CMR

Solutions - IAT-1 18MAT41 19-05-2021

Q1 Derive CR equations in cartesian form; $u_x = v_y$ & $u_y = -v_x$ (statement) $\rightarrow 1$

Proof:- Let $f(z) = u + iv$ be analytic at any point $z = x + iy$

$\because$ $f(z) = u(x,y) + iv(x,y)$ is analytic it is differentiable

$$\Rightarrow f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ exists and is unique} \rightarrow 1$$

But we know $f(z) = u(x,y) + iv(x,y)$ and $\delta z = \delta x + i\delta y$

& $f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$

$$\Rightarrow f'(z) = \lim_{\delta z \to 0} \frac{\{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)\} - \{u(x,y) + iv(x,y)\}}{\delta z}$$

$$\Rightarrow f'(z) = \lim_{\delta z \to 0} \left(\frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta z} \right) + i \lim_{\delta z \to 0} \frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta z} \rightarrow (1) \rightarrow 2$$

$\because$ $\delta z \to 0$ we can have the following possibilities

Case i :- Let $\delta y = 0$ and $\delta x \to 0$ then $\delta z \to 0$ implies $\delta x \to 0$

$$\therefore (1) \Rightarrow f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x \longrightarrow (2) \rightarrow 2$$

Case ii :- Let $\delta x = 0$ and $\delta y \to 0$, then $\delta z \to 0$ implies $i\delta y \to 0$ or $\delta y \to 0$

$$\text{then } (1) \Rightarrow f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

$$\Rightarrow f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = v_y - iu_y \quad \left(\because \frac{1}{i} = \frac{-i^2}{i} = -i \right)$$

$\longrightarrow (3)$ $\rightarrow 1$

from (2) & (3) $f'(z) = u_x + iv_x = v_y - iu_y$

equating real and imaginary parts we get

$$u_x = v_y \text{ and } v_x = -u_y \longrightarrow (1)$$

These are CR eqns in cartesian form.

Q2) Given eqn is $pv^\gamma = k$

$\Rightarrow \log pv^\gamma = \log k \Rightarrow \log p + \log v^\gamma = \log k \Rightarrow \log p + \gamma \log v = \log k$

put $X = \log p$; $Y = \log v$; $a = \gamma$ $b = \log k \longrightarrow (1)$

$\Rightarrow X + aY = b$ The Normal eqns are $\sum X + a \sum Y = 6b$

$\sum X^2 + a \sum XY = b \sum X \longrightarrow (1)$

Table :-

p	v	$X = \log p$	$Y = \log v$	X^2	XY
0.5	1.62	-0.6931	0.4824	0.4804	-0.3344
1.0	1.00	0	0	0	0
1.5	0.75	0.4055	-0.2877	0.1644	-0.1167
2.0	0.62	0.6931	-0.4780	0.4804	-0.3313
2.5	0.52	0.9163	-0.6539	0.8396	-0.5992
3.0	0.46	1.0986	-0.7765	1.2069	-0.8531
		2.4204	-1.7134	3.1717	-2.2347

$\longrightarrow (3)$

So the Normal eqns are $2.4204 + a(-1.7134) = 6b$

$3.1717 + a(-2.2347) = 2.4204b \longrightarrow (1)$

Solving these eqns for a and b and writing γ for a and $k = e^b$ we get $\gamma = \cancel{1.2776}\ 1.4225$ & $k = \cancel{1.1039}\ 0.9970$ $\therefore$ $\boxed{pv^{\cancel{1.2775}\ 1.4225} = \cancel{1.1039}\ 0.997} \longrightarrow (1)$

Q3. The eqn is $R = a + bV + cV^2 \Rightarrow$ Normal eqns are
$\Sigma R = 6a + b\Sigma V + c\Sigma V^2$
$\Sigma RV = a\Sigma V + b\Sigma V^2 + c\Sigma V^3$
$\Sigma RV^2 = a\Sigma V^2 + b\Sigma V^3 + c\Sigma V^4$ → 2

Table

V	R	VR	V^2	V^2R	V^3	V^4
20	5.5	110	400	2200	8000	160000
40	9.1	364	1600	14560	64000	2560000
60	14.9	894	3600	53640	216000	12960000
80	22.8	1824	6400	145920	512000	40960000
100	33.3	3330	10000	333000	1000000	100000000
120	46.0	5520	14400	662400	1728000	207360000
$\Sigma V = 420$	$\Sigma R = 131.6$	$\Sigma VR = 12042$	$\Sigma V^2 = 36400$	$\Sigma V^2R = 12,11,720$	$\Sigma V^3 = 35,28,000$	ΣV^4 364000000 ~~361552000~~

→ 3

The Normal eqns are
$131.6 = 6a + 420b + 36400c$
$12042 = 420a + 36400b + 3528000c$
$1211720 = 36400a + 3528000b + 364000000c$ ~~361552000c~~ → 1

Solving these 3 linear eqns in a, b & c we get.

