USN 1CR16EC078

Third Semester B.E. Degree Examination, Dec.2017/Jan.2018

Engineering Electromagnetics

Time: 3 hrs. Max. Marks: 80

Note: Answer any FIVE full questions, choosing one full question from each module.

Module-1

1 ✓a. State and explain Coulomb's law in vector form. (05 Marks)

b. Find the electric field $\bar{E}$ at origin, if the following charge distributions are present in free space:
   i) Point charge 12 nc at P(2, 0, 6).
   ii) Uniform line charge of linear charge density 3 nc/m at x = 2, y = 3.
   iii) Uniform surface charge of density $P_s = 0.2\ nc/m^2$ at x = 2. (06 Marks)

c. Define volume charge density. Also find the total charge within each of the indicated volumes.
   i) $0 \le \rho \le 0.1,\ 0 \le \phi \le \pi,\ 2 \le z \le 4;\ \rho_v = \rho^2 z^2 \sin(0.6\phi)$
   ii) Universe : $\rho_v = \dfrac{e^{-2r}}{r^2}$ (05 Marks)

OR

2 a. Define Electric flux and flux density. (04 Marks)

b. Given a 60 μC point charge located at the origin, find the total electric flux passing through:
   i) That portion of the sphere r = 26 cm bounded by $0 < \theta < \dfrac{\pi}{2}$ and $0 < \phi < \dfrac{\pi}{2}$.
   ii) The closed surface defined by ρ = 26 cm and z = ±26 cm.
   iii) The plane z = 26 cm. (07 Marks)

c. Derive the expression for $\bar{E}$ due to infinite line charge of charge density $\rho_L\,(c/m)$. (05 Marks)

Module-2

3 a. State and prove Gauss law for point charge. (05 Marks)

b. State and prove divergence theorem. (05 Marks)

c. In each of the following parts, find value for div $\bar{D}$ at the point specified.
   i) $\bar{D} = (2xyz - y^2)\bar{a}_x + (x^2z - 2xy)\bar{a}_y + x^2y\bar{a}_z\ c/m^2$ at $P_A(2, 3, -1)$.
   ii) $\bar{D} = 2\rho z^2 \sin^2\phi\,\bar{a}_\rho + \rho z^2 \sin 2\phi\,\bar{a}_\phi + 2\rho^2 z \sin^2\phi\,\bar{a}_z\ c/m^2$ at $P_B(\rho = 2, \phi = 110°, z = -1)$. (06 Marks)

OR

4 a. Define potential difference and absolute potential. (04 Marks)

b. A point charge of 6 nc is located at origin in free space, find potential of point p, if p is located at (0.2, -0.4, 0.4) and
   i) V = 0 at infinity
   ii) V = 0 at (1, 0, 0)
   iii) V = 20 V at (-0.5, 1, -1) (06 Marks)

✓c. Derive point form of continuity equation for current. (06 Marks)

Module-3

5 ✓a. Derive the expression for Poisson's and Laplace's equation. ✓ (05 Marks)

b. Two plates of parallel plate capacitors are separated by distance 'd' and maintained at potential zero and V_0 respectively. Assuming negligible fringing effect, determine potential at any point between the plates. (06 Marks)

✓c. State and prove uniqueness theorem. (05 Marks)

OR

6 a. State and explain Biot-Savart law. (06 Marks)

b. Find the magnetic flux density at the centre '0' of a square of sides equal to 5m and carrying 10 amperes of current. (06 Marks)

c. At a point p(x, y, z), the components of vector magnetic potential $\bar{A}$ are given as $A_x = 4x + 3y + 2z$, $A_y = 5x + 6y + 3z$ and $A_z = 2x + 3y + 5z$. Determine $\bar{B}$ at point P. (04 Marks)

Module-4

7 a. Derive Lorentz force equation. (05 Marks)

b. ✓Derive an expression for the force on a differential current element placed in a magnetic field. (06 Marks)

c. ✓A conductor 4m long lies along the y-axis with a current of 10 amps in the $\bar{a}_y$ direction. Find the force on the conductor if the field is $\bar{B} = 0.005\,\bar{a}_x$ Telsa. (05 Marks)

OR

8 a. Define: i) Magnetization, ii) Permeability. (04 Marks)

b. Find the magnetization in a magnetic material where
   i) $\mu = 1.8 \times 10^5$ (H/m) and 120 (A/m)
   ii) $\mu_r = 22$, there are 8.3×10^{28} atoms/m³ and each atom has a dipole moment of 4.5×10^{-27} (A/m²) and
   iii) B = 300 μT and $\chi_m = 15$ (06 Marks)

c. Discuss the boundary conditions at the interface between two media of different permeabilities. (06 Marks)

