USN

15EE45

# Fourth Semester B.E. Degree Examination, June/July 2018
## Electromagnetic Field Theory

Time: 3 hrs. Max. Marks: 80

*Note: Answer any FIVE full questions, choosing one full question from each module.*



### Module-1

1 a. Two points A and B have the following orientations.
A(2.614, 7.369, –3.079) and B (3.162, 7.023, –2.318)
Check whether $\overline{AB}$ is a unit vector. **(05 Marks)**

b. Given two points, C(–3, 2, 1) and D(r = 5, θ = 20°, ϕ = –70°)
Find (i) The spherical coordinates of C
(ii) The rectangular coordinates of D
(iii) The distance from C to D. **(06 Marks)**

c. Two point charges $Q_1$ = 100 μC and $Q_2$ = 100 μC are located at points $(-1, 1, -3)_m$ and $(3, 1, 0)_m$ respectively. Find the X, Y & Z components of the forces on $Q_1$. **(05 Marks)**

OR

2 a. Determine the electric field intensity at a point 'A' located at distance 0.3m and 0.4m respectively from charges $Q_1$ and $Q_2$ spaced 0.5m apart. Given $Q_1 = 1\times10^{-9}$ C and $Q_2 = 8\times10^{-10}$ C. **(06 Marks)**

b. State and prove Gauss Divergence theorem. **(06 Marks)**

c. If $\overline{D} = 9x^3\hat{a}_x + 5y^2\hat{a}_y + 2z\hat{a}_z$ c/m², find the charge density at the point (1, 5, 9)m. **(04 Marks)**

### Module-2

3 a. Prove that electric field intensity is expressed as negative gradient of scalar potential. **(05 Marks)**

b. Prove that the potential at a point P due to a charge disc at distance 'r' is $\dfrac{Q}{4\pi \in_0 r}$ V. **(06 Marks)**

c. A parallel plate capacitor consists of 3 dielectric layers if
$\in_1 = 1,\ d_1 = 0.4$ mm
$\in_2 = 2,\ d_2 = 0.6$ mm
$\in_3 = 1,\ d_3 = 0.8$ mm
and the area of cross section is 20 cm², find its capacitance C. **(05 Marks)**

OR

4 a. Find the electric field strength at the point (1, 2, –1) given the potential $V = 3x^2y + 2yz^2 + 3xyz$. **(05 Marks)**

b. An electric field of strength 3 V/m in air enters a dielectric medium. The orientation of electric fields with respect to boundary in air and dielectric are 30 and 60 respectively. Find the relative permeability of the dielectric. Also find the electric field strength in the dielectric. **(06 Marks)**

c. Determine the capacitance of a capacitor consisting of two parallel plates 30cm × 30cm surface area separated by 5 mm in air. What is the total energy stored by the capacitor is capacitor is charged to a potential difference of 500 V? What is the energy density? **(05 Marks)**

## Module-3

5 a. Derive Poisson's and Laplace's equations. Write Laplace's equations in cylindrical and spherical coordinate system. (06 Marks)

b. State and explain uniqueness theorem. (05 Marks)

c. Given vector field $\bar{E} = (12yx^2 - 6z^2x)\hat{a}_x + (4x^3 + 18zy^2)\hat{a}_y + (6y^3 - 6zx^2)\hat{a}_z$. Check for Laplace or Poisson's field. (05 Marks)

**OR**

6 a. State Biot-Savart's law, Ampere's circuital law and Stoke's theorem. (06 Marks)

b. A single turn circular coil of 50 meter in diameter carries a current of $28\times10^4$ Amps. Determine the magnetic field intensity $\bar{H}$ at a point on the axis of coil and 100 m from the coil. The $\mu_r$ of the free space is unity. (05 Marks)

c. Verify whether the vector field $\bar{F} = y^2z\hat{a}_x + z^2x\hat{a}_y + x^2y\hat{a}_z$ is irrotational or solenoidal. (05 Marks)

## Module-4

7 a. Obtain the expression of Energy stored in a magnetic field. (05 Marks)

b. Derive Lorentz force equation and mention the applications of its solution. (06 Marks)

c. Derive the boundary conditions at the boundary between two magnetic media of different permeabilities. (05 Marks)

**OR**

8 a. Derive the expression for the inductance of a solenoid. (05 Marks)

b. Calculate the inductance of a 10 m long co-axial cable filled with a material for which $\epsilon_r = 18$, $\sigma = 0$, $\mu_r = 80$. The external and internal diameters of the cable are 1 mm and 4 mm respectively. (06 Marks)

c. Find the maximum torque on an 85 turn rectangular coil 0.2m by 0.3m carrying a current 2A in a field B = 6.5 J. (05 Marks)

## Module-5

9 a. State and explain Poynting theorem with derivation. (08 Marks)

b. Determine the propagation constant at 500 kHz for a medium in which $\mu_r = 1$, $\epsilon_r = 15$, $\sigma = 0$. At what velocity will an electromagnetic wave travel in this medium? (08 Marks)

**OR**

10 a. A uniform plane wave $E_y = 10\sin(2\pi10^8t - \beta x)$ is travelling in x-direction in free space. Find the phase constant, phase velocity and the expression for $H_z$. Assume $E_z = 0 = H_y$. (08 Marks)

b. Explain skin depth and skin effect. Derive an expression for skin depth. (08 Marks)

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1) a) $A(2.614, 7.369, -3.079)$ and

$B(3.162, 7.023, -2.318)$

$\vec{AB} = 0.548\,\vec{a_x} - 0.346\,\vec{a_y} + 0.761\,\vec{a_z}$

$|\vec{AB}| = 0.9995 \simeq 1$

AB is a unit vector

1) b) $C(-3, 2, 1)$

$D(r=5, \theta=20^\circ, \phi=-70^\circ)$

i) spherical co-ordinates of C.

$r = \sqrt{x^2+y^2+z^2} = \sqrt{9+4+1} = 3.741$

$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = 74.49^\circ$

$\phi = \tan^{-1}(y/x) = -33.69^\circ$

$C(r=3.741,\ \theta=74.49^\circ,\ \phi=-33.69^\circ)$

ii) Rectangular co-ordinates of D.

$x = r\sin\theta\cos\phi = 0.58488$

$y = r\sin\theta\sin\phi = -1.60696$

$z = r\cos\theta = 4.69846.$

$D(0.58488, -1.60696, 4.69846)$

(iii) Distance b/w C & D

$$\vec{CD} = (0.58488 + 3)\vec{a_x} + (-1.60696 - 2)\vec{a_y} + (4.69846 - 1)\vec{a_z}$$

$$\vec{CD} = 3.58488\,\vec{a_x} - 3.60696\,\vec{a_y} + 3.69846\,\vec{a_z}$$

$$|\vec{CD}| = 6.288 \text{ m.}$$

1) c)

$Q_1 = 100\,\mu C$ $\quad \vec{R} \quad$ $Q_2 = 100\,\mu C$

$(-1, 1, -3)$ $\quad \vec{F_1} \quad$ $(3, 1, 0)$

$\vec{R} = -4\vec{a_x} - 3\vec{a_z}$

$|\vec{R}| = 5$

$$\vec{F_1} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 |\vec{R}|^3} \cdot \vec{R}$$

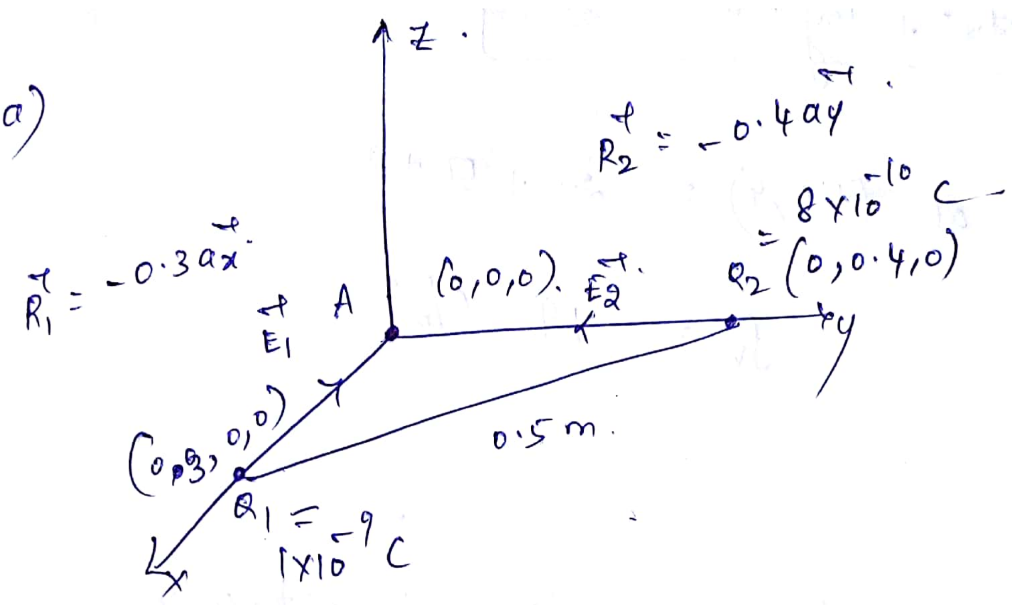
$$= \frac{100\times10^{-6} \times 100\times10^{-6} \times 9\times10^{9} \times (-4\vec{a_x} - 3\vec{a_z})}{(5)^3}$$

$$= \frac{90}{125}(-4\vec{a_x} - 3\vec{a_z})$$

$$\boxed{\vec{F_1} = -2.88\,\vec{a_x} - 2.16\,\vec{a_z}\ \text{N}}$$

$$\boxed{\begin{aligned} F_x &= -2.88 \text{ N} \\ F_y &= 0 \text{ N} \\ F_z &= -2.16 \text{ N} \end{aligned}}$$