$a = \cancel{3.148}$ 4.35; $b = +0.00241$; $c = 0.0029$ → 1

Q4. The Correlation Coefficient r is given by $r = \frac{\Sigma XY}{\sqrt{\Sigma X^2}\sqrt{\Sigma Y^2}}$

where $X = x - \bar{x}$ $Y = y - \bar{y}$

For the given data $\bar{x} = 58.57$ → 1

$\bar{y} = 40.28$ → 1

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	X^2	Y^2	XY
55	35	−3.57	−5.28	12.7449	27.8784	18.8496
56	38	−2.57	−2.28	6.6049	5.1984	5.8596
58	38	−0.57	−2.28	0.3249	5.1984	1.2996
59	39	0.43	−1.28	0.1849	1.6384	−0.5504
60	44	1.43	3.72	2.0449	13.8384	5.3196
60	43	1.43	2.72	2.0449	7.3984	3.8896
62	45	3.43	4.72	11.7649	22.2784	16.1896
				35.7143	83.4288	50.8572 → ③

$\bar{x} = \frac{410}{7} = 58.57$ $\quad \Sigma y = 282$ $\quad \bar{y} = \frac{282}{7} = 40.28$

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2}\sqrt{\Sigma Y^2}} \rightarrow ① = \frac{50.8572}{\sqrt{35.7143}\sqrt{83.4288}}$$

$$= \frac{50.8572}{5.9761 \times 9.1339} = \cancel{\frac{50.8572}{5.9761 \times 9.1339}} = \frac{50.8572}{54.5851} = 0.9317 \rightarrow ①$$

Q5 Reg line of y on x is $y = 0.3x + 2.8$; Reg line of x on y is $x = 0.7y + 5.2$

$\Rightarrow b_{yx} = 0.3$ & $b_{xy} = 0.7$ → ① $\Rightarrow r = \sqrt{b_{yx} \times b_{xy}}$ → ① $= \sqrt{0.3 \times 0.7} = +0.458 \approx 0.46$ → ①

$\bar{x}$ & $\bar{y}$ are obtained by solving

$$\left.\begin{aligned} \bar{y} - 0.3\bar{x} &= 2.8 \\ 0.7\bar{y} - \bar{x} &= -5.2 \end{aligned}\right\} \rightarrow ②$$

Solving these two equations for $\bar{x}$ & $\bar{y}$ we get

$$\left.\begin{aligned} \bar{x} &= 9.06 \\ \bar{y} &= 5.52 \end{aligned}\right\} \rightarrow ②$$

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Q6 rank correlation Coefficient $\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$ → ②

Here $n=10$; $\sum d^2 = (3-5)^2 + (8-9)^2 + \cdots (5-6)^2 = 24$ → ③

$\therefore \rho = 1 - \frac{6(24)}{10(10^2-1)} = 1 - \frac{144}{990} = 1 - 0.14545$

$= 0.8545$ → ②

Q7 Given $f(z) = z^n$; To prove f is analytic & to find $\frac{d}{dz}(f(z))$

Let $w = f(z) = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$ → ②

$\Rightarrow u(r,\theta) = r^n \cos n\theta$ & $v(r,\theta) = r^n \sin n\theta$.

$\Rightarrow u_r = nr^{n-1}\cos n\theta$ | $v_r = nr^{n-1}\sin n\theta$

$u_\theta = r^n(-\sin n\theta)n$ | $v_\theta = r^n \cdot \cos n\theta \cdot n$ → ②

$\therefore ru_r = r\,n r^{n-1}\cos n\theta = nr^n \cos n\theta = v_\theta$

$rv_r = r\,nr^{n-1}\sin n\theta = nr^n \sin n\theta = -u_\theta$

$\Rightarrow$ CR eqns are satisfied $\Rightarrow f(z) = z^n$ is analytic → ①

Now $\frac{dw}{dz} = f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(nr^{n-1}\cos n\theta + i\,nr^{n-1}\sin n\theta)$ → ①

$= nr^{n-1}\left[e^{-i\theta}(\cos n\theta + i\sin n\theta)\right]$

$= nr^{n-1}\left[e^{-i\theta}(e^{in\theta})\right]$

$= nr^{n-1}e^{i(n-1)\theta} = n(re^{i\theta})^{n-1} = nz^{n-1}$ → ①

$\Rightarrow f'(z) = nz^{n-1}$

Q8 To find $f(z) = u + iv$ where $f(z)$ is analytic and

$$u(x,y) = x \sin x \cosh y - y \cos x \sinh y$$

$\because f(z) = u + iv$ is analytic u & v satisfy CR eqns $\rightarrow$ (1) ie $u_x = v_y$, $u_y = -v_x$

$$\Rightarrow u_x = \cosh y\,(x \cdot \cos x + \sin x) - y \sinh y(-\sin x)$$

$$\Rightarrow u_x = \cosh y\,(x \cos x + \sin x) + y \sinh y \sin x \rightarrow (1)$$

and $u_y = x \sin x \cdot \sinh y - \cos x(y \cdot \cosh y + \sinh y)$

But we know $u_y = -v_x$

$$\Rightarrow v_x = \cos x\,(y \cosh y + \sinh y) - x \sin x \sinh y \rightarrow (1)$$

Also $f'(z) = u_x + i v_x \rightarrow (1)$

$$= \cosh y(x \cos x + \sin x) + y \sinh y \sin x + i\left(\cos x\,(y \cosh y + \sinh y) - x \sin x \sinh y\right)$$

By Milne-Thompson method put $x = z$ & $y = 0 \rightarrow (1)$

$$\Rightarrow f'(z) = \cosh 0\,(z \cos z + \sin z) + 0 \cdot \sinh 0 \sin z + i\left(\cos z\,(0 \cdot \cosh 0 + \sinh 0) - z \sin z \sinh 0\right)$$

$$= \{1\,(z \cos z) + \sin z\} + i\{\cos z(0) - 0\}$$

$$f'(z) = z \cos z + \sin z \rightarrow (1)$$

$$\Rightarrow f(z) = \int (z \cos z + \sin z)\,dz = \int z \cos z\,dz + \int \sin z\,dz$$

$$= z(\sin z) - (1)(-\cos z) + (-\cos z) + c$$

$$\Rightarrow f(z) = z \sin z + c. \rightarrow (1)$$