Module-5

9 a. State and explain Faraday's law of electromagnetic induction. (04 Marks)

b. Find the frequency at which conduction current density and displacement current are equal in a medium with $\sigma = 2 \times 10^{-4}$ ℧/m and $\varepsilon_r = 81$. (06 Marks)

c. List Maxwell's equations in point form and integral form. (06 Marks)

OR

10 a. Obtain solution of the wave equation for a uniform plane wave in free space. (06 Marks)

b. State and prove Poynting theorem. (06 Marks)

c. The depth of penetration in a certain conducting medium is 0.1 m and the frequency of the electromagnetic wave is 1.0 MHz. Find the conductivity of the conducting medium. (04 Marks)

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1.(a) State and explain Coulomb's law in vector form. Express the result in Cartesian coordinates.

<u>Soln.</u> The force b/w two very small charged objects separated in vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the dist. b/w them.

i.e. $F = \dfrac{k\, Q_1 Q_2}{R^2}$

Q_1 & Q_2 → +ve or −ve quantities of charge
<u>unit coulomb</u> (C)

R → separation in m.

k → constant of proportionality.

$k = \dfrac{1}{4\pi\epsilon_0}$

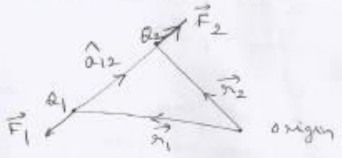
where, ϵ_0 → permittivity of free space.

$\epsilon_0 = 8.854 \times 10^{-12}$ F/m

$= \dfrac{1}{36\pi} \times 10^{-9}$ F/m.

F → Force in Newton.

1.(a) <u>Vector form of Coulomb's law</u>



$\vec{r_1} \rightarrow$ locates Q_1

$\vec{r_2} \rightarrow$ locates Q_2

Q_1, Q_2 of same sign, F_2 in the direction as indicated.

$F_2 \rightarrow$ force ~~ed~~ exerted on Q_2 by Q_1.

$\hat{a_{12}} \rightarrow$ unit vector along $\vec{R_{12}}$.

Then, the vector form of coulomb's law is

$$\vec{F_2} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2} \hat{a_{12}}$$

where, $$\hat{a_{12}} = \frac{\vec{R_{12}}}{|\vec{R_{12}}|} = \frac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|}$$

$|\vec{R_{12}}| = R =$ Distance b/w the ~~the~~ two charges.

let, $\vec{F_1}$ be the force exerted by Q_1 on Q_2.

$$\vec{r_1} = \vec{r_2} + \vec{R_{21}}$$

$$\Rightarrow \vec{R_{21}} = \vec{r_1} - \vec{r_2} = -(\vec{r_2} - \vec{r_1})$$

$$\therefore \hat{a}_{12} = -\hat{a}_{21}.$$

1.(a) $$\therefore F_1 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{21}^2}(-\hat{a_{21}}) = -\vec{F_2}.$$

$\Rightarrow$ Coulomb's law is a mutual force.

<u>Important observations:</u>

i) charges should be point charges and stationary in nature.

ii) Should consider the signs of the charges to decide whether the force will be attractive or repulsive.

iii) Coulomb's law is linear.

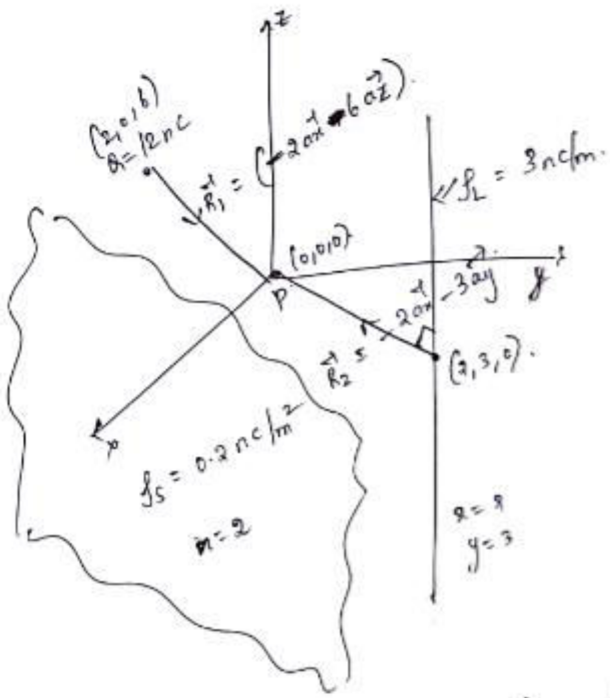
i.e. if $\vec{F_2} = -\vec{F_1}$

then, $n\vec{F_2} = -n\vec{F_1}$

where n is a scalar.

iv) Force on a charge in the presence of several other charges is the sum of the forces on that charge due to each of the other charges acting alone.

