30/6/2021

IAT-2 Solution

18DIPMAT41 Additional Mathematics II

1)

Weight less than (x)	40	60	80	100	120
No of students (y)	250	370	470	540	590

The forward difference table is given by

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	250				
		120			
60	370		-20		
		100		-10	
80	470		-30		20
		70		10	
100	540		-20		
		50			
120	590				

$$p = \frac{x - x_0}{h} = \frac{70-40}{20} = \frac{30}{20} = 1.5$$

$\Delta y_0 = 120$, $\Delta^2 y_0 = -20$, $\Delta^3 y_0 = -10$, $\Delta^4 y_0 = 20$.

The Newton's forward interpolation formula is,

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$f(70) = 250 + 1.5\,(120) + \frac{1.5\,(1.5-1)}{2}\,(-20) + \frac{(1.5)(1.5-1)(1.5-2)}{6}\,(-10) + \frac{(1.5)\,(1.5-1)\,(1.5-2)\,(1.5-3)}{24}\,(20)$$

$$= 250 + 180 - 7.5 + 0.625 + 0.46875$$

$$= 423.59375$$

The number of students with weight less than 70 is 424. But the number of students with weight less than 60 is 370.

Hence number of students with weight between 60 and 70 is $424 - 370 = \underline{\underline{54}}$

2) $\dfrac{\partial^2 z}{\partial x \partial y} = e^{-2y} \cos 3x$ ——— ① , $\dfrac{\partial z}{\partial y} = 0$ when $x = 0$

& $z = 0$ when $y = 0$

Integrate ① wrt 'x'.

$$\int \frac{\partial^2 z}{\partial x \partial y} dx = \int e^{-2y} \cos 3x \, dx$$

$$\frac{\partial z}{\partial y} = e^{-2y} \frac{\sin 3x}{3} + f(y) \text{ ——— ②}$$

Given that $\dfrac{\partial z}{\partial y} = 0$ when $x = 0$

$$② \Longrightarrow 0 = e^{-2y} \frac{\sin 0}{3} + f(y)$$

$$\Longrightarrow f(y) = 0 \text{ ——— ③}$$

③ in ② gives $\dfrac{\partial z}{\partial y} = e^{-2y} \dfrac{\sin 3x}{3}$ ——— ④

Integrating ④ wrt y,

$$\int \frac{\partial z}{\partial y} dy = \frac{\sin 3x}{3} \int e^{-2y} dy$$

$$z = \frac{\sin 3x}{3} \frac{e^{-2y}}{(-2)} + g(x)$$

$$z = -\frac{\sin 3x}{6} e^{-2y} + g(x) \quad \text{——— (7)}$$

Given that $z=0$ when $y=0$

$$0 = -\frac{\sin 3x}{6} e^{0} + g(x)$$

$$\Rightarrow g(x) = \frac{\sin 3x}{6} \quad \text{——— (8)}$$

(8) in (7) $\Rightarrow$

$$z = -\frac{\sin 3x}{6} e^{-2y} + \frac{\sin 3x}{6}$$

$$z = \frac{\sin 3x}{6}\left(1 - e^{-2y}\right)$$ is the solution.

3) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, $y=0$, $y'=3$ for $x=0$.

$(D^2+D-2)y = 0$

The auxiliary equation is given by

$m^2+m-2=0$

$m^2+2m-m-2=0$

$m(m+2) - 1(m+2) = 0$

$m = 1$ and $m = -2$.

$\therefore$ The general solution is given by

$y = Ae^{-2x} + Be^{x}$ ——— (1)

Given that $y = 0$ for $x = 0$

$0 = Ae^{0} + Be^{0}$

$A + B = 0$ ——— (2)

From (1), $y' = Ae^{-2x}(-2) + Be^{x}$

$3 = -2A + B$ ——— (3)

Solving (2) & (3),

$$\begin{array}{r} A + \cancel{B} = 0 \\ -2A + \cancel{B} = 3 \\ (+) \quad (-) \quad (-) \\ \hline 3A = -3 \end{array} \Rightarrow A = -1 \Rightarrow B = 1$$

$\therefore y = -e^{-2x} + e^{x}$ is the solution.

