But in case of gases the cohesive forces are small and molecular momentum transfer predominates. With the increase in temperature, molecular momentum transfer increases and hence viscosity increases. The

4		A 30 cm diameter pipe, conveying water, branches into two pipes of diameters 20 am and 15 cm respectively. If the average velocity in the 30 cm pipe is 2.5 m/s, find the discharge in this pipe. Also determine the velocity in 15 cm pipe if the average velocity in the 20 cm diameter pipe is 2 m/s.	10	CO3	PO1	L3

$D_1 = 30 \text{ cm} = 0.30 \text{ m}$

∴ $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times .3^2 = 0.07068 \text{ m}^2$

$V_1 = 2.5 \text{ m/s}$

$D_2 = 20 \text{ cm} = 0.20 \text{ m}$

∴ $A_2 = \frac{\pi}{4} (.2)^2 = \frac{\pi}{4} \times .4 = 0.0314 \text{ m}^2$,

$V_2 = 2 \text{ m/s}$

$D_3 = 15 \text{ cm} = 0.15 \text{ m}$

∴ $A_3 = \frac{\pi}{4} (.15)^2 = \frac{\pi}{4} \times 0.225 = 0.01767 \text{ m}^2$

Find (*i*) Discharge in pipe 1 or Q_1

(*ii*) Velocity in pipe of dia. 15 cm or V_3

Let Q_1, Q_2 and Q_3 are discharges in pipe 1, 2 and 3 respectively.
Then according to continuity equation

$$Q_1 = Q_2 + Q_3 \quad \ldots(1)$$

(*i*) **The discharge Q_1** in pipe 1 is given by

$$Q_1 = A_1V_1 = 0.07068 \times 2.5 \text{ m}^3/\text{s} = \mathbf{0.1767\ m^3/s.\ Ans.}$$

(*ii*) **Value of V_3**

$$Q_2 = A_2V_2 = 0.0314 \times 2.0 = 0.0628 \text{ m}^3/\text{s}$$

Substituting the values of Q_1 and Q_2 in equation (1)

$$0.1767 = 0.0628 + Q_3$$

$\therefore$ $$Q_3 = 0.1767 - 0.0628 = 0.1139 \text{ m}^3/\text{s}$$

But $$Q_3 = A_3 \times V_3 = 0.01767 \times V_3 \quad \text{or} \quad 0.1139 = 0.01767 \times V_3$$

$\therefore$ $$V_3 = \frac{0.1139}{0.01767} = \mathbf{6.44\ m/s.\ Ans.}$$

5	The velocity vector in a fluid flow is given by $V = 4x^3i - 10x^2yj + 2tk$ .Find the velocity and acceleration of a fluid particle at (2,1,3) and t=1	10	CO3	PO1	L3

Solution. The velocity components u, v and w are $u = 4x^3$, $v = -10x^2y$, $w = 2t$
For the point (2, 1, 3), we have $x = 2$, $y = 1$ and $z = 3$ at time $t = 1$.
Hence velocity components at (2, 1, 3) are

$$u = 4 \times (2)^3 = 32 \text{ units}$$
$$v = -10(2)^2(1) = -40 \text{ units}$$
$$w = 2 \times 1 = 2 \text{ units}$$

$\therefore$ Velocity vector V at (2, 1, 3) = $32i - 40j + 2k$

or Resultant velocity $= \sqrt{u^2 + v^2 + w^2}$

$$= \sqrt{32^2 + (-40)^2 + 2^2} = \sqrt{1024 + 1600 + 4} = \mathbf{51.26\ units.\ Ans.}$$

Acceleration is given by equation (5.6)

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Now from velocity components, we have

$$\frac{\partial u}{\partial x} = 12x^2, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = -20xy, \frac{\partial v}{\partial y} = -10x^2, \frac{\partial v}{\partial z} = 0\ \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = 0, \frac{\partial w}{\partial y} = 0, \frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 2.1$$

Substituting the values, the acceleration components at (2, 1, 3) at time $t = 1$ are

$$a_x = 4x^3(12x^2) + (-10x^2y)(0) + 2t \times (0) + 0$$
$$= 48x^5 = 48 \times (2)^5 = 48 \times 32 = 1536 \text{ units}$$

$$a_y = 4x^3(-20xy) + (-10x^2y)(-10x^2) + 2t(0) + 0$$
$$= -80x^4y + 100x^4y$$
$$= -80(2)^4(1) + 100(2)^4 \times 1 = -1280 + 1600 = 320 \text{ units.}$$
$$a_z = 4x^3(0) + (-10x^2y)(0) + (2t)(0) + 2.1 = 2.0 \text{ units}$$

∴ Acceleration is $A = a_x i + a_y j + a_z k = \mathbf{1536i + 320j + 2k}$. **Ans.**

or Resultant $A = \sqrt{(1536)^2 + (320)^2 + (2)^2}$ units

$= \sqrt{2359296 + 102400 + 4}$ = **1568.9 units. Ans.**

6		A fluid flow field is given by $V = x^2yi + y^2zj - (2xyz + yz^2)k$ Prove that is a case of possible steady incompressible flow. Calculate the velocity, acceleration at point (2,1,3)	10	CO3	PO1	L3

Solution. For the given fluid flow field $u = x^2y$ ∴ $\frac{\partial u}{\partial x} = 2xy$

$v = y^2z$ ∴ $\frac{\partial v}{\partial y} = 2yz$

$w = -2xyz - yz^2$ ∴ $\frac{\partial w}{\partial z} = -2xy - 2yz.$

For a case of possible steady incompressible fluid flow, the continuity equation (5.4) should be satisfied.

i.e.,
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Substituting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2xy + 2yz - 2xy - 2yz = 0$$