1(b)



$$\vec{E} = \vec{E_P} + \vec{E_L} + \vec{E_S} = \frac{Q}{4\pi\varepsilon_0 |\vec{R_1}|^3} \cdot \vec{R_1} + \frac{\rho_L}{2\pi\varepsilon_0 |\vec{R_2}|^2} \cdot \vec{R_2} + \frac{\rho_S}{2\varepsilon_0} \cdot \vec{a_n}$$

$$\vec{E} = \frac{12\times10^{-9} \times 9\times10^{9} \times (-2\vec{a_x} - 6\vec{a_z})}{(\sqrt{4+36})^3} + \frac{3\times10^{-9} \times 18\times10^{9} \times (-2\vec{a_x} - 3\vec{a_y})}{(\sqrt{4+9})^2}$$

$$+ \frac{0.2\times10^{-9} \times 36\pi\times10^{9}}{2} (-\vec{a_x})$$

$$\boxed{\vec{E} = -20.4545\,\vec{a_x} - 12.46\,\vec{a_y} - 2.56\,\vec{a_z} \ \ V/m}$$

1.(c) (i)

$$Q_{enc} = \iiint_V \rho_v \, dv = \int_{z=2}^{4}\int_{\phi=0}^{\pi}\int_{\rho=0}^{0.1} \rho^2 z^2 \sin 0.6\phi \;\; \rho \, d\rho \, d\phi \, dz$$

$$= \int_0^{0.1} \rho^3 \, d\rho \times \int_0^{\pi} \sin 0.6\phi \, d\phi \times \int_2^4 z^2 \, dz$$

$$= \left[\frac{\rho^4}{4}\right]_0^{0.1} \left[-\frac{\cos 0.6\phi}{0.6}\right]_0^{\pi} \times \left[\frac{z^3}{3}\right]_2^4$$

$$= \frac{(0.1)^4}{4} \times \left[\frac{0.309}{0.6} + \frac{1}{0.6}\right] \times \left[\frac{1}{3}\left(4^3 - 2^3\right)\right]$$

$$\boxed{Q_{enc} = 1.018 \text{ mC}}$$

$$Q_{enc} = \iiint_V \rho_v \, dv$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} \frac{e^{-2r}}{r^2} \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{\infty} e^{-2r} \, dr \times \int_0^{\pi} \sin\theta \, d\theta \times \int_0^{2\pi} d\phi$$

$$= \left[\frac{e^{-2r}}{-2}\right]_0^{\infty} \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= -\frac{1}{2}(0 - 1) \times (1+1) \times 2\pi$$

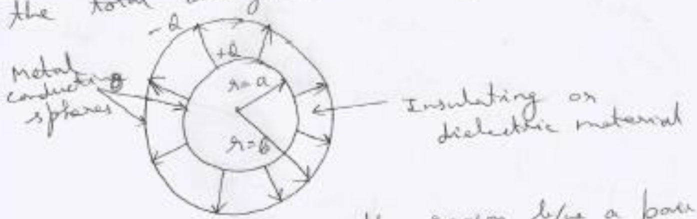
$$= 2\pi$$

$$\boxed{Q_{enc} = 6.28 \text{ C}}$$

2.(a)

soln.

Gauss's law states that the electric flux passing through any closed surface is equal to the total charge enclosed by that surface.



The electric flux in the region b/w a pair of charged concentric spheres. The direction and mag. of D are not functions of the dielectric b/w the spheres.

The mathematical form of Gauss's law is,

$$\psi = \oint_s \vec{D_s} \cdot \vec{ds} = \text{charge enclosed} = Q.$$

where, $Q = \int_{vol} \rho_v \, dv$.

$\psi \rightarrow$ total flux passing through the closed surface.

