IAT - II

18MAT 41 - Solution

CMR

2 Marks

1) If $f(z) = u + iv$ is analytic then both u and v are harmonic

(Harmonic property)

(option c)

2) If $f(z) = u + iv$ then which of the following is true

option - a — If $f(z)$ is analytic then u & v are orthogonal

(by using orthogonal property, here converse is not true)

3) If a transformation is conformal then

option c → Both Magnitude and sense are preserved

(Definition of Conformal transformation)

4) $y = -c$ (a constant) under the transformation $w = z^2$ is transformed in to a parabola symmetrical about the real axis with vertex $(-c_1^2, 0)$

(option - b) → Refer discussion of $w = z^2$

5) The straight line parallel to y axis in z plane maps onto a circle with center origin and radius r in w-plane under the transformation

$(w = e^z)$ $\boxed{\text{option\_b}}$ (here parallel to Y axis Means $x = c$ constant)

6) A circle with centre zero & radius r mapped in to ellipse with foci $(\pm 2, 0)$ under the transformation $w = z + \frac{1}{z}$

$\boxed{\text{(option-b)}}$

7) A circle with center zero & radius r mapped in to circle with centre origin & radius r^2 under the transformation $w = z^2$

$\boxed{\text{Option C}}$

8) The straight line $y=c$ in z plane maps on to a straight line passing through origin in w plane under the transformation $w=e^z$

$\boxed{\text{option - b}}$

9) The harmonic property in polar form is

$\boxed{\text{Ans d-}}$ None of the above

(Reason → If u is harmonic in polar then formula is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0)$$

10) Which of the following is false?

$\boxed{\text{option - b}}$ Bilinear transformations are not conformal if $ad-bc \neq 0$ is false

Reason : By definition Bilinear transformations are conformal if $ad-bc \neq 0$.

11) $\phi = u^2 + v^2$

$\phi_x = 2uu_x + 2vv_x = 2(uu_x + vv_x)$

$\phi_{xx} = 2(u u_{xx} + u_x^2 + v v_{xx} + v_x^2)$ — ①

Similarly $\phi_{yy} = 2(u u_{yy} + u_y^2 + v v_{yy} + v_y^2)$ — ②

adding ① & ② $\phi_{xx} + \phi_{yy} = 2(u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy})$

$+ u_x^2 + v_x^2 + u_y^2 + v_y^2)$

Since $f(z)$ is analytic, u & v are harmonic

$\Rightarrow u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$ & $u_x = v_y$ & $u_y = -v_x$

(CR equations)

$\therefore \phi_{xx} + \phi_{yy} = 2(2u_x^2 + 2v_x^2) = 4(u_x^2 + v_x^2)$

—x—

$\boxed{\phi_{xx} + \phi_{yy} = 4|f'(z)|^2}$ option c

12) $u = y + e^x \cos y$

$u_x = e^x \cos y$ & $u_y = 1 - e^x \sin y$

$u_x = v_y \Rightarrow v_y = e^x \cos y$

Integrating w.r.to y $v = e^x \int \cos y \, dy + f(x)$

$= e^x \cdot \sin y + f(x)$ — ①

$u_y = -v_x \Rightarrow v_x = e^x \sin y - 1$

Integrating w.r.to x ~~$v = \sin y e^x$~~ $v = \int (\sin y e^x - 1) dx + g(y)$

$v = \sin y e^x - x + g(y)$ — ②

Comparing ① & ② $\boxed{v = e^x \sin y - x + c}$ option c

$g(y) = 0$; $f(x) = -x$.

14) $w = \frac{1+iz}{1-iz}$ To find image of $|z|<1$

$w(1-iz) = (1+iz) \Rightarrow z = i\frac{1-w}{1+w}$

If $|z|<1 \Rightarrow |i|\left|\frac{1-w}{1+w}\right|<1$

$\Rightarrow |1-w|^2 < |1+w|^2$

$= |1-(u+iv)|^2 < |1+(u+iv)|^2$

$\Rightarrow (1-u)^2 + v^2 < (1+u)^2 + v^2$ (expanding by ~~taking~~ real & Imaginary)

$\Rightarrow 1+u^2-2u+v^2 < 1+u^2+2u+v^2$

$\Rightarrow -4u < 0 \Rightarrow \boxed{u>0}$ $\boxed{\text{option c}}$

13) $u = \frac{\cos 2\theta}{r^2}$, $r \neq 0$ is harmonic since.

$\boxed{\text{option A}}$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Proof: $\frac{\partial u}{\partial r} = -\frac{2}{r^3}\cos 2\theta$; $\frac{\partial u}{\partial \theta} = -\frac{2}{r^2}\sin 2\theta$

$\frac{\partial^2 u}{\partial r^2} = \frac{6}{r^4}\cos 2\theta$; $\frac{\partial^2 u}{\partial \theta^2} = \left(-\frac{4}{r^2}\right)\cos 2\theta$.

$$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{6}{r^4}\cos 2\theta - \frac{2}{r^4}\cos 2\theta - \frac{4}{r^4}\cos 2\theta$$

$= 0$ Hence Harmonic

(15) Find the Bilinear Transformation which maps the points $0, 1, \infty$ onto the points $-5, -1, 3$ respectively

Soln: Here $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$, so that $1/z_3 = 0$ and $w_1 = -5$, $w_2 = -1$ and $w_3 = 3$

The Reqd transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(z-z_1)(z_2/z_3-1)}{(z/z_3-1)(z_2-z_1)}$$

