USN :

CMR Institute of Technology, Bangalore
DEPARTMENT OF MECHANICAL ENGINEERING
II - INTERNAL ASSESSMENT

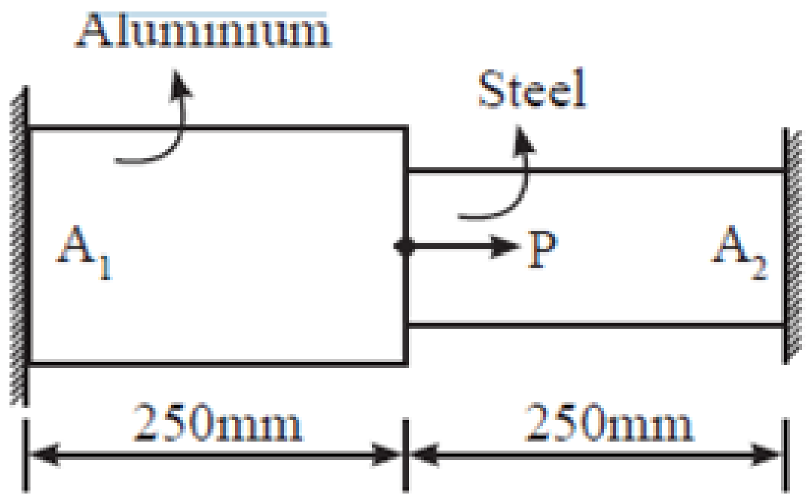
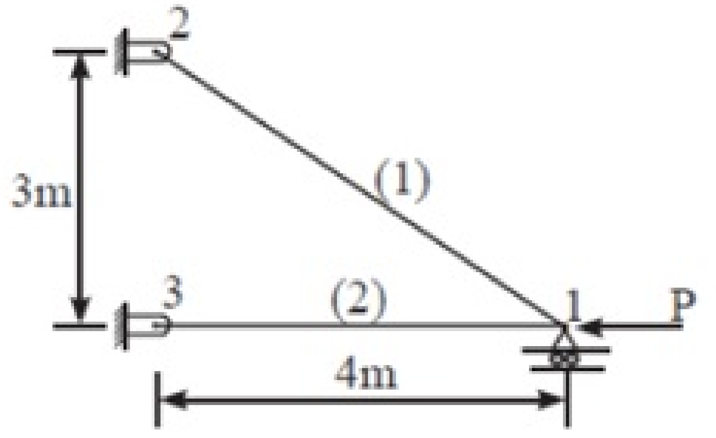
Semester: 6-CBCS 2018
Subject: FINITE ELEMENT METHODS (18ME61)
Faculty: Mr Prashanth Hatti

Date: 22 Jun 2021
Time: 09:00 AM - 10:00 AM
Max Marks: 50

Instructions to Students :

Answer all questions

Answer All Questions

Q.No			Marks	CO	PO	BT/CL
1	a	What are the properties of shape function?	6	CO3	PO1,PO2,PO3,PO9	L1
	b	Determine the nodal displacements, stress and reaction forces for the bar shown in figure, using penalty approach of handling boundary condition P = 4000 N, $A_1 = 1600mm^2$ $A_2 = 800mm^2$, $E_{Al} = 80GPa, E_{St} = 210GPa$ 	14	CO4	PO1,PO2,PO3,PO9,PO12	L3
2	a	Evaluate the integral $\int_{-2}^{3} (x^2 + 11x - 32)\, dx$ Using two point and three point Gauss Quadrature method.	6	CO3	PO1,PO2,PO9	L2
	b	For the two-bar truss shown in fig. Determine the nodal displacements element stresses and support reactions. A force of P = 1000 KN is applied at node 1. Assume E = 210GPa and $A = 600mm^2$ for each element 	14	CO4	PO1,PO2,PO3,PO9,PO12	L3
3		Derive an expression for stiffness matrix for one dimensional bar element	10	CO3	PO1,PO2,PO3,PO9	L2

1.a

PROPERTIES OF SHAPE FUNCTION

1. The Value of Shape function at the Corresponding node is unit & at all other nodes are zero

$$N_1 = \frac{x_2 - x}{x_2 - x_1}$$

At node 1, $x = x_1$, $N_1 = 1$, $N_2 = 0$

At node 2, $x = x_2$, $N_2 = 1$, $N_1 = 0$

2. Sum of Shape function is unity.

$$N_1 + N_2 = 1.$$

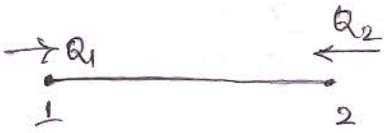
$$\frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1} = 1$$