The direction of D at a pt. is the direction of the flux lines at that point, and the mag. is given by the no. of flux lines crossing a surface normal to the lines divided by the surface area

$$\therefore \vec{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\,\hat{a_r} \quad (\text{inner sphere})$$

$$\vec{D}\Big|_{r=b} = \frac{-Q}{4\pi b^2}\,\hat{a_r} \quad (\text{outer sphere})$$

$a \leq r \leq b$

$$\vec{D} = \frac{Q}{4\pi r^2}\,\hat{a_r}$$

Shrink inner sphere, smaller and smaller. we reach point charge

$$\vec{D} = \frac{Q}{4\pi r^2}\,\hat{a_r}.$$

Q lines of flux are symmetrically directed outward from the pt. and pass through an imaginary spherical surface of area $4\pi r^2$.

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{a_r}.$$

$\therefore \boxed{D = \epsilon_0 E} \rightarrow$ free space.

2.(b)

solution a) Flux = flux density × Surface area

$\psi = \vec{D} \cdot$ Surface area

at $r = 26\,cm$, $\vec{D} = \dfrac{Q}{4\pi r^2} \cdot \vec{a_r}$

$$\vec{D} = \frac{60\mu}{4\pi \cdot (26\times10^{-2})^2} \cdot \vec{a_r}$$

Surface area, $\vec{S} = \iint \vec{ds_r} = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} r^2 \sin\theta \, d\theta \, d\phi \cdot \vec{a_r}$

$$= (26\times10^{-2})^2 \times \left[-\cos\theta\right]_0^{\pi/2} \cdot \pi/2 \;\; \vec{a_r}$$

$$\vec{S} = (0.26)^2 \times \pi/2 \cdot \vec{a_r}$$

$$\psi = \frac{60\mu}{4\pi \times (0.26)^2} \cdot (0.26)^2 \times \pi/2$$

$$\boxed{\psi = 7.5\,\mu C}$$

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b) closed surface ⇒ total flux = Q_{enc}

$\psi = 60\,\mu c$

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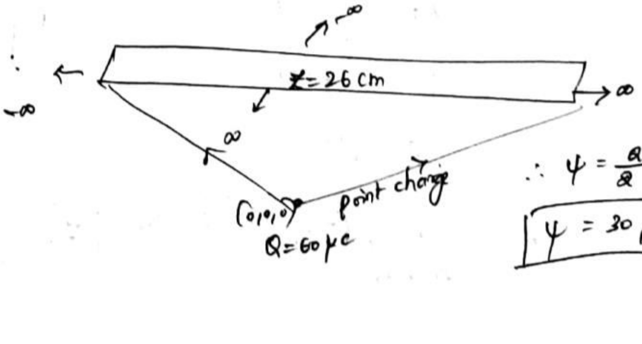
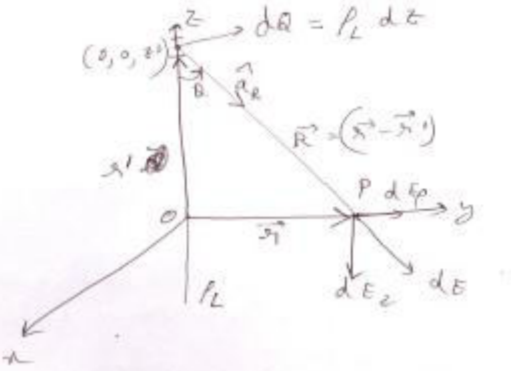
c) plane $z = 26\,cm$



$\therefore \psi = \dfrac{Q}{2}$

$\boxed{\psi = 30\,\mu c}$

2.(c)



– consider a general point $P(0, y, 0)$ on the y-axis to determine the field.

– we consider incremental charge,

$$dQ = \rho_L dz'.$$

– Aim to find incremental field.

$$\therefore d\vec{E} = \frac{\rho_L\, dz' (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |r - r'|^3}.$$

where, $\vec{r} = y\hat{a}_y = \rho\hat{a}_\rho$

$\vec{r}' = z'\hat{a}_z$.

$$\therefore \vec{r} - r' = (\rho\hat{a}_\rho - z'\hat{a}_z)$$

$$\therefore d\vec{E} = \frac{\rho_L\, dz' (\rho\hat{a}_\rho - z'\hat{a}_z)}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

we know only ρ component is present

$$dE_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho\, dz'}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho \, dz'}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

let, $z' = \rho \cot\theta$

$\therefore dz' = -\rho \,\mathrm{cosec}^2\theta \, d\theta$

$z' = \rho \frac{\cos\theta}{\sin\theta}$

$\infty = \frac{\cos 0}{\sin 0} \Rightarrow$ for $z' = \infty$, $\theta = 0^\circ$