2) a)




$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0 |\vec{R_1}|^3}\cdot\vec{R_1} + \frac{Q_2}{4\pi\varepsilon_0 |\vec{R_2}|^3}\cdot\vec{R_2}$$

$$= \frac{1\times\cancel{10^{-9}}\times 9\times\cancel{10^{9}}\times(-0.3\vec{a_x})}{(0.3)^3} + \frac{8\times10^{-10}\times9\times10^{9}\times(-0.4\vec{a_y})}{(0.4)^3}$$

$$= \frac{9}{(0.3)^3}\times(-0.3\vec{a_x}) + \frac{7.2}{(0.4)^3}(-0.4\vec{a_y})$$

$$\boxed{\vec{E} = -100\vec{a_x} - 45\vec{a_y}}\ \text{V/m}$$

$$\boxed{|\vec{E}| = 109.65\ \text{V/m}}$$

2 b)



## Application of Gauss' law:

Differential Volume element.

No symmetry.

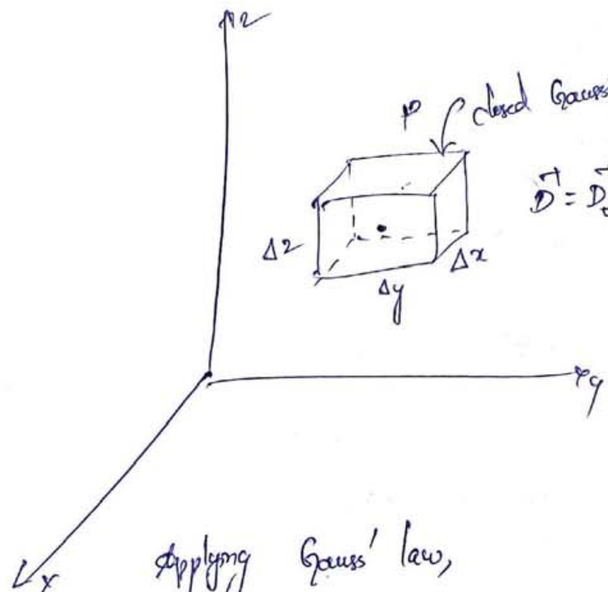
Chose a small gaussian surface → almost $\vec{D}$ is constant over that surface.

Result becomes correct only when volume $\Delta v \to 0$ (shrinks).

We will not obtain $\vec{D}$, obtain the valuable information about the way $\vec{D}$ varies in the region.

⇓

one of Maxwell's four equations (base to all electromagnetic theory).




$$\vec{D} = \vec{D}_0 = D_{x0}\vec{a_x} + D_{y0}\vec{a_y} + D_{z0}\vec{a_z}.$$

Applying Gauss' law,

$$\oint_S \vec{D}\cdot d\vec{s} = Q$$

$$\text{Divergence of } \vec{D} = \text{div } \vec{D} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{D} \cdot d\vec{s}}{\Delta v}$$

Divergence of the vector flux density $\vec{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

$$\boxed{\text{div } \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}}$$ Rectangular

$$\boxed{\text{div } \vec{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}}$$ cylindrical

$$\boxed{\text{div } \vec{D} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \, D_\theta) + \frac{1}{r \sin\theta} \frac{\partial D_\phi}{\partial \phi}}$$ Spherical

(7) MAXWELL's FIRST EQUATION OF ELECTROSTATICS:

1) $\text{div}\,\vec{D} = \lim_{\Delta v \to 0} \dfrac{\oint_S \vec{D}\cdot d\vec{s}}{\Delta v}$ (Definition of divergence)

2) $\text{div}\,\vec{D} = \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z}$ (Result of applying the definition to a differential volume element (in Rectangular co-ordinates))

3) $\text{div}\,\vec{D} = \rho_v$

Gauss's law

flux leaving any closed surface

$$\boxed{\oint_S \vec{D}\cdot d\vec{s} = Q}$$

charge enclosed

Gauss' law per unit volume,

$$\frac{\oint_S \vec{D}\cdot d\vec{s}}{\Delta v} = \frac{Q}{\Delta v}$$

As the volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{\oint_s \vec{D}\cdot d\vec{s}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

Divergence

volume charge density

$$\boxed{\operatorname{div} \vec{D} = \rho_v .}$$

(i)

First of Maxwell's four equations

Point form of Gauss's law (or) Maxwell's first equation

Statement:

The electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there.

Gauss's law → $\boxed{\oint_s \vec{D}\cdot d\vec{s} = Q = \int_v \rho_v \, dv .}$

(ii) Integral form of Maxwell's first equation

Divergence theorem: (For Electric flux density).

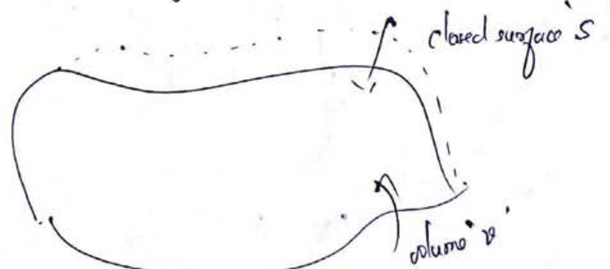
Gauss' law:

$$\oint_S \vec{D}\cdot d\vec{s} = Q = \int_V \rho_v \, dv.$$

$$\boxed{\oint_S \vec{D}\cdot d\vec{s} = \int_V \nabla\cdot\vec{D}\, dv.}$$ ← Divergence theorem.

Statement:

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.




Divergence of flux density throughout a volume

=

net flux crossing the enclosing surface.

2) c) $\vec{D} = 9x^3 \vec{a_x} + 5y^2 \vec{a_y} + 2z \vec{a_z}$ C/m$^2$

$\rho_v = \vec{\nabla} \cdot \vec{D}$

$\rho_v$ at $(1, 5, 9) = ?$

$$\rho_v = \frac{\partial}{\partial x}(9x^3) + \frac{\partial}{\partial y}(5y^2) + \frac{\partial}{\partial z}(2z)$$

$$\boxed{\rho_v = 27x^2 + 10y + 2 \quad C/m^3}$$

$\rho_v$ at $(1, 5, 9) = 27 + 50 + 2$

$\rho_v = 79$ C/m$^3$

3 a)

## Potential Gradient :

1.c.

① $\vec{E} \rightarrow V = -\int \vec{E}\cdot d\vec{l}$.

② $\int \rho_v \, dv \Rightarrow V = \int_v \frac{\rho_v \, dv}{4\pi\epsilon_0 r}$.

In practical problems,

neither $\vec{E}$, nor $\rho_v$ is known.

Description of two equipotential surfaces

Eg: two ||el conductors of circular cross sections at potentials 100 V & -100 V.

To find 1) capacitance between conductors.