Hence the velocity field $V = x^2yi + y^2zj - (2xyz + yz^2)k$ is a possible case of fluid flow. **Ans.**

Velocity at (2, 1, 3)

Substituting the values $x = 2$, $y = 1$ and $z = 3$ in velocity field, we get

$$V = x^2yi + y^2zj - (2xyz + yz^2)k$$
$$= 2^2 \times 1i + 1^2 \times 3j - (2 \times 2 \times 1 \times 3 + 1 \times 3^2)k$$
$$= \mathbf{4i + 3j - 21k}. \textbf{ Ans.}$$

and Resultant velocity $= \sqrt{4^2 + 3^2 + (-21)^2} = \sqrt{16 + 9 + 441} = \sqrt{466}$ = **21.587 units. Ans.**

Acceleration at (2, 1, 3)

The acceleration components a_x, a_y and a_z for steady flow are

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$u = x^2y, \frac{\partial u}{\partial x} = 2xy, \frac{\partial u}{\partial y} = x^2 \text{ and } \frac{\partial u}{\partial z} = 0$$

$$v = y^2 z,\ \frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = 2yz,\ \frac{\partial v}{\partial z} = y^2$$

$$w = -2xyz - yz^2, \frac{\partial w}{\partial x} = -2yz,\ \frac{\partial w}{\partial y} = -2xz - z^2,\ \frac{\partial w}{\partial z} = -2xy - 2yz.$$

Substituting these values in acceleration components, we get acceleration at (2, 1, 3)

$$a_x = x^2 y\,(2xy) + y^2 z\,(x)^2 - (2xyz + yz^2)\,(0)$$
$$= 2x^3y^2 + x^2y^2z$$
$$= 2\,(2)^3 1^2 + 2^2 \times 1^2 \times 3 = 2 \times 8 + 12$$
$$= 16 + 12 = 28 \text{ units}$$
$$a_y = x^2 y\,(0) + y^2 z\,(2yz) - (2xyz + yz^2)\,(y^2)$$
$$= 2y^3z^2 - 2xy^3z - y^3z^2$$
$$= 2 \times 1^3 \times 3^2 - 2 \times 2 \times 1^3 \times 3 - 1^3 \times 3^2 = 18 - 12 - 9 = -3 \text{ units}$$
$$a_z = x^2 y\,(-2yz) + y^2 z\,(-2xz - z^2) - (2xyz + yz^2)\,(-2xy - 2yz)$$
$$= -2x^2y^2z - 2xy^2z^2 - y^2z^3 + [4x^2y^2z + 2xy^2z^2 + 4xy^2z^2 + 2y^2z^3]$$
$$= -2 \times 2^2 \times 1^2 \times 3 - 2 \times 2 \times 1^2 \times 3^2 - 1^2 \times 3^3$$
$$+ [4 \times 2^2 \times 1^2 \times 3 + 2 \times 2 \times 1^2 \times 3^2 + 4 \times 2 \times 1^2 \times 3^2 + 2 \times 1^2 \times 3^3]$$
$$= -24 - 36 - 27 + [48 + 36 + 72 + 54]$$
$$= -24 - 36 - 27 + 48 + 36 + 72 + 54 = 123$$

∴ Acceleration $= a_x i + a_y j + a_z k =$ **28i – 3j + 123k. Ans.**

or Resultant acceleration $= \sqrt{28^2 + (-3)^2 + 123^2} = \sqrt{784 + 9 + 15129}$

$= \sqrt{15922} =$ **126.18 units. Ans.**

7		The stream function for a two-dimensional flow is given by $\psi = 2xy$. Calculate the velocity at point (2,3). Find the potential function.	10	CO3	PO1	L3

Solution. Given : $\psi = 2xy$

The velocity components u and v in terms of ψ are

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y}(2xy) = -2x$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y.$$

At the point P (2, 3), we get $u = -2 \times 2 = -4$ units/sec

$v = 2 \times 3 = 6$ units/sec

∴ Resultant velocity at $P = \sqrt{u^2 + v^2} = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52} = 7.21$ units/sec.

Velocity Potential Function ϕ

We know $\dfrac{\partial \phi}{\partial x} = -u = -(-2x) = 2x$ ...(*i*)

$\dfrac{\partial \phi}{\partial y} = -v = -2y$ ...(*ii*)

Integrating equation (*i*), we get

$$\int d\phi = \int 2x\,dx$$

or $\phi = \dfrac{2x^2}{2} + C = x^2 + C$ ...(*iii*)

where C is a constant which is independent of x but can be a function of y.

Differentiating equation (*iii*) w.r.t. 'y', we get $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial C}{\partial y}$

But from (*ii*), $\frac{\partial \phi}{\partial y} = -2y$

$\therefore$ $\frac{\partial C}{\partial y} = -2y$

Integrating this equation, we get $C = \int -2y\,dy = -\frac{2y^2}{2} = -y^2$

Substituting this value of C in equation (*iii*), we get $\boldsymbol{\phi = x^2 - y^2}$. **Ans.**

SCHEME OF EVALUATION

Question Number	*Max Marks*	*Split-Up*	*Marks Distribution*
1	10	Diagram	2
		Derivation	8
2(a)	5	Lagrangian	2.5
		Eulerian	2.5
2(b)	5	Velocity Expression	2
		Acceleration Expression	3
3	10	Definition	3
		Diagram	2
		Variation of viscosity with temperature	5
4	10	Steps	2
		Answer	8
5	10	Finding Velocity with resultant	4
		Finding Acceleration with resultant	6
6	10	Proof	3
		Finding Velocity with resultant	3
		Finding Acceleration with resultant	4
7	10	Finding Velocity with resultant	3
		Finding Potential Function	7