$z' = -\frac{\cos(\pi)}{\sin(\pi)} \Rightarrow z' = -\infty$, $\theta = \pi$

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0} \int_{\pi}^{0} \frac{\rho \cdot (-\rho \,\mathrm{cosec}^2\theta \, d\theta)}{(\rho^2 + \rho^2 \cot^2\theta)^{3/2}}$$

$$= \frac{\rho_L}{4\pi\epsilon_0} \int_{\pi}^{0} \frac{-\rho^2 \,\mathrm{cosec}^2\theta \, d\theta}{(\rho^2)^{3/2} (1 + \cot^2\theta)^{3/2}}$$

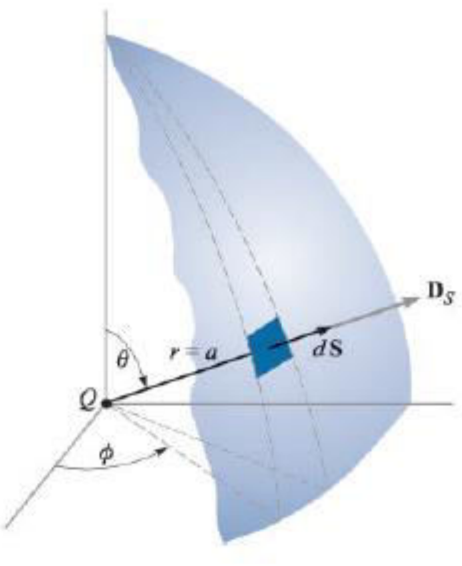
$$= \frac{\rho_L}{4\pi\epsilon_0 \rho} \int_{\pi}^{0} \frac{-\mathrm{cosec}^2\theta \, d\theta}{\mathrm{cosec}^3\theta}$$

$$= \frac{\rho_L}{4\pi\epsilon_0 \rho} \int_{\pi}^{0} -\sin\theta \, d\theta = \frac{\rho_L}{4\pi\epsilon_0 \rho} \left[+\cos\theta \right]_{\pi}^{0}$$

$$= \frac{\rho_L}{4\pi\epsilon_0 \rho} [1 + 1] = \frac{2\rho_L}{4\pi\epsilon_0 \rho}$$

$$= \frac{\rho_L}{2\pi\epsilon_0 \rho}$$

3.(a)



Let's place a point charge Q at the origin of a spherical co-ordinate system.

- Let the closed surface be a sphere of radius a.

- The electric field intensity of the point charge,

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{a}_r$$

Now, $\vec{D} = \epsilon_0 E$.

$$\therefore\ \vec{D} = \frac{Q}{4\pi r^2}\,\hat{a}_r$$

At the surface of the sphere,

$$\vec{D}_s = \frac{Q}{4\pi a^2}\,\hat{a}_r.$$

The differential area element in spherical co-ordinates,

$$ds = r^2 \sin\theta\, d\theta\, d\phi = a^2 \sin\theta\, d\theta\, d\phi.$$

$$\text{or}\quad \vec{ds} = a^2 \sin\theta\, d\theta\, d\phi\, \hat{a}_r.$$

The integrand is,

$$\vec{D}_s \cdot \vec{ds} = \frac{Q}{4\pi a^2} \cdot a^2 \sin\theta\, d\theta\, d\phi\, \hat{a}_r \cdot \hat{a}_r$$

$$= \frac{Q}{4\pi} \sin\theta\, d\theta\, d\phi.$$

... board is chosen

$$\therefore \int_{\phi=0}^{2\pi} \frac{Q}{4\pi}\left(-\cos\theta\right)_0^{\pi} d\phi$$

$$= \int_0^{2\pi} \frac{Q}{4\pi}(+1+1)\, d\phi$$

$$= \frac{Q}{2\pi} \cdot 2\pi = Q.$$

$\therefore$ we obtain the result that Q coulombs of electric flux are crossing the surface, as we should since the enclosed charge is Q coulombs.

3.(b)

Divergence theorem: (For Electric flux density).