2) charge or current distribution on conductors

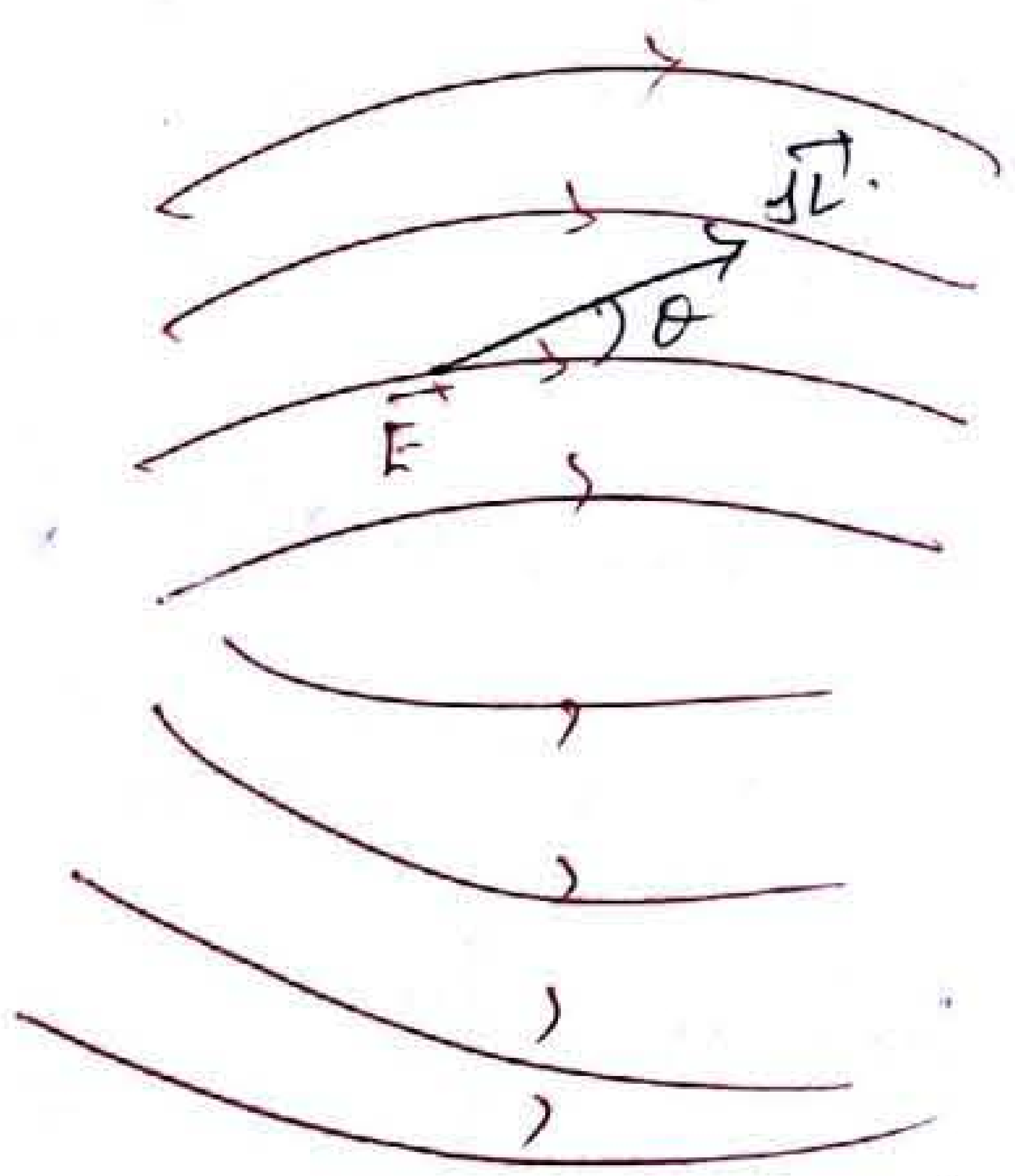
from which

3) Losses may be calculated.

Simple method to find $\vec{E}$ from potential:

$$V = -\int \vec{E}\cdot d\vec{l}.$$

applied to a very small length element, $\vec{\Delta L}$ along which $\vec{E}$ is constant.




$$\boxed{\Delta V = -\vec{E}\cdot\vec{\Delta L}}$$

$\vec{E}$ & $V$ both change as we move from point to point

$$\vec{\Delta L} = \Delta L \cdot \vec{a_L}$$

$$\Delta V = -|\vec{E}| \cdot \Delta L \cdot \cos\theta$$

$$\boxed{\Delta V = -E \, \Delta L \cos\theta}$$

V may be interpreted as a function $V(x, y, z)$.

Consider derivative of $\frac{dV}{dL} = ?$

$V \leftarrow$ unique function of end point $(x, y, z)$

$\downarrow$

$V \leftarrow$ single valued function

$\vec{E} \leftarrow$ conservative field.

$$\boxed{\frac{dV}{dL} = -E\cos\theta}$$

In which dirn should $\vec{\Delta L}$ be placed to obtain maximum value of $\Delta V = ?$

$$\boxed{\left.\frac{dV}{dL}\right|_{max} = E}$$

where $\cos\theta = -1$

$\theta = \pi$

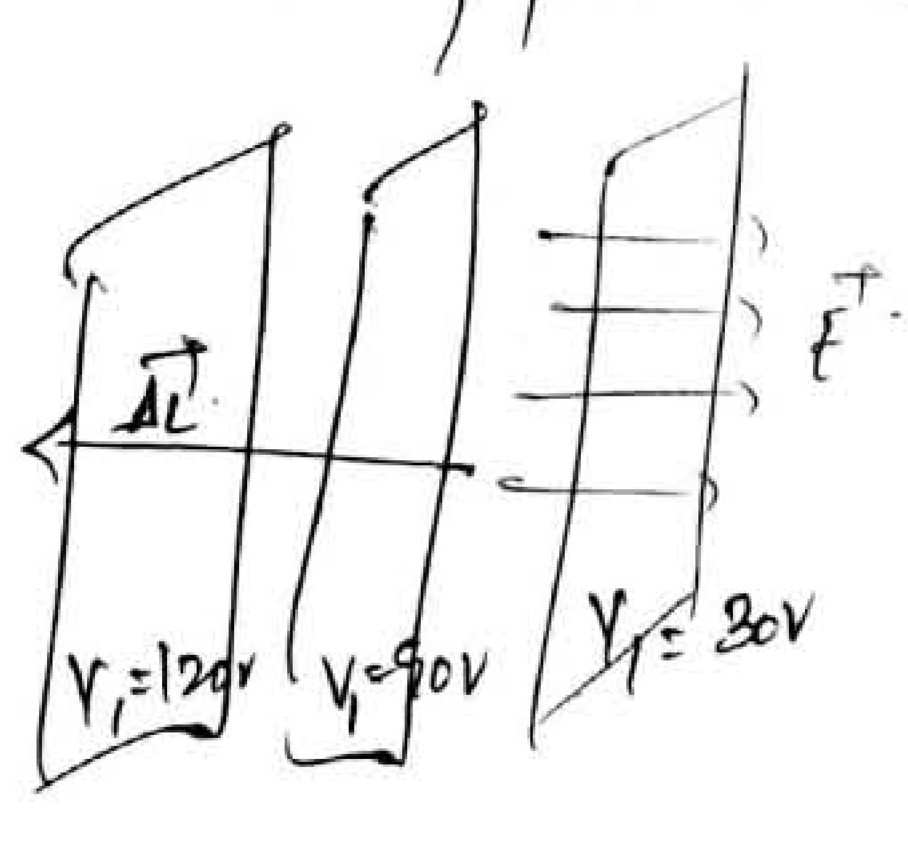
Relationship b/w $V$ & $\vec{E}$

1) magnitude of $\vec{E}$ is given by maximum value of rate of change of potential with distance

2) This maximum value is attained when the direction of $\vec{\Delta L}$ is opposite to $\vec{E}$. Con the direction of $\vec{E}$ is opposite to the direction in which the potential is increasing the most rapidly.

for equipotential surfaces,

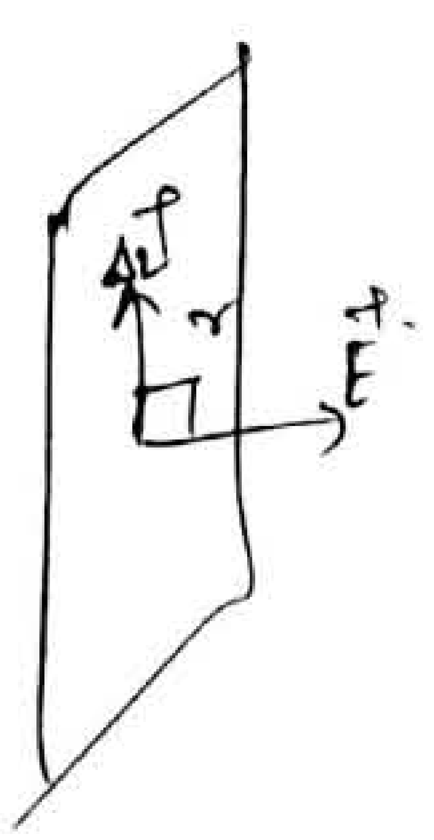
(i)




$\vec{E} \perp^r$ to the surface

$\vec{E}$ & $\vec{\Delta L}$ are oppositely directed

(ii)




Definition of equipotential surface

$$\Delta V = -\vec{E}\cdot\vec{\Delta L} = 0$$

$$\boxed{\vec{E} = -\left(\left.\frac{dV}{dL}\right|_{max}\right)\vec{n}}$$

direction of

1) $\vec{\delta L}$ in which maximum increase in potential

$\downarrow$

$\vec{a_L}$

in terms of potential field (rather than $\vec{E}$).

2) $\vec{a_N}$ ← unit normal vector to the equipotential surface & directed towards higher potentials

$$\boxed{\vec{E} = -\left.\frac{dV}{dL}\right|_{max} \vec{a_N}}$$

$\downarrow$

max space rate of change of V.

$$\left.\frac{dV}{dL}\right|_{max} \vec{a_N} = \text{grad } V. \quad (\underline{\text{Gradient}})$$

$$\boxed{\vec{E} = -\text{grad } V.}$$

$V(x,y,z) = dV = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}.$

$dV = -\vec{E}\cdot\vec{dL} = -E_x dx - E_y dy - E_z dz.$

$E_x = -\frac{\partial V}{\partial x}$

$E_y = -\frac{\partial V}{\partial y}$

$E_z = -\frac{\partial V}{\partial z}.$

$$\boxed{-\nabla V = -\text{grad } V = \vec{E} = -\left(\frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}\right)}$$

$$\boxed{\vec{E} = -\nabla V}$$

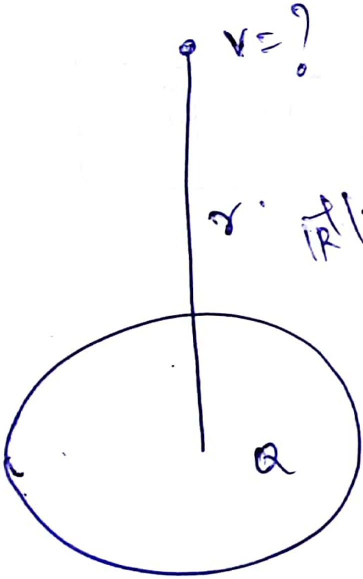
Rectangular ⇒ $\boxed{\nabla V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}.}$