3. For a 1D element, derivative of a shape function is a Constant.

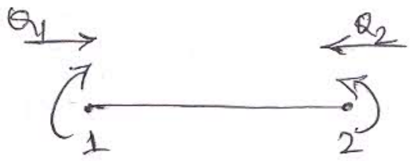
$$\frac{dN_1}{dx} = -\frac{1}{x_2 - x_1} = -\frac{1}{l_e}$$

4. Shape function are generally derived in natural Coordinates.

5. No. of Shape function for an element is equal to the no. of nodes in the element provided it has only one DOF.

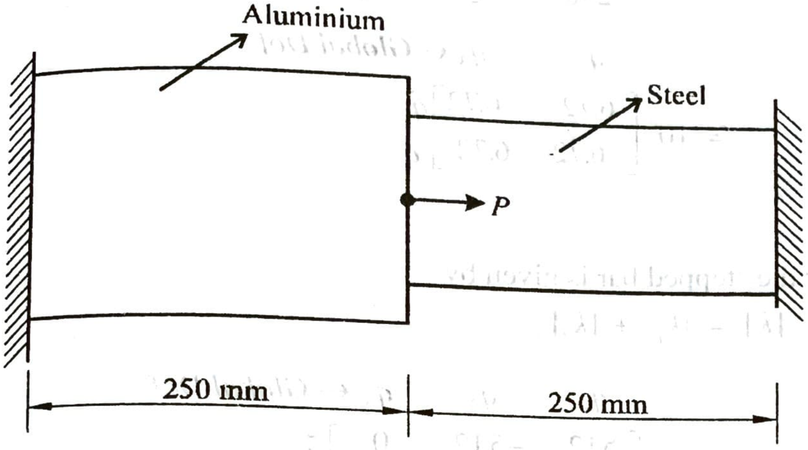


This rule holds good



This rule is not valid.

1.b



$P = 4000$ N

$A_1 = 1600$ mm^2

$A_2 = 800$ mm^2

$E_{Al} = 80$ GPa

$E_{steel} = 210$ GPa

Fig. E 4.6

Solution :

(i) FE model

The FE model of the stepped bar is as shown in figure 4.6a.

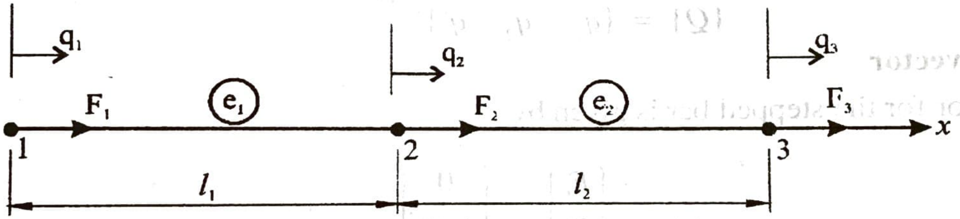


Fig. E 4.6a

Comparing both figures

Data: $A_1 = 1600$ mm^2; $A_2 = 80$ mm^2 ; $E_1 = 80$ GPa $= 80 \times 10^3$ MPa ; $E_2 = 210$ GPa $= 2.1 \times 10^5$ MPa; $l_1 = 250$ mm ; $l_2 = 250$ mm ; $F_2 = 4000$ N

(ii) Elemental stiffness matrix

The stiffness matrix for the bar element is given by

$$[k_e] = \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For element 1

$$[k_1] = \frac{1600 \times 80 \times 10^3}{250} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \\ \end{array}\!\!\begin{array}{cc|c} q_1 & q_2 & \\ \left[\begin{matrix} 5.12 \\ -5.12 \end{matrix}\right. & \left.\begin{matrix} -5.12 \\ 5.12 \end{matrix}\right] & \begin{matrix} q_1 \\ q_2 \end{matrix} \end{array}$$

For element 2

$$[k_2] = \frac{800 \times 2.1 \times 10^5}{250}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{array}{cc} q_2 & q_3 \leftarrow \textit{Global Dof} \end{array}$$

$$= 10^5 \begin{array}{c} \\ \end{array}\left[\begin{array}{cc} 6.72 & -6.72 \\ -6.72 & 6.72 \end{array}\right]\begin{array}{l} \downarrow \\ q_2 \\ q_3 \end{array}$$