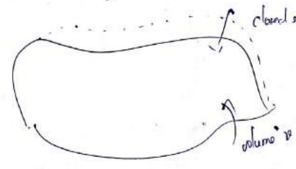
Gauss' law:

$$\oint_S \vec{D}\cdot d\vec{s} = Q = \int_V \rho_v \, dv.$$

$$\boxed{\oint_S \vec{D}\cdot d\vec{s} = \int_V \nabla\cdot\vec{D}\, dv.} \leftarrow \text{Divergence theorem.}$$

Statement:

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of that vector field throughout the volume enclosed by the closed surface.



Divergence of flux density throughout a volume

=

net flux crossing the enclosing surface.

Differential Volume element.

No Symmetry.

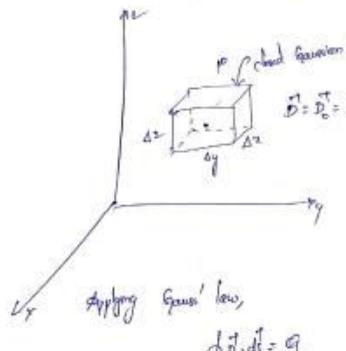
Choose a small gaussian surface → almost $\vec{D}$ is constant over that surface.

Result becomes correct only when volume $\Delta v \to 0$ (shrinks).

we will not obtain $\vec{D}$, obtain the valuable information about the way $\vec{D}$ varies in the region.

⇓

one of Maxwell's four equations (base to all electromagnetic theory)



$$\vec{D} = \vec{D}_0 = D_{x0}\vec{a_x} + D_{y0}\vec{a_y} + D_{z0}\vec{a_z}.$$

Applying Gauss' law,

$$\oint_S \vec{D}\cdot d\vec{s} = Q$$

$$\int_{front} + \int_{back} + \int_{left} + \int_{Right} + \int_{top} + \int_{bottom} \vec{D}\cdot d\vec{s} = Q.$$

Surface element is very small → $\vec{D}$ is constant over the surface.

$$\int_{front} \vec{D}\cdot d\vec{s} = \vec{D}_{front}\cdot \Delta\vec{S}_{front}$$

$$= \vec{D}_{front}\cdot \Delta y \Delta z \,\vec{a_x}$$

$$= D_{x,front}\cdot \Delta y \Delta z$$

Front face ⇒ distance of $\frac{\Delta x}{2}$ from P.

$$\boxed{D_{x,\text{front}} = D_{x0} + \frac{\Delta x}{2} \cdot \frac{\partial D_x}{\partial x}}$$

Rate of change of D_x wrt x
∴ D_x varies with y & z.

value of D_x at P.

Expansion of Taylor series.

$$\boxed{\int_{\text{front}} = \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}\right) \Delta y \, \Delta z}$$

Integral over the back surface

$$\int_{\text{back}} = \vec{D}_{\text{back}} \cdot \Delta \vec{S}_{\text{back}}$$

$$= \vec{D}_{\text{back}} \cdot (-\Delta y \, \Delta z \, \vec{a}_x)$$

$$\int_{\text{back}} = -D_{x,\text{back}} \, \Delta y \, \Delta z$$

$$\int_{\text{back}} = -\left[D_{x0} + \left(-\frac{\Delta x}{2}\right) \frac{\partial D_x}{\partial x}\right] \Delta y \, \Delta z$$

$$\boxed{\int_{\text{back}} = \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}\right) \Delta y \, \Delta z}$$

$$\boxed{\int_{\text{front}} + \int_{\text{back}} = \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z}$$

Exactly in the same way

$$\int_{\text{right}} + \int_{\text{left}} = \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

$$\int_{\text{top}} + \int_{\text{bottom}} = \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

$$\therefore \oint_S \vec{D} \cdot d\vec{s} = \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z + \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z + \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z \qquad (2)$$

$$\boxed{\oint_S \vec{D} \cdot d\vec{s} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \Delta x \, \Delta y \, \Delta z = Q}$$

It is an approximation which is better when Δv becomes smaller.

∴ $\Delta v \to 0$. where $\Delta v = \Delta x \, \Delta y \, \Delta z$

$$\boxed{\text{Charge enclosed in volume } \Delta v = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \cdot \Delta v}$$

We now obtain the exact relationship by $\Delta v \rightarrow 0$.