Substituting z_1, z_2, $1/z_3$ and w_1, w_2, w_3 this yields

$$\frac{(w+5)(-4)}{(w-3)(4)} = \frac{(z-0)(-1)}{(-1)(1-0)} \quad \text{or} \quad -\frac{w+5}{w-3} = z$$

or $w - 5 = -z(w-3)$

$$\boxed{w = \frac{3z-5}{z+1}}$$

Ans- B
L3
CO2

(18) If the Bilinear Transformation is $w = \frac{1-z}{z+1}$ what are the Invariant points

Soln:

$$w = \frac{1-z}{z+1}$$

The invariant points got by setting $w = z$ in the transformation

$$z = \frac{1-z}{1+z} \quad \text{or} \quad z(1+z) = 1-z$$

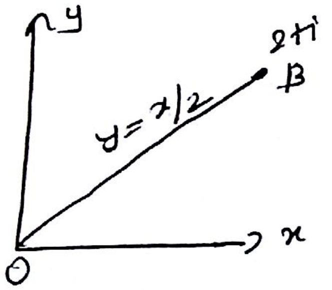
(or) $z^2 + 2z - 1 = 0$

$$z = -\frac{2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

Ans A
CO2
L3

(16) Evaluate $I = \int_{0}^{2+i} (\bar{z})^2 dz$ along the Straight line $y = x/2$

Sol: The parametric equation of the line $y = x/2$ may be taken as $x = 2t$, $y = t$, so that $z = x + iy = (2+i)t$. As z varies from 0 to $2+i$ the parameter t varies from 0 to 1.



$\therefore$ along the line $y = x/2$, from origin $(z=0)$ to the point $B(z = 2+i)$

$$I = \int_{z=0}^{2+i} (\bar{z})^2 dz = \int_{z=0}^{2+i} (x - iy)^2 (dx + i\,dy)$$

$$= \int_{t=0}^{1} (2t - it)^2 (2dt + i\,dt)$$

$$= (2-i)^2 (2+i) \int_{t=0}^{1} t^2 dt$$

$$= (2-i)^2 (2+i) \left. \frac{t^3}{3} \right|_0^1 =$$

$$\boxed{I = \int_{z=0}^{2+i} (\bar{z})^2 dz = \frac{1}{3}(10 - 5i) = \frac{5}{3}(2 - i)}$$

Ans-B
L3
Co2

(17) Evaluate $I = \int_C |z|^2 dz$ where C is the line Joining the points $(1,1)$ to $(0,1)$

Soln: $z = x+iy \Rightarrow |z| = \sqrt{x^2+y^2} \Rightarrow |z|^2 = x^2+y^2$

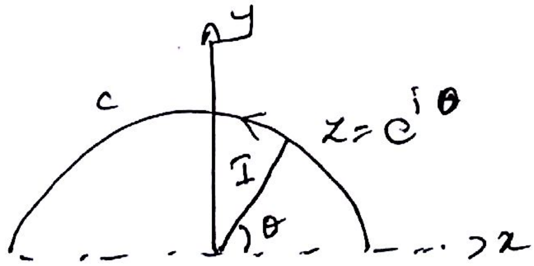
$\int_C |z|^2 dz = \int_1^0 (x^2+1)\, dx$ along $(1,1)$ to $(0,1)$ when $y=1$, $1 \le x \le 0$, $dy=0$

$= -\left[x^3/3 + x \right]_1^0 = -\left[1/3 + 1 \right] = -4/3$

$\boxed{\text{Ans-b. CO2 L3}}$

(18) Evaluate $\int_C (z-z^2)\,dz$ where C is the upper half of the $|z|=1$ where the angle increasing from 0 to π

Soln: The given curve C is the upper-half of the circle $|z|=1$. Therefore on C, we have $z = re^{i\theta}$ with $r=1$ and θ increasing from 0 to π.



$\int_C (z-z^2)\,dz = \int_0^\pi (e^{i\theta} - e^{2i\theta})\, ie^{i\theta} d\theta = i\int_0^\pi (e^{i2\theta} - e^{3i\theta})\, d\theta$

$= i\left[\frac{1}{2i}\left[e^{2\pi i} - 1\right] - \frac{1}{3i}(e^{3\pi i} - 1) \right]$

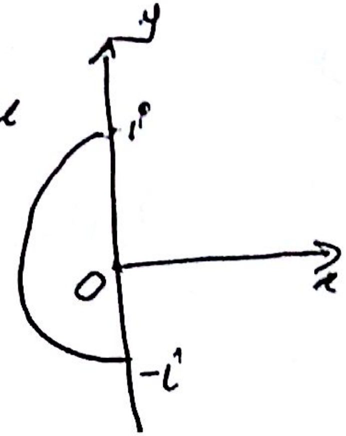
$= \frac{1}{2}(\cos 2\pi + i\sin 2\pi - 1) - \frac{1}{3}(\cos 3\pi + i\sin 3\pi - 1)$

$\int_C (z-z^2)\,dz = \frac{2}{3}$

$\boxed{\text{Ans } 2/3 \text{ CO2 L3}}$

(20) Evaluate $\int_C |z|\, dz$ in the following case when c is the left half of the circle $|z|=1$ from $-i$ to i

<u>Soln</u>: The left-hand of the circle $|z|=1$ from $-i$ to $z=i$. If c is this circular arc. we have on c, $z=e^{i\theta}$ where θ decreases from $2\pi/2$ to $\pi/2$.



$$\int_C |z|\, dz = \int_{\theta=3\pi/2}^{\pi/2} |e^{i\theta}| (ie^{i\theta} d\theta)$$

$$= i\int_{3\pi/2}^{\pi/2} e^{i\theta}\, d\theta$$

$$= i(\cos\pi/2 + i\sin\pi/2) - (\cos 3\pi/2 + i\sin 3\pi/2)$$

$$= i(1+1) = 2i$$