Cylindrical ⇒ $\boxed{\nabla V = \frac{\partial V}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a_\phi} + \frac{\partial V}{\partial z}\vec{a_z}.}$

~~Cartesian~~ Spherical ⇒ $\boxed{\nabla V = \frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi}}$

# 3 b)

3 b)




$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 |\vec{R}|^3}\cdot\vec{R}$$

$$|\vec{R}| = r$$

$$V = \int_L \vec{E}\cdot d\vec{l}$$

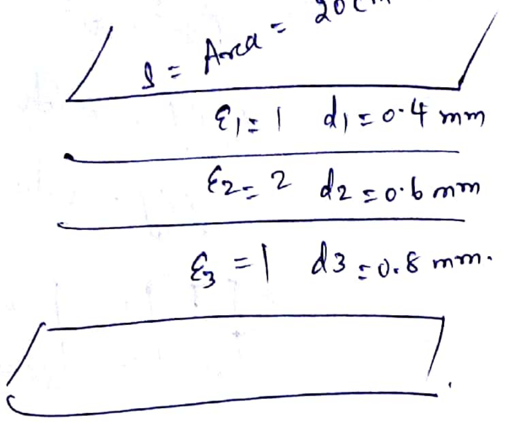
$$= \int_L \frac{Q}{4\pi\varepsilon_0 |\vec{R}|^2}\cdot \vec{a}_R \cdot d\vec{l}$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2}\cdot \int_0^r dl$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2}\cdot r$$

$$\boxed{V = \frac{Q}{4\pi\varepsilon_0 r}}$$

3) c)




$$C_1 = \frac{\varepsilon_1 \varepsilon_0 S}{d_1} = \frac{1\times 20\times10^{-4}\times\varepsilon_0}{0.4\times10^{-3}} = \frac{20\times\varepsilon_0}{4} = 5\times 8.854\times10^{-12}$$

$$\boxed{C_1 = 44.27\ pF}$$

$$C_2 = \frac{\varepsilon_2 S}{d_2} = \frac{\varepsilon_0\times2\times20\times10^{-4}}{0.6\times10^{-3}} = \frac{40\times8.854\times10^{-12}}{6} \quad \boxed{C_2 = 59.027\ pF}$$

$$C_3 = \frac{\varepsilon_3 S}{d_3} = \frac{\varepsilon_0\times1\times20\times10^{-4}}{0.8\times10^{-3}} = \frac{20\,\varepsilon_0}{8} = 22.135\ pF$$

$$\boxed{C_3 = 22.135\ pF}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$\frac{1}{C} = \frac{1}{44.27} + \frac{1}{59.027} + \frac{1}{22.135}$$

$$\boxed{C = 11.805\ pF}$$

4) a) $\vec{E} = -\vec{\nabla} V$

$V = 3x^2y + 2yz^2 + 3xyz$

$$\vec{E} = -\vec{\nabla}V = -\left[\frac{\partial}{\partial x}(3x^2y + 2yz^2 + 3xyz)\,\vec{a_x} + \frac{\partial}{\partial y}(3x^2y + 2yz^2 + 3xyz)\,\vec{a_y} + \frac{\partial}{\partial z}(3x^2y + 2yz^2 + 3xyz)\,\vec{a_z}\right]$$

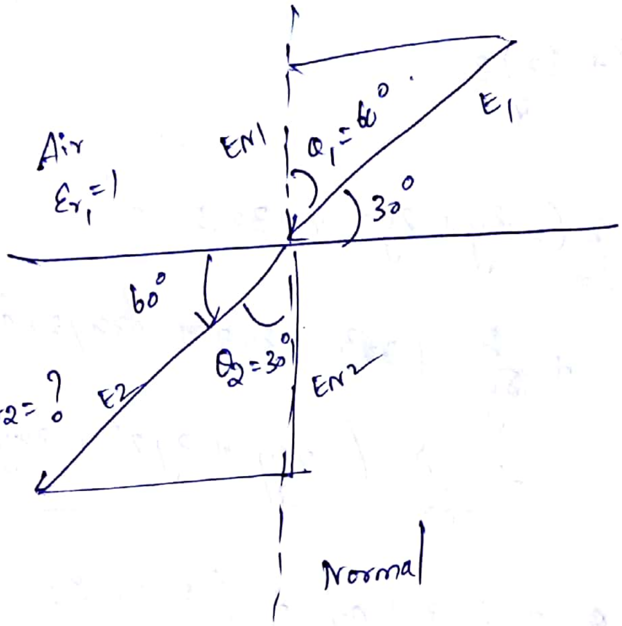
$$\boxed{\vec{E} = -\left[(6xy + 3yz)\,\vec{a_x} + (3x^2 + 2z^2 + 3xz)\,\vec{a_y} + (4yz + 3xy)\,\vec{a_z}\right]} \text{ V/m}$$

$$\vec{E}_{\text{at }(1,2,-1)} = -\left[6\times1\times2 + 3(2)(-1)\right]\vec{a_x} - \left[3\times1^2 + 2\times(-1)^2 + 3(1)(-1)\right]\vec{a_y} - \left[4\times2\times(-1) + 3(1)(2)\right]\vec{a_z}$$

$$= -[12 - 6]\,\vec{a_x} - [\cancel{8}3 + 2 - 3]\,\vec{a_y} - [-8 + 6]\,\vec{a_z}$$

$$\boxed{\vec{E} = -6\vec{a_x} - 2\vec{a_y} + 2\vec{a_z}} \text{ V/m.}$$

4) b)




$$\boxed{D_{N1} = D_{N2}}$$

$$\boxed{\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2}}$$

$$\boxed{\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1}}$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}} \Rightarrow \frac{\tan 60^\circ}{\tan 30^\circ} = \frac{\varepsilon_0 \varepsilon_{r1}}{\varepsilon_0 \varepsilon_{r2}}$$

$$\varepsilon_{r2} = \frac{\tan 30^\circ}{\tan 60^\circ}$$

$$\boxed{\varepsilon_{r2} = 0.3333}$$

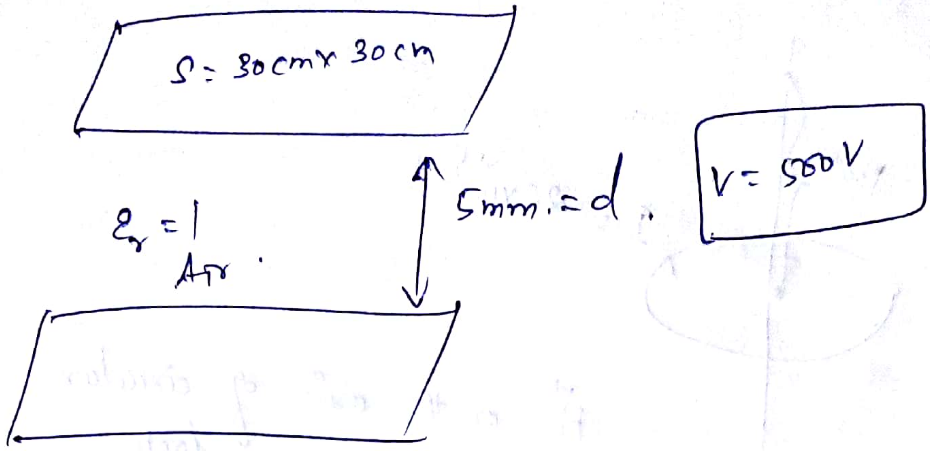
$$\cos\theta_1 = \frac{E_{N1}}{E_1}$$

$$\cos\theta_2 = \frac{E_{N2}}{E_2}$$

$$\frac{\cos\theta_1}{\cos\theta_2} = \frac{E_{N1}}{E_1} \times \frac{E_2}{E_{N2}} = \frac{E_2}{E_1} \times \frac{\varepsilon_{r2}}{\varepsilon_{r1}} \Rightarrow \frac{0.5}{0.866} = \frac{0.333}{1} \times \frac{E_2}{3.}$$

$$\frac{\cos 60^\circ}{\cos 30^\circ} = \frac{E_2}{E_1} \times \frac{0.3333}{1} = \frac{E_2}{3} \times 0.3333 \Rightarrow \boxed{E_2 = 5.1966 \text{ V/m}}$$

4) c)




$$C = \frac{\varepsilon S}{d} = \frac{\varepsilon_0 \times 30\times10^{-2}\times 30\times10^{-2}}{5\times10^{-3}} = \frac{900\times 8.854\times10^{-12}\times10^{-1}}{5}$$

$$\boxed{C = 159.372\ pF}$$

Energy stored $= W_E = \frac{1}{2}CV^2$

$$= \frac{1}{2}\times 159.372\times10^{-12}\times(500)^2$$

$$= 19.9215\times10^{-6}\ J$$

$$\boxed{W_E = 19.9215\ \mu J}$$

Energy density $= \frac{dW_E}{dV} = \frac{W_E}{\text{volume}} = \frac{19.9215\times10^{-6}}{30\times30\times10^{-4}\times5\times10^{-3}} = 44.27\ m\ J/m^3$

$$\boxed{\text{Energy density} = 44.27\times10^{-3}\ J/m^3}$$

5 a) Poisson's & Laplace's equations:

Point form of Gauss's law, $\nabla \cdot \vec{D} = \rho_v$.