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \frac{\oint \vec{D}\cdot d\vec{s}}{\Delta v} = \frac{Q}{\Delta v}.$$

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_s \vec{D}\cdot d\vec{s}}{\Delta v} = \left(\lim_{\Delta V \to 0} \frac{Q}{\Delta v}\right) \rightarrow \text{volume charge density}$$

$$\boxed{\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{D}\cdot d\vec{s}}{\Delta v}} \rightarrow (1)$$

&

$$\boxed{\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v.} \rightarrow (2)$$

Divergence of $\vec{D}$ = $\text{div}\,\vec{D} = \lim_{\Delta v \to 0} \dfrac{\oint_s \vec{D}\cdot d\vec{s}}{\Delta v}$

Divergence of the vector flux density $\vec{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

$$\boxed{\text{div}\,\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}} \quad \text{Rectangular}$$

$$\boxed{\text{div}\,\vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}} \quad \text{cylindrical}$$

$$\boxed{\text{div}\,\vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}} \quad \text{spherical}$$

MAXWELL's FIRST EQUATION OF ELECTROSTATICS:

1) $\text{div}\,\vec{D} = \lim_{\Delta v \to 0} \dfrac{\oint_s \vec{D}\cdot d\vec{s}}{\Delta v}$ (Definition of divergence)

2) $\text{div}\,\vec{D} = \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z}$ (Result of applying the definition to a differential volume element (in Rectangular co-ordinates))

3) $\text{div}\,\vec{D} = \rho_v$

Gauss's law — flux leaving any closed surface

$$\boxed{\oint_s \vec{D}\cdot d\vec{s} = Q} \leftarrow \text{charge enclosed}$$

Gauss' law per unit volume,

$$\frac{\oint_s \vec{D}\cdot d\vec{s}}{\Delta v} = \frac{Q}{\Delta v}$$

As the volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{\oint \vec{D} \cdot \vec{ds}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$ volume charge density

Divergence

$$\boxed{\operatorname{div} \vec{D} = \rho_v.}$$

First of Maxwell's four equations

Point form of Gauss's law (or) Maxwell's first equation

Statement:

The electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there.

Gauss's law → $\boxed{\oint_S \vec{D} \cdot \vec{ds} = Q = \int_V \rho_v \, dv.}$

(or) Integral form of Maxwell's first equation

3.(c)

3) c) Find $\operatorname{div} \vec{D}$.

i) $\vec{D} = (2xyz - y^2)\vec{a_x} + (x^2 z - 2xy)\vec{a_y} + x^2 y \, \vec{a_z}$ C/m^2

at P_A $(2,3,-1)$

$$\vec{\nabla} \cdot \vec{D} = \frac{\partial}{\partial x}(2xyz - y^2) + \frac{\partial}{\partial y}(x^2 z - 2xy) + \frac{\partial}{\partial z}(x^2 y).$$

$$\vec{\nabla} \cdot \vec{D} = 2yz - 2x + 0.$$

$$\vec{\nabla} \cdot \vec{D} = 2yz - 2x \quad C/m^3$$

$$\vec{\nabla} \cdot \vec{D} \text{ at } P_A (2,3,-1) = 2(3)(-1) - 2(2)$$

$$= -6 - 4$$

$$\boxed{\vec{\nabla} \cdot \vec{D} = -10 \quad C/m^3}$$

(iii) $\vec{D} = 2\rho z^2 \sin^2\phi \, \vec{a_\rho} + \rho z^2 \sin 2\phi \, \vec{a_\phi} + 2\rho^2 z \sin^2\phi \, \vec{a_z}$ C/m²

at $P_B(\rho = 2, \phi = 110^\circ, z = -1)$

$$\vec{\nabla}\cdot\vec{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho \cdot 2\rho z^2 \sin^2\phi\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\rho z^2 \sin 2\phi\right) + \frac{\partial}{\partial z}\left(2\rho^2 z \sin^2\phi\right).$$

$$= \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(2\rho^2 z^2 \sin^2\phi\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\rho z^2 \sin 2\phi\right) + \frac{\partial}{\partial z}\left(2\rho^2 z \sin^2\phi\right)$$

$$= \frac{1}{\cancel{\rho}}\left(4\cancel{\rho} z^2 \sin^2\phi\right) + \frac{1}{\cancel{\rho}}\cdot \cancel{\rho} z^2 \cos 2\phi \cdot 2 + 2\rho^2 \sin^2\phi$$

$$\boxed{\vec{\nabla}\cdot\vec{D} = 4z^2\sin^2\phi + 2z^2\cos 2\phi + 2\rho^2\sin^2\phi} \text{ C/m}^3$$