(iii) Global stiffness matrix

The global stiffness matrix for the stepped bar is given by

$$[k] = [k_1] + [k_2]$$

$$\begin{array}{ccc} q_1 & q_2 & q_3 \leftarrow \textit{Global Dof} \end{array}$$

i.e.,

$$[k] = 10^5 \left[\begin{array}{ccc} 5.12 & -5.12 & 0 \\ -5.12 & 11.84 & -6.72 \\ 0 & -6.72 & 6.72 \end{array}\right]\begin{array}{l} \downarrow \\ q_1 \\ q_2 \\ q_3 \end{array}$$

Global Nodal displacement vector

The Nodal displacement vector for the stepped bar is given by

$$\{Q\} = \{q_1 \quad q_2 \quad q_3\}^T$$

Global load vector

The load vector for the stepped bar is given by

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4000 \\ 0 \end{Bmatrix}$$

(iv) Equilibrium condition

The equilibrium condition for the stepped bar is given by

$$[k]\ \{Q\} = \{F\}$$

i.e.,

$$10^5 \begin{bmatrix} 5.12 & -5.12 & 0 \\ -5.12 & 11.84 & -6.72 \\ 0 & -6.72 & 6.72 \end{bmatrix}\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4000 \\ 0 \end{Bmatrix}$$

(iv) Applying Boundary Condition

From figure (E4.6) boundary conditions are $q_1 = q_3 = 0$ (fixed ends). Using Penalty method of handling boundary conditions, i.e.,

$$C = \max |k_{ij}| \times 10^4$$

$$\therefore \quad C = 11.84 \times 10^5 \times 10^4 = 11.84 \times 10^9 \text{ N/mm}$$

This value of C is to be added to the k_{11} and k_{33} where nodes are fixed.

∴ Global stiffness matrix becomes

$$[k] = 10^5 \begin{bmatrix} 118405.12 & -5.12 & 0 \\ -5.12 & 11.84 & -6.72 \\ 0 & -6.72 & 118406.72 \end{bmatrix}$$

∴ Global force vector changes to

$$\{F\} = \begin{Bmatrix} F_1 + Ca_1 \\ F_2 \\ F_3 + Ca_3 \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ F_2^1 \\ F_3^1 \end{Bmatrix}$$

Where, a_1 and a_3 are the spring displacement and equal to q_1, q_3 respectively. But $q_1 = q_3 = 0$ (fixed support).

$$\therefore \quad a_1 = a_3 = 0$$

$$\therefore \quad \{F\} = \begin{Bmatrix} 0 \\ 4000 \\ 0 \end{Bmatrix}$$

Thus, equilibrium equation can be written as,

$$10^5 \begin{bmatrix} 118405.12 & -5.12 & 0 \\ -5.12 & 11.84 & -6.72 \\ 0 & -6.72 & 118406.72 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4000 \\ 0 \end{Bmatrix}$$

$$q_1 = 1.46093 \times 10^{-7} \text{ mm}$$

$$q_2 = 3.3785 \times 10^{-3} \text{ mm}$$

$$q_3 = 1.91745 \times 10^{-7} \text{ mm}$$

∴ The nodal displacement vector

$$\{Q\} = \{1.46093 \times 10^{-7} \quad 3.3785 \times 10^{-3} \quad 1.91745 \times 10^{-7}\}^T \text{ mm}$$ **Answer**

(vi) Stress in each element (member)

Stress in a bar element is given by

$$\sigma = \frac{E_e}{l_e} [-1 \quad 1] \begin{Bmatrix} q_i \\ q_{i+1} \end{Bmatrix} \qquad \text{where } i = e$$

For element 1

$$\sigma_1 = \frac{80\times10^3}{250}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}1.46093\times10^{-3}\\3.3785\times10^{-3}\end{Bmatrix}\begin{matrix}q_1\\q_2\end{matrix} = 1.081\ \text{MPa}$$ **Answer**

For element 2

$$\sigma_2 = \frac{2.1\times10^5}{250}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}3.3785\times10^{-3}\\1.91745\times10^{-7}\end{Bmatrix}\begin{matrix}q_2\\q_3\end{matrix} = 2.8378\ \text{MPa}$$ **Answer**

(vii) Reaction forces

The reaction forces at the supports is given by

$$R_1 = -Cq_1 = -1729.74\ \text{N}$$
$$R_3 = -Cq_3 = -2270.26\ \text{N}$$

2.a

Evaluate the integral: $I=\int_{-2}^{3}\left(x^2+11x-32\right)dx$ **using one, two and three point gauss Quadrature. Also, find the exact solution for comparison of accuracy**

Solution:

The existing limits of integration should be changed from [–2, +3] to [–1, +1].