4) $y'' - 4y' + 13y = e^{3x}\cosh 2x + 2^x$

The auxiliary equation is given by

$$(m^2 - 4m + 13) = 0.$$

$$m = \frac{4 \pm \sqrt{16 - 4(13)}}{2(1)} = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

$$\therefore y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

$$y_p = \frac{e^{3x}\cosh 2x + 2^x}{D^2 - 4D + 13}$$

$$= e^{3x}\left(\frac{e^{2x} + e^{-2x}}{2}\right)\frac{1}{D^2 - 4D + 13} + \frac{2^x}{D^2 - 4D + 13}$$

$$= \frac{e^{5x} + e^{x}}{2(D^2 - 4D + 13)} + \frac{(e^{\log 2})^x}{D^2 - 4D + 13}$$

$$= \frac{e^{5x}}{2(18)} + \frac{e^{x}}{2(10)} + \frac{e^{\log 2x}}{(\log 2)^2 - 4(\log 2) + 13}$$

$$= \frac{e^{5x}}{36} + \frac{e^{x}}{20} + \frac{e^{\log 2x}}{(\log 2)^2 - 4\log 2 + 13}$$

$$\therefore y = e^{2x}\left(c_1 \cos 3x + c_2 \sin 3x\right) + \frac{e^{5x}}{36} + \frac{e^{x}}{20}$$

$$+ \frac{e^{\log 2x}}{(\log 2)^2 - 4\log 2 + 13}.$$

5) $(D^2+1)y = \sin x \sin 2x$

$m^2+1 = 0 \implies m = \pm i$

$$y_c = A\cos x + B\sin x$$

$$y_p = \frac{\cos(2x-x) - \cos 3x}{2} = \frac{\cos x - \cos 3x}{2}$$

$$\therefore y_p = \frac{\cos x}{2(D^2+1)} - \frac{\cos 3x}{2(D^2+1)}$$

$$= \frac{x}{4(1)}\sin x - \frac{\cos 3x}{2(-9+1)}$$

$$= \frac{x}{4}\sin x + \frac{\cos 3x}{16}$$

$$\therefore y = A\cos x + B\sin x + \frac{x}{4}\sin x + \frac{\cos 3x}{16}.$$

AE is $m^2+3m+2=0 \Rightarrow (m+1)(m+2)=0 \Rightarrow m=-1,-2$, $CF = c_1e^{-x}+c_2e^{-2x}$

$$PI = \frac{1}{D^2+3D+2}\cos^2 x = \frac{1}{D^2+3D+2}\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+3D+2}e^{0x}\right]+\frac{1}{2}\left[\frac{1}{D^2+3D+2}\cos 2x\right]$$

$$= \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\left[\frac{1}{-4+3D+2}\cos 2x\right] = \frac{1}{4}+\frac{1}{2}\left[\frac{1}{3D-2}\cos 2x\right]$$

$$= \frac{1}{4}+\frac{1}{2}\left[\frac{3D+2}{9D^2-4}\cos 2x\right] = \frac{1}{4}+\frac{1}{2}\left[\frac{(3D+2)\cos 2x}{9(-4)-4}\right]$$

$$= \frac{1}{4}-\frac{1}{80}\left[-6\sin 2x+2\cos 2x\right] = \frac{1}{4}+\frac{1}{40}\left[3\sin 2x-\cos 2x\right]$$

Therefore the complete solution of the given equation is $y = C.F + P.I$

$$y = c_1e^{-x}+c_2e^{-2x}+\frac{1}{4}+\frac{1}{40}\left[3\sin 2x-\cos 2x\right]$$

Example 9 : Find the cubic polynomial which takes the following values. Hence or otherwise evaluate f(4)

x	0	1	2	3
f(x)	1	2	1	10

Solution : The difference table is

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$y_0 = 1$			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
$x_n = 3$	$y_n = 10$			

$h = 1, \quad x_0 = 0 \quad , \quad p = \frac{x-0}{h} = x$

using newton's forward interpolation formula we get,

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}$$

$$\text{i.e } f(x) = 1 + x.1 + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{6}\times 12 \;=\; 2x^3 - 7x^2 + 6x + 1.$$

Which is the required polynomial. To compute f(4) we take

$x_n = 3, \quad x = 4$ so that $p = \frac{x - x_n}{h} = \frac{4-3}{1} = 1$

using Newton's backward interpolation formula, we get

$$f(4) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$= 10 + 1(9) + \frac{1(1+1)}{2}(10) + \frac{1(1+1)(1+2)}{6}(12) = 41$$

verification : we have $f(x) = 2x^3 - 7x^2 + 6x + 1$, put $x = 4$, we get

$f(4) = 128 - 7(16) + 24 + 1 = 128 - 112 + 25 = 41$

8) $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

Integrating wrt x,

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{18x^2y^2}{2} + \frac{\cos(2x-y)(-1)}{2} = f(y)$$

Integrating again wrt x,

$$\frac{\partial z}{\partial y} + \frac{18x^3y^2}{6} - \frac{\sin(2x-y)}{4} = xf(y) + g(y)$$

Integrating wrt y,

$$z + \frac{3x^3y^3}{3} - \frac{\cos(2x-y)}{4} = xf'(y) + g'(y) + h(x).$$

$\therefore$

$$z + x^3y^3 - \frac{\cos(2x-y)}{4} = xf'(y) + g'(y) + h(x)$$

is the required solution.