& $\vec{D} = \varepsilon \vec{E}$.

$\vec{E} = -\nabla V$

$\nabla \cdot \vec{D} = \nabla \cdot \varepsilon \vec{E} = \varepsilon \, \nabla \cdot (-\nabla V) = \rho_v$. In homogeneous region, $\varepsilon$ is constant

$\Rightarrow \boxed{\nabla \cdot \nabla V = -\dfrac{\rho_v}{\varepsilon}}$

Double $\nabla$ operation,

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}$$

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial}{\partial z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \nabla V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\therefore \boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

Poisson's equations
In rectangular co-ordinates.

If $\rho_v = 0$ (zero volume charge density)

$\Downarrow$

but point charges,
line charges &
surface charge density . exist at singular locations
$\downarrow$
as source of field

$\Rightarrow$ $\boxed{\nabla^2 V = 0}$ Laplace's equation

$\nabla^2 \leftarrow$ Laplacian of V.

In rectangular co-ordinates, Laplace's equation is

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.}$$

Laplacian in cylindrical co-ordinates is

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}}$$

In spherical co-ordinates

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}}$$

# Uniqueness theorem:



5 b) Laplace's equation →

wherever volume charge density is zero, every conceivable configuration of electrodes or conductors produce a field for which $\nabla^2 V = 0$

Solution of Laplace's equation subject to certain boundary conditions

Every physical problem contains atleast one conducting boundary (usually has two or more).

Potentials on these boundaries are $V_0, V_1, \ldots$ Equipotential surface.

**Uniqueness theorem:**

If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is unique

there is only one solution

Theorem applies to any solution of Laplace and Poisson's equations.

## Proof of uniqueness theorem:

Proved by contradiction.

Assume two solutions of Laplace's equation; $V_1$ and $V_2$

$$\therefore \nabla^2 V_1 = 0 \quad \& \quad \nabla^2 V_2 = 0.$$

$$\boxed{\nabla^2 (V_1 - V_2) = 0}$$

Each solution must satisfy the boundary conditions.

At boundary potential values are given by $V_b$.

Value of $V_1$ on boundary → $V_{1b}$

Value of $V_2$ on boundary → $V_{2b}$

} Equipotential conducting boundary.

$$\therefore V_{1b} = V_{2b} = V_b.$$

$$\therefore \boxed{V_{1b} - V_{2b} = 0}$$

Vector identity,

$$\boxed{\nabla\cdot(V\vec{D}) = V(\nabla\cdot\vec{D}) + \vec{D}\cdot\nabla V}$$ for any scalar $V$ & vector $\vec{D}$.

Let $V_1-V_2$ as the scalar & $\nabla(V_1-V_2)$ as the vector

$$\nabla\cdot\left((V_1-V_2)\,\nabla(V_1-V_2)\right) = V_1-V_2\,(\nabla\cdot\nabla(V_1-V_2)) + \nabla(V_1-V_2)\cdot\nabla(V_1-V_2).$$

Integrate throughout the volume enclosed by the boundary surfaces

$$\int_{vol} \nabla\cdot(\underbrace{(V_1-V_2)\nabla(V_1-V_2)}_{\text{Vector}})\,dv = \int_{vol}\left[V_1-V_2\,(\nabla\cdot\nabla(V_1-V_2)) + \nabla(V_1-V_2)\cdot\nabla(V_1-V_2)\right]dv$$

$$= \int_{vol}(V_1-V_2)\left[\nabla\cdot\nabla(V_1-V_2)\right]dv + \int_{vol}\left[\nabla(V_1-V_2)\cdot\nabla(V_1-V_2)\right]dv$$

$$= \int_{vol}(V_1-V_2)\left[\underbrace{\nabla\cdot\nabla(V_1-V_2)}_{\nabla^2(V_1-V_2)}\right]dv + \int_{vol}\left[\nabla(V_1-V_2)\right]^2 dv$$

Apply divergence theorem to L.H.S. of equation

$$\boxed{\int_{vol}\nabla\cdot\left((V_1-V_2)\nabla(V_1-V_2)\right)dv = \oint_S (V_1-V_2)\nabla(V_1-V_2)\cdot d\vec{s}} = 0.$$

closed surface encloses equipotential boundary

$V_{1b} - V_{2b} = 0 \Rightarrow V_{1b} = V_{2b}$

$\Downarrow$

$V_1 - V_2 = 0$

$$\boxed{\int_{vol}\left[\nabla(V_1-V_2)\right]^2 dv = 0}$$

① Integral can be zero ~~when~~ everywhere

② Integrand is positive in some regions & negative in some other regions, their contribution is zero

$\left[\nabla(V_1-V_2)\right]^2 \leftarrow$ cannot be negative.

$\Downarrow$

$$\left[\nabla(V_1-V_2)\right]^2 = 0$$

$\Downarrow$

$$\nabla(V_1-V_2) = 0$$

Gradient of $V_1 - V_2$ is zero everywhere.

$\Downarrow$

$V_1 - V_2$ is constant with any co-ordinates

$$\boxed{V_1 - V_2 = \text{constant}}$$

If this constant is zero, uniqueness theorem is proved.

If we consider a point on the boundary

$$V_1 - V_2 = V_{1b} - V_{2b} = 0.$$

$$\therefore \boxed{V_1 = V_2}$$

$\Downarrow$

giving two identical solutions.

Uniqueness theorem also applies to Poisson's equation

$$\nabla^2 V_1 = -\frac{\rho_v}{\varepsilon} \quad \text{and} \quad \nabla^2 V_2 = -\frac{\rho_v}{\varepsilon}.$$

$$\nabla^2 (V_1 - V_2) = 0$$

Boundary conditions require that $V_{1b} - V_{2b} = 0$.

5 c) $\nabla^2 V = 0$

$\vec{E} = -\vec{\nabla} V$

$(12yx^2 - 6z^2x)\hat{a}_x + (4x^3 + 18zy^2)\hat{a}_y + (6y^3 - 6zx^2)\hat{a}_z = -\nabla V$

$\nabla V = (-12yx^2 + 6z^2x)\hat{a}_x - (4x^3 + 18zy^2)\hat{a}_y + (-6y^3 + 6zx^2)\hat{a}_z$

$\nabla^2 V = \nabla \cdot \nabla V = \nabla \cdot \left[(-12yx^2 + 6z^2x)\hat{a}_x - (4x^3 + 18zy^2)\hat{a}_y + (-6y^3 + 6zx^2)\hat{a}_z\right]$

$= \frac{\partial}{\partial x}(-12yx^2 + 6z^2x) + \frac{\partial}{\partial y}(-4x^3 - 18zy^2) + \frac{\partial}{\partial z}(-6y^3 + 6zx^2)$

$= -24xy + 6z^2 - 36zy + 6x^2 \neq 0$

as $\nabla^2 V \neq 0$, the laplace equation is not satisfie

Thus there exist $\rho_v$ as region is not free of charge

6 a) state and explain Biot-Savart's law for a small differential current elements?

Biot - Savart law states that,

i) The magnetic field intensity $d\vec{H}$ produced at a point P due to a differential element $Id\vec{L}$ is

1) Proportional to the product of I and differential length $d\vec{l}$.
2) The sine of angle b/w the element and the line joining the point P to the element.
3) and inversely proportional to the square of the distance R b/w point P and the element

mathematically the Biot-Savart law can be stated as

$$d\vec{H} \propto \frac{Idl\sin\theta}{R^2}$$

$$dH = \frac{KIdl\sin\theta}{R^2}$$

$K \rightarrow$ Constant of proportionality

in SI unit $K = \frac{1}{4\pi}$

$$d\vec{H} = \frac{Idl\sin\theta}{4\pi R^2}$$

$$d\vec{H} = \frac{I dl \sin\theta}{4\pi R^2}$$

let us express this equation in vector form,

$dl$ = magnitude of vector length $\vec{dl}$ and

$\hat{a}_R$ = unit vector in the direction from differential current

$$\vec{dL} \times \hat{a}_R = dL[\hat{a}_R] \sin\theta = dl \sin\theta$$

$$d\vec{H} = \frac{I\vec{dL} \times \hat{a}_R}{4\pi R^2} \quad A/m$$

$$\hat{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R}$$

$$d\vec{H} = \frac{I\vec{dl} \times \vec{R}}{4\pi R^3} \quad A/m$$

ii) state and explain ampere's circuital law?