Given $a=-2$; $b=3$

Mapping function, $$x = \frac{(b-a)}{2}\xi+\frac{(b+a)}{2}=\frac{(5\xi+1)}{2}\ ;\ dx=2.5d\xi\ ;$$

Thus, the given equation can be written as

$$I=\int_{-2}^{3}\left(x^2+11x-32\right)dx = 2.5\int_{-1}^{1}\left(\left(\frac{(5\xi+1)}{2}\right)^2+11\left(\frac{(5\xi+1)}{2}\right)-32\right)d\xi$$

(i) Exact Solution: $I_{\text{exact}} = -120.83333$

(ii) One Point Formula:

$$I = \int_{-1}^{1}f(\xi)d\xi=w_1 f(\xi_1)$$

For one point formula in Gauss Quadrature integration, weight, $w_1=2$, and Gauss Point, $\xi_1=0$.

Thus,

$$I_1 = 2\times2.5\times\left[\left(\frac{(5\times0+1)}{2}\right)^2+11\left(\frac{(5\times0+1)}{2}\right)-32\right]=-131.25$$

Thus, **Percentage of error** = 8.62%

(iii) Two Point Formula:

Here, for two point formula in Gauss Quadrature integration, weights and Gauss points are

$$w_1=w_2=1,\ \xi_1=\frac{1}{\sqrt{3}},\ \xi_2=-\frac{1}{\sqrt{3}}.$$

Thus,

$$I = \int_{-1}^{1}f(\xi)d\xi=w_1 f(\xi_1)+w_2 f(\xi_2)$$

$$I_2 = 1\times 2.5\times\left[\left(\frac{\left(5\times\frac{1}{\sqrt{3}}+1\right)}{2}\right)^2 + 11\left(\frac{\left(5\times\frac{1}{\sqrt{3}}+1\right)}{2}\right) - 32\right]$$

$$+1\times 2.5\times\left[\left(\frac{\left(5\times\frac{-1}{\sqrt{3}}+1\right)}{2}\right)^2 + 11\left(\frac{\left(5\times\frac{-1}{\sqrt{3}}+1\right)}{2}\right) - 32\right]$$

$$= -120.83325$$

Thus, **Percentage of error** $= 6.62 \times 10^{-05}$

(iv) Three Point Formula:

For three point formula in Gauss Quadrature integration, weights and Gauss points are $w_1 = 0.00089,\ \xi_1 = 0\ ;\ w_2 = 0.5556, \xi_2 = +0.7746, w_3 = 0.5556, \xi_3 = --0.7746$

$$I = \int_{-1}^{1} f(\xi)d\xi = w_1.f(\xi_1) + w_2.f(\xi_2) + w_3.f(\xi_3)$$

$$I_3 = 0.8889\times 2.5\times\left[\left(\frac{(50+0+)}{2}\right)^2 + 11\left(\frac{(5\times 0+1)}{2}\right) - 32\right]$$

$$+0.5556\times 2.5\times\left[\left(\frac{(5\times 0.7746+1)}{2}\right)^2 + 11\left(\frac{(5\times 0.7746+1)}{2}\right) - 32\right]$$

$$+0.5556\times 2.5\times\left[\left(\frac{(5\times(-0.7746)+1)}{2}\right)^2 + 11\left(\frac{(5\times(-0.7746)+1)}{2}\right) - 32\right]$$

$$I_3 = 2.5\times(-23.3336 + 0.4100 - 25.4120) = -120.839$$

Thus, **Percentage of error** $= 4.69 \times 10^{-03}$. However, difference of results will approach to zero, if few more digits after decimal points are taken in calculation.

2.b

Nodal Data

Node No.	x (mm)	y (mm)
1	4000	0
2	0	3000
3	0	0

Element Connectivity Table

Element NO	Initial Node	Final Node	length of element l_e (mm)	l	m
1	2	1	5000	−0.8	−0.6
2	1	3	4000	−1	0

Element Stiffness matrix

$$K = \frac{EA}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

For element 1

$$K_1 = \frac{210\times 10^3 \times 600}{5000}\begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

$$K_1 = 10^3 \begin{array}{c} \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 16.128 & -12.096 & -16.12 & 12.096 \\ -12.096 & 9.072 & 12.096 & -9.072 \\ -16.128 & 12.096 & 16.128 & -12.096 \\ 12.096 & -9.072 & -12.096 & 9.072 \end{array} \right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

For element 2

$$K_2 = \frac{210 \times 10^3 \times 600}{4000} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 10^3 \begin{array}{c} \\ \left[\begin{array}{cccc} 1 & 2 & 5 & 6 \\ 31.5 & 0 & -31.5 & 0 \\ 0 & 0 & 0 & 0 \\ -31.5 & 0 & 31.5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$$

Global stiffness matrix

$$K = 10^3 \begin{array}{c} \\ \left[\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 47.628 & -12.096 & -16.128 & 12.096 & -31.5 & 0 \\ -12.096 & 9.072 & 12.096 & -9.072 & 0 & 0 \\ -16.128 & 12.096 & 16.128 & -12.096 & 0 & 0 \\ 12.096 & -9.072 & -12.096 & 9.072 & 0 & 0 \\ -31.5 & 0 & 0 & 0 & 31.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Equilibrium eqn.

$$[K][q] = [F]$$

$$10^3 \begin{bmatrix} 47.628 & -12.096 & -16.128 & 12.096 & -31.5 & 0 \\ 12.096 & 9.072 & 12.096 & -9.072 & 0 & 0 \\ -16.128 & 12.096 & 16.128 & -12.096 & 0 & 0 \\ 12.096 & -9.072 & -12.096 & 9.072 & 0 & 0 \\ -31.5 & 0 & 0 & 0 & 31.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} -10^6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Nodal displacement

$$q_1 = -20.996 \text{ mm} //$$

$$q_2 = q_3 = q_4 = q_5 = q_6 = 0 //$$

Stress

$$\sigma = \frac{E}{l_e} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$= \frac{210 \times 10^3}{5000} \begin{bmatrix} 0.8 & -0.6 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} -20.996 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_1 = -705.46 \text{ N/mm}^2 //$$

$$\sigma_2 = \frac{210\times10^3}{4000}\begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}-20.996\\0\\0\\0\end{bmatrix}$$

$$= -1102.96\ N/mm^2$$

Reaction

$$R_2 = K_{21}\,q_1 = -12.096\times10^3\times -20.996 = 253.96\ kN$$

$$R_3 = K_{31}\,q_1 = -16.126\times10^3\times -20.996 = 338.62\ kN$$

$$R_4 = K_{41}\,q_1 = -253.96\ kN$$

$$R_5 = K_{51}\,q_1 = -31.5\times10^3\times -20.996 = 661.37\ kN$$

3

DERIVATION OF STIFFNESS MATRIX

Strain Energy for 3D element is given by

$$U_e = \frac{1}{2} \int_V \sigma^T \epsilon \cdot dV$$

Strain Energy for 1D element is given by

$$U_e = \frac{1}{2} \int_{l_e} \sigma^T \epsilon \cdot A \cdot dx \quad \rightarrow ①$$

Strain is given by $\epsilon = B \cdot q$

Where $B \rightarrow$ Strain Displacement matrix

$$B = \frac{1}{l_e} [-1 \quad 1]$$

$q \rightarrow$ Nodal displacement

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Using Hooke's law

$$\frac{\sigma}{\epsilon} = E \Rightarrow \sigma = E \epsilon = E B \cdot q$$

Substituting this in eqn ① we get

$$U_e = \frac{1}{2} \int_{l_e} [E B q]^T \cdot B \cdot q \cdot A \cdot dx$$

$$U_e = \frac{1}{2} \int_{l_e} q^T B^T \cdot E \cdot B \cdot q \cdot A \cdot dx$$

Relation b/w natural & Cartesian Coordinate is

$$\xi = \frac{2(x - x_1)}{x_2 - x_1} - 1$$

$$\frac{d\xi}{dx} = \frac{2}{x_2 - x_1} = \frac{2}{l_e}$$

$$dx = \frac{l_e}{2} \, d\xi.$$

$$\therefore \; U_e = \frac{1}{2} \int_{-1}^{+1} q^T B^T E . B . q . A . \frac{l_e}{2} \, d\xi$$

$$= \frac{1}{2} q^T \left[\frac{EAl_e}{2} \int_{-1}^{+1} B^T . B . d\xi \right] q$$

$$U_e = \frac{1}{2} q^T K q$$

Stiffness matrix for an element

$$K = \frac{EAl_e}{2} \int_{-1}^{+1} B^T . B . \, d\xi$$

$$B = \frac{1}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \quad ; \quad B^T = \frac{1}{l_e} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$K = \frac{EAl_e}{2} \int_{-1}^{+1} \frac{1}{l_e} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{1}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} d\xi$$

$$= \frac{EAl_e}{2l_e^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{-1}^{+1} d\xi$$

$$= \frac{EAl_e}{2l_e^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left[\xi \right]_{-1}^{+1}$$

$$= \frac{EA}{2l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left[1 - (-1) \right]$$

$$\boxed{K = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \; //$$