Ampere's circuital law states that,
The line integral of magnetic field intensity $\vec{H}$ around a closed path is exactly equal to the direct current enclosed by that path

$$\oint \vec{H} \cdot \vec{dl} = I$$

Proof: $\vec{dl} = r\,d\phi\,\hat{a}_\phi$

$$\vec{H} = \frac{I}{2\pi r}\hat{a}_\phi$$

$$\vec{H} \cdot \vec{dl} = \frac{I}{2\pi r}\hat{a}_\phi \cdot r\,d\phi\,\hat{a}_\phi$$

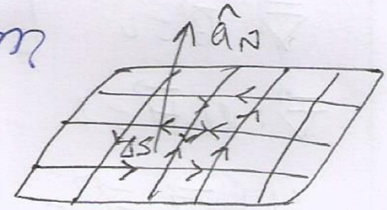
$$= \frac{I}{2\pi r} r\,d\phi = \frac{I}{2\pi} d\phi$$

$$\oint \vec{H} \cdot \vec{dl} = \int_0^{2\pi} \frac{I}{2\pi} d\phi = \frac{I}{2\pi}[\phi]_0^{2\pi} = I$$

I = current carried by conductor

iii) state and prove stoke's theorem

$$\oint \vec{H} \cdot \vec{dl} = \iint (\vec{\nabla} \times \vec{H}) \cdot \vec{ds}$$




each small box is incremental surface area
The surface area S is broken down into small area $\Delta S$.

applying the defination of curl to one of the incremental area,

$$\frac{\oint \vec{H}\cdot \vec{dl}}{\Delta s} = (\vec{\nabla}\times\vec{H}).\hat{a}_N$$

$\hat{a}_N$ is the unit vector in the direction of right hand normal of $\Delta s$

evaluating the circulating for every $\Delta s$ comprising S and sum of the results.

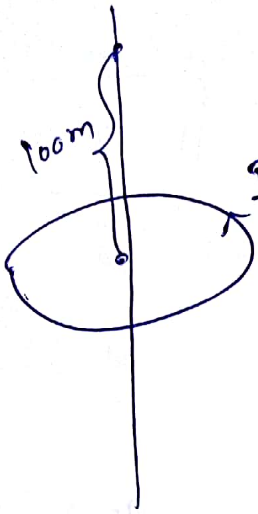
→ every interior wall is ~~co~~ covered in each direction

→ so some cancellation will occur.

→ only outside boundary no cancellation

$$\oint \vec{H}\cdot\vec{dL} = \int_S (\vec{\nabla}\times\vec{H}).\vec{ds}.$$

6) b)




$I = 28 \times 10^{+4}$ A.

$\vec{H}$ on the axis of circular loop

$$\vec{H} = \frac{I \rho^2}{2(\rho^2 + h^2)^{3/2}} \vec{a_z}.$$

$\rho = \frac{50}{2}$ m

$\boxed{\rho = 25 \text{ m}}$

(i) $h = 100$ m.

$$\vec{H} = \frac{28 \times 10^4 \times (25)^2}{2(25^2 + 100^2)^{3/2}} \vec{a_z}$$

$$\boxed{\vec{H} = 79.894 \; \vec{a_z} \; A/m}$$

6) c)

Irrotational $\rightarrow$ $\vec{\nabla} \times \vec{F} = 0$

Solenoidal $\rightarrow$ $\vec{\nabla} \cdot \vec{F} = 0$.

$$\vec{F} = y^2 z \, \vec{a_x} + z^2 x \, \vec{a_y} + x^2 y \, \vec{a_z}.$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(y^2 z) + \frac{\partial}{\partial y}(z^2 x) + \frac{\partial}{\partial z}(x^2 y)$$

$$= 0 + 0 + 0$$

$\boxed{\vec{\nabla} \cdot \vec{F} = 0}$ $\Rightarrow$ $\vec{F}$ is solenoidal Field

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$

$$= \vec{a_x}\left[(x^2 - 2zx)\right] - \vec{a_y}\left[2xy - y^2\right] + \vec{a_z}\left[z^2 - 2yz\right].$$

$\boxed{\vec{\nabla} \times \vec{F} \neq 0} \rightarrow \vec{F}$ is not irrotational.

(i) $\vec{F}$ is rotational field.

7 a)

## Potential energy and forces on magnetic materials :

Total energy stored in magnetic field,

$$\boxed{W_H = \frac{1}{2}\iiint_V \vec{B}\cdot\vec{H}\, dv} \quad J/m^3$$

$$\boxed{\text{Energy density } \frac{dW_H}{dv} = \frac{1}{2}\vec{B}\cdot\vec{H}} \quad J/m^3$$

(or $\boxed{\frac{dW_H}{dv} = \frac{1}{2}\frac{(\vec{B})^2}{\mu} = \frac{1}{2}\mu(\vec{H})^2}$ //

### Forces on magnetic material:

$$dW_H = F\,dL \Rightarrow \frac{1}{2}\frac{B^2}{\mu}\cdot dv = F\,dL$$

$\boxed{dv = S\,dL}$

$S$ → surface area.

$$\frac{1}{2}\frac{B^2}{\mu} S\,dL = F\,dL$$

$$\boxed{F = \frac{1}{2}\frac{B^2}{\mu} S} \quad N. \; //.$$

$S$ → surface area

7 b) Electric field causes a force to be exerted on a stationary or moving charge

Steady magnetic field → exerts force only on a moving charge
(produced by moving charges)

Force on a moving charge:

Electric force on a charged particle, $\boxed{\vec{F} = Q\vec{E}}$ → (1)

Same dxn as $\vec{E}$ for a positive charge.

If the charge is in motion, the above equation gives the force at any point in its trajectory.

Force on a Charged particle is in motion in a magnetic field of flux density, $\vec{B}$

$\boxed{\vec{F} = Q\vec{v} \times \vec{B}}$ → (2) direction of force is ⊥r to both $\vec{v}$ and $\vec{B}$

$\vec{F}$ applied ⊥r to the dxn in which charge is moving.

↓

can never change its velocity.

$\boxed{\text{Acceleration vector is always normal to velocity vector}}$

Kinetic energy of particle remains unchanged.

Steady magnetic field is incapable of transferring energy to a moving charge

Electric field → exerts force on particle which is independent of the dxn of progressing charge

& effects an energy transfer between field and particle in general.

Force on a moving particle arising from combined electric & magnetic field.

(by superposition)

$$\boxed{\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})} \rightarrow \text{Lorentz force equation} \quad (3)$$

↓

Solution is required in determining

1) electron orbits in magnetron
2) Proton paths in cyclotron
3) plasma characteristics in a magneto hydro dynamic (MHD) generator.

In general charged particle motion in combined electric and magnetic fields

## 7 c) Magnetic boundary conditions:

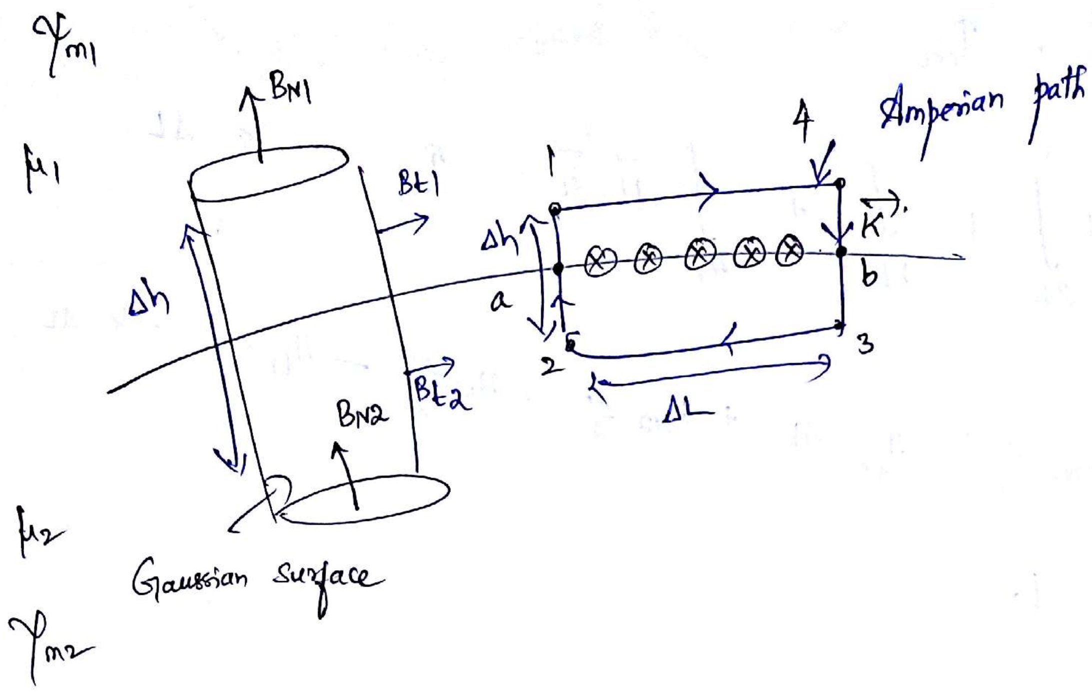
$\vec{B}_1 = \vec{B}_{t1} + \vec{B}_{N1}$

$\vec{B}_2 = \vec{B}_{t2} + \vec{B}_{N2}$

$$\boxed{\begin{aligned}\vec{B} &= \mu \vec{H} \\ \vec{M} &= \chi_m \vec{H}\end{aligned}}$$

$\vec{H}_1 = \vec{H}_{t1} + \vec{H}_{N1}$

$\vec{H}_2 = \vec{H}_{t2} + \vec{H}_{N2}$




① Gauss's law for Magnetic fields:

$$\phi = \oint_S \vec{B} \cdot \vec{ds} = 0$$

$$\Rightarrow \int_{top} + \int_{bottom} + \int_{lateral} \vec{B} \cdot \vec{ds} = 0$$

$$\Rightarrow B_{N1} \cdot \Delta S - B_{N2} \cdot \Delta S + B_{t1} \underbrace{\frac{\Delta h}{2}}_{0} 2\pi r + B_{t2} \underbrace{\frac{\Delta h}{2}}_{0} 2\pi r = 0$$

We are obtaining Conditions at the boundary.

$\therefore \Delta h \longrightarrow 0.$

$$B_{N1} \cdot \Delta S - B_{N2} \Delta S = 0$$

$$\boxed{B_{N1} = B_{N2}} \quad \Rightarrow \mu_1 H_{N1} = \mu_2 H_{N2}$$

$$\Rightarrow \boxed{H_{N1} = \frac{\mu_2}{\mu_1} H_{N2}}$$

$$\frac{M_{N1}}{\chi_{m1}} = \frac{\mu_2}{\mu_1} \frac{M_{N2}}{\chi_{m2}} \Rightarrow \boxed{M_{N1} = \frac{\chi_{m1}}{\chi_{m2}} \cdot \frac{\mu_2}{\mu_1} M_{N2}}$$

② Ampere's circuital Law:

$$\oint_L \vec{H} \cdot d\vec{L} = I_{enc}$$

$$\int_{14} + \int_{4b} + \int_{b3} + \int_{32} + \int_{2a} + \int_{a1} \vec{H} \cdot d\vec{L} = K \cdot \Delta L$$

$$H_{t1} \cdot \Delta L + \left(-H_{N1} \cancel{\frac{\Delta h}{2}}\right) + \left(-H_{N2} \cdot \cancel{\frac{\Delta h}{2}}\right) - H_{t2}\, \Delta L + H_{N2} \cancel{\frac{\Delta h}{2}} + H_{t1} \cancel{\frac{\Delta h}{2}} = K \cdot \Delta L$$

$0 \quad 0 \quad 0 \quad 0$

At the boundary $\quad \Delta h \longrightarrow 0$

$$(H_{t1} - H_{t2}) \cdot \Delta L = K \cdot \Delta L$$

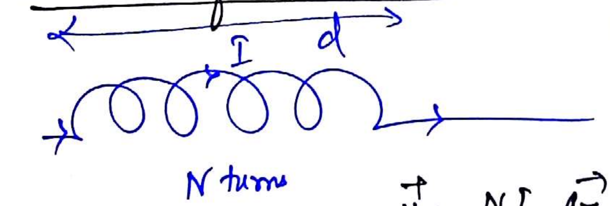
$$\boxed{H_{t1} - H_{t2} = K} \Rightarrow \boxed{\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K}$$

$\Downarrow$

$$\frac{M_{t1}}{\chi_{m1}} - \frac{M_{t2}}{\chi_{m2}} = K \Rightarrow \boxed{M_{t2} = \frac{M_{t1}}{\chi_{m1}} \cdot \chi_{m2} - K \chi_{m2}}$$

## 8 a)

**Inductance of a solenoid:**




$$L = \frac{N\phi}{I}.$$

$$\vec{H} = \frac{NI}{d}\,\vec{a_z}.$$

$$\vec{B} = \mu \frac{NI}{d}\,\vec{a_z}$$

$$\phi = \iint_S \vec{B}\cdot\vec{ds} = \iint_S \frac{\mu NI}{d}\,.\,\vec{a_z}\,.\,\vec{ds_z} = \frac{\mu NI}{d}\iint_S ds$$

Surface area 'S'

$$\phi = \frac{\mu NI}{d} S$$

$$\boxed{L = \frac{\mu N^2 I S}{dI}} \Rightarrow \boxed{L = \frac{\mu N^2 S}{d}}\ H$$

8) b) Inductance of co-axial cable.

$$\boxed{L = \frac{\mu d}{2\pi} \ln(b/a)} \ H.$$

$\mu_r = 80$

$d = 10m.$

$b = \frac{4 \times 10^{-3}}{2} m$

$a = \frac{1 \times 10^{-3}}{2} m.$

$$L = \frac{\cancel{4}^{2}\cancel{\pi} \times 10^{-7} \times 80 \times 10 \times}{\cancel{2\pi}} \ln\left(\frac{4 \times \cancel{10^{-3}}}{1 \times \cancel{10^{-3}}}\right)$$

$$= 800 \times 2 \times 10^{-7} \ln(4)$$

$$\boxed{L = 221.80 \ \mu H}$$

8) c) $N = 85$

Surface area $= S = 0.2 \times 0.3 \ m^2$

$I = 2A$.

$B = 6.5 \ T$

Torque $= \vec{m} \times \vec{B}$.

$m = IS = 2 \times 0.2 \times 0.3 = 0.12 \ Am^2$.

$B = 6.5$

$T = 0.12 \times 6.5 = 0.78 \ Nm$

$$\boxed{T = 0.78 \ Nm}$$

9 a)

# Poynting's theorem & Wave power:

## Poynting's theorem.

It states that the net power flowing out of a given volume is equal to the time rate of decrease in stored energy within the volume minus conduction losses.

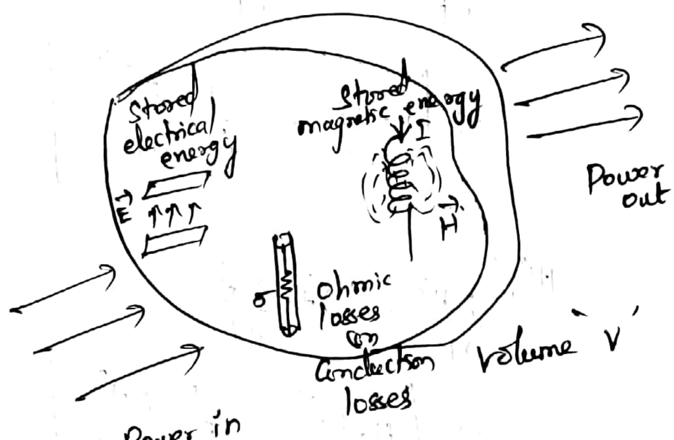



Fig. Illustration of power balance for EM fields.

**Proof:**

Maxwell's equations

$$\vec{\nabla}\cdot\vec{D} = \rho_v$$

$$\vec{\nabla}\times\vec{E} = -\mu\frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t}.$$

Vector identity:

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot\vec{\nabla}\times\vec{E} - \vec{E}\cdot\vec{\nabla}\times\vec{H}$$

Apply Maxwell's equations into this vector identity,

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{H}\cdot\mu\frac{\partial \vec{H}}{\partial t} - \vec{E}\cdot\left[\sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t}\right]$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\mu\,\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} - \sigma E^2 - \varepsilon\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}.$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\sigma E^2 - \varepsilon\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} - \mu\cdot\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}$$

$\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = ?$

$\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} = ?$

Let

$$\frac{\partial\vec{E}^2}{\partial t} = 2\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \boxed{\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial\vec{E}^2}{\partial t}}$$

Ill$^{rly}$

$$\frac{\partial H^2}{\partial t} = 2\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}$$

$$\Rightarrow \boxed{\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} = \frac{1}{2}\frac{\partial H^2}{\partial t}.}$$

Point form.

$$\boxed{\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\sigma E^2 - \frac{1}{2}\cdot\varepsilon\frac{\partial E^2}{\partial t} - \frac{1}{2}\mu\frac{\partial H^2}{\partial t}.}$$

Integrating over the given volume,

$$\iiint_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dv = -\iiint_V \sigma E^2 dv - \frac{1}{2}\iiint_V \varepsilon\frac{\partial E^2}{\partial t}dv - \frac{1}{2}\iiint_V \mu\frac{\partial H^2}{\partial t}dv$$

Apply Divergence theorem

Divergence theorem,

$$\iiint_V(\vec{\nabla}\cdot\vec{A})\,dv = \oiint_S \vec{A}\cdot\vec{ds}$$

∴ The equation becomes        Poynting theorem.

$$\boxed{\underbrace{\oint\!\!\!\oint_S (\vec{E}\times\vec{H})\cdot\vec{ds}}_{\text{net power flowing out of the volume}} = \underbrace{-\sigma\iiint_V E^2 dv}_{\text{Ohmic losses (or) Conduction losses}} \underbrace{- \frac{d}{dt}\left[\frac{1}{2}\iiint_V \varepsilon E^2 dv\right]}_{\text{Rate of decrease in stored electric energy}} \underbrace{- \frac{d}{dt}\left[\frac{1}{2}\iiint_V \mu H^2 dv\right]}_{\text{Rate of decrease in stored magnetic energy}}}$$

Hence proved.

w.k.t.

Electric potential energy, $\boxed{W_E = \frac{1}{2}\iiint_V \varepsilon E^2 dv}$

and

Magnetic potential energy, $\boxed{W_H = \frac{1}{2}\iiint_V \mu H^2 dv}$

Power flow of an electromagnetic wave

$$\boxed{P = \oint\!\!\!\oint_S (\vec{E}\times\vec{H})\cdot\vec{ds}} \quad W$$

where $\boxed{\vec{E}\times\vec{H} = \vec{P}}$ = Poynting vector = Power density vector

(or)

$\boxed{\vec{E}\times\vec{H} = \vec{S}}$ = Poynting vector $(W/m^2)$

9) b) $f = 500$ KHz

$\mu_r = 1$

$\varepsilon_r = 15$

$\sigma = 0$ (loss less medium).

i) $\gamma = ?$

ii) $v_p = ?$

$j\gamma = \alpha + j\beta$

Attenuation Constant if ($\sigma = 0$)

$\boxed{\alpha = 0}$

$$\beta = \omega\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}$$

$$\beta = 2\pi \times 500 \times 10^3 \sqrt{4\pi \times 10^{-7} \times 1 \times 8.854 \times 10^{-12} \times 15}$$

$$\beta = \omega\sqrt{\mu_0 \varepsilon_0} \cdot \sqrt{\mu_r \varepsilon_r}$$

$$= \frac{2\pi \times 5200 \times 10^3}{3 \times 10^8} \times \sqrt{1 \times 15}$$

$$= 0.004055$$

Phase constant $\boxed{\beta = 40.55 \times 10^{-3} \ \text{rad/m}}$

$$j\gamma = j\beta \Rightarrow \boxed{\gamma = 40.55 \times 10^{-3} \ \text{rad/m}}$$ Propagation constant

$$v_p = \frac{\omega}{\beta} = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{\mu_0 \varepsilon_0}\sqrt{\mu_r \varepsilon_r}}$$

$$= \frac{3 \times 10^8}{\sqrt{1 \times 15}}$$

Velocity of the wave $\boxed{v_p = 77.459 \times 10^6 \ \text{m/s}}$

10) a) $\vec{E} = 10 \sin(2\pi \times 10^8 t - \beta x)\, \vec{a_y}$.

Free space.

$E_z = H_y = 0$.

i) $\vec{H} = ?$

ii) $\beta = ?$ $\qquad \omega = 2\pi \times 10^8$ rad/s

iii) $v_p = ?$

ii) $\beta = \omega \sqrt{\mu_0 \varepsilon_0}$.

$$= \frac{2\pi \times \cancel{10^8}}{3 \times \cancel{10^8}}$$

$$\boxed{\beta = 2.094 \text{ rad/m}}$$

(iii) $v_p = \dfrac{\omega}{\beta} = \dfrac{2\pi \times 10^8}{2.094} = 3 \times 10^8$ m/s.

$\boxed{v_p = 3 \times 10^8 \text{ m/s}}$ free space velocity

(i) $H_{z0} = ?$ $\qquad E_{y0} = 10$.

$$\frac{E_{y0}}{H_{z0}} = \eta_0 = 120\pi\ \Omega$$

$$H_{z0} = \frac{E_{y0}}{120\pi} = \frac{10}{120\pi} = 0.0265 \text{ A/m}$$

$$\boxed{\vec{H} = 0.0265 \sin(2\pi \times 10^8 t - 2.094 x)\, \vec{a_z}} \text{ A/m}$$

10 b) Wave equation: for Good Conductors and Skin effect

$$\boxed{\nabla^2 \vec{E_s} = -\gamma^2 \vec{E_s}} \rightarrow (1)$$

from Maxwell's equations,

$$\vec{\nabla} \times \vec{E_s} = -j\omega\mu \vec{H_s}$$
$$\vec{\nabla} \times \vec{H_s} = \sigma \vec{E_s} + j\omega\varepsilon \vec{E_s}$$

$$\vec{\nabla} \cdot \vec{D_s} = \rho_v$$
$$\vec{\nabla} \cdot \vec{B} = 0$$

For dielectric medium (no free charges, $\rho_v = 0$)

$$\vec{\nabla} \cdot \vec{E_s} = 0.$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E_s} = -j\omega\mu \vec{\nabla} \times \vec{H_s}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E_s}) - \nabla^2 \vec{E_s} = -j\omega\mu \left[\sigma \vec{E_s} + j\omega\varepsilon \vec{E_s}\right]$$

$$-\nabla^2 \vec{E_s} = -j\omega\mu\sigma \vec{E_s} + \omega^2\mu\varepsilon \vec{E_s}$$

$$\boxed{\nabla^2 \vec{E_s} = -\left(-j\omega\mu\sigma \vec{E_s} + (\omega^2\mu\varepsilon)\vec{E_s}\right)} \rightarrow (2)$$

$$\boxed{\gamma^2 = -j\omega\mu(\sigma + j\omega\varepsilon)}$$

Comparing (1) & (2)

$$-\gamma^2 = -(-j\omega\mu\sigma + \omega^2\mu\varepsilon) \Rightarrow -\gamma^2 = j\omega\mu\sigma - \omega^2\mu\varepsilon$$

$$\sqrt{-\gamma^2} = \sqrt{j\omega\mu\sigma - \omega^2\mu\varepsilon}$$

$$\pm j\gamma = \pm\sqrt{j\omega\mu\sigma - \omega^2\mu\varepsilon} = \pm\sqrt{j\omega\mu\sigma + j^2\omega^2\mu\varepsilon}$$

$$j\gamma = \sqrt{(j\omega)^2\mu\varepsilon\left[\frac{\sigma}{\omega\varepsilon j} + 1\right]}$$

$$\boxed{j\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]}}$$

(or)

$$\boxed{j\gamma = j\omega\sqrt{\mu\varepsilon}\left[\sqrt{1 - j\frac{\varepsilon''}{\varepsilon'}}\right]}$$

For good conductors,

$\sigma \simeq \infty.$

$\frac{\sigma}{\omega\varepsilon} >> 1$

$$j\gamma = j\omega\sqrt{\mu\varepsilon'}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon'}}$$

$$j\gamma = j\omega\sqrt{\mu\varepsilon'}\sqrt{-j\frac{\sigma}{\omega\varepsilon'}}$$

$$j\gamma = j\omega\sqrt{\mu\varepsilon'}\sqrt{\frac{\sigma}{\omega\varepsilon'}}\sqrt{-j}$$

$$\sqrt{-j} = \sqrt{1\angle -90^\circ} = 1\angle -45^\circ$$

$$\sqrt{-j} = \frac{1-j}{\sqrt{2}}$$

$$j\gamma = j\sqrt{\omega\mu\sigma}\cdot\sqrt{-j}$$

$$= j\sqrt{\omega\mu\sigma}\,\frac{(1-j)}{\sqrt{2}}$$

$$= \frac{j\sqrt{\omega\mu\sigma}}{\sqrt{2}} + \frac{\sqrt{\omega\mu\sigma}}{\sqrt{2}}$$

$$j\gamma = \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}} = \alpha + j\beta$$

$$\boxed{\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}}$$

Intrinsic impedance,

$$\eta = \sqrt{\mu/\varepsilon} = \sqrt{\frac{\mu}{\varepsilon' - j\varepsilon''}} = \sqrt{\frac{\mu}{\varepsilon'\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)}} = \sqrt{\frac{\mu}{\varepsilon'\left(1 - j\frac{\sigma}{\omega\varepsilon'}\right)}}$$

$$= \sqrt{\frac{\mu}{\varepsilon' - j\frac{\sigma}{\omega}}}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon' + \frac{\sigma}{j\omega}}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon'}}$$

$\sigma >> \omega\varepsilon$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$\sqrt{j} = \sqrt{1 \angle 90^\circ} = 1 \angle 45^\circ$$

$$\sqrt{j} = \frac{1+j}{\sqrt{2}}$$

$$= \frac{(1+j)}{\sqrt{2}} \cdot \sqrt{\frac{\omega\mu}{\sigma}}$$

$$= \sqrt{\frac{\omega\mu}{2\sigma}} + j\sqrt{\frac{\omega\mu}{2\sigma}}$$

In polar form

$$\boxed{\eta = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}} \Rightarrow \boxed{\eta = \sqrt{\frac{2\pi f\mu}{\sigma}} \angle 45^\circ} \Rightarrow \vec{E} \text{ leads } \vec{H} \text{ by } 45^\circ.$$

Solution of Wave equation:

$$\vec{E} = \vec{E_s}\, e^{j\omega t}$$

$$\vec{E} = \left(E_{x0}\, e^{-j\gamma z} e^{j\omega t} + \underbrace{E_{x0}'\, e^{j\gamma z} e^{j\omega t}}_{\text{Backward wave}}\right) \vec{a_x}$$

Consider only forward wave

$$\vec{E} = E_{x0}\, e^{-\alpha z} \cdot e^{-j\beta z} e^{j\omega t}.$$

$$\boxed{\vec{E} = E_{x0}\, e^{-\alpha z} \cos(\omega t - \beta z)\, \vec{a_x}}$$

$$\vec{H} = \frac{E_{x0}}{\eta} e^{-\alpha z} \cos(\omega t - \beta z)\, \vec{a_y} \qquad \vec{H} = \frac{E_{x0}}{|\eta| \angle\theta_\eta} e^{-\alpha z} \cos(\omega t - \beta z)\, \vec{a_y}.$$

$$\boxed{\vec{H} = \frac{E_{x0}}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45^\circ)\, \vec{a_y}}$$

As $\vec{E}$ (or $\vec{H}$) travels in a conducting medium, its amplitude is attenuated by a factor $e^{-\alpha z}$.

The distance through which the amplitude of wave decreases to a factor $e^{-1}$ (about 37% of its original value) is called skin depth or depth of penetration of the medium.

$$E_0 e^{-\alpha z} = E_0 e^{-1}$$

$z = \delta \leftarrow$ depth of penetration

$$e^{-\alpha\delta} = e^{-1}$$

$$\alpha\delta = 1$$

$$\boxed{\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}}$$

Skin depth is the measure of the depth to which an EM wave can penetrate the medium