$\vec{\nabla}\cdot\vec{D}$ at $P_B(2, 110^\circ, -1)$

$$= 4(-1)^2\sin^2(110^\circ) + 2(-1)^2\cos(2\times110^\circ) + 2(2)^2\sin^2(110^\circ)$$

$$= 3.532 + (-1.532) + 7.0641$$

$$\boxed{\vec{\nabla}\cdot\vec{D} = 9.0641 \text{ C/m}^3}$$

4.(a)

Potential Difference: Potential difference V is defined as the work done (by an external source) in moving a unit +ve charge from one point to another in an electric field.

$$\text{Potential difference} = V = -\int_{\text{init}}^{\text{final}} \vec{E} \cdot d\vec{l}.$$

$V_{AB} \rightarrow$ potential diff. b/w points A and B and is the work done in moving a unit charge from B to A.

Unit of potential difference: Joules/coulomb. commonly Volt.

Absolute potential: The absolute potential of a point is defined as the work done in moving a unit +ve charge from infinity to that point.

The potential diff. b/w two points located at $r = r_A$ and $r = r_B$ in the field of a point charge Q placed at the origin,

$$V_{AB} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = V_A - V_B.$$

$V = 0$ at infinity. We let, $r = r_B$ recede to infinity.

$\therefore$ The potential at r_A becomes, $V_A = \frac{Q}{4\pi\epsilon_0 r_A}$

This expression defines the potential at any

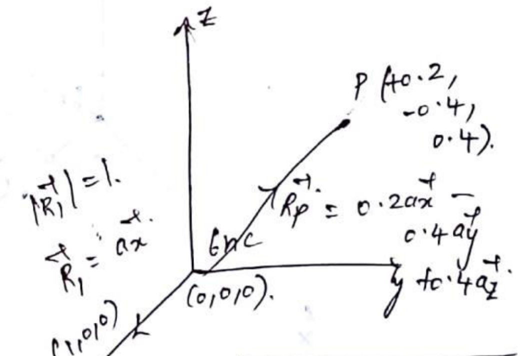
4) 5) A point charge of 6nC is located at origin in free space. Find the potential of point P, if P is located at $(0.2, -0.4, 0.4)$ and.

(i) $V = 0$ at infinity

(ii) $V=0$ at $(1,0,0)$

(iii) $V = 20V$ at $(-0.5, 1, -1)$.

$$V = \frac{Q}{4\pi\varepsilon |\vec{R}|}. \qquad V_{AB} = \frac{Q}{4\pi\varepsilon_0 |\vec{R_A}|} - \frac{Q}{4\pi\varepsilon_0 |\vec{R_B}|}.$$



(i). $V_{P\infty} = V_p - V_\infty.$

$$= \frac{Q}{4\pi\varepsilon_0 |\vec{R_p}|} - V_\infty.$$

$$= \frac{6\times10^{-9} \times 9\times10^{9}}{0.6} - \cancel{\frac{1}{\infty}} \to 0.$$

$$|\vec{R_p}| = \sqrt{(0.2)^2 + (0.4)^2 + (0.4)^2}$$

$$= \sqrt{0.04 + 0.16 + 0.16}$$

$$|\vec{R_p}| = 0.6$$

$V_{p\infty} = 90\ V$

$$\boxed{V_p = 90V}$$

(ii) $$V_{P1} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{R_p}|} - \frac{1}{|\vec{R_1}|}\right].$$

$$= 9\times10^{9} \times 6\times10^{-9}. \left[\frac{1}{0.6} - \frac{1}{1}\right]$$

$$= 9\times6\times\left[\frac{10}{6} - 1\right] = 9\times6\times\left[\frac{4}{6}\right] \quad \boxed{V_{P1} = 36\ V}$$

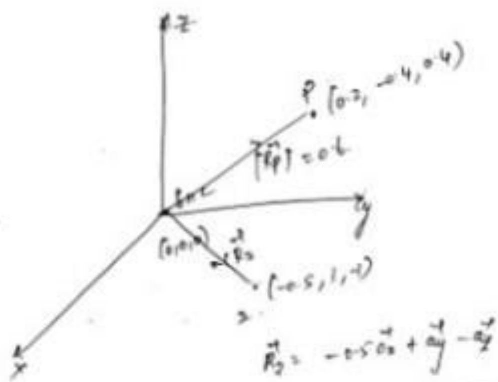
$V_{p1} = V_p - V_1$

$V_p = V_{p1} + V_1$

$= 36 + 0$

$\boxed{V_p = 36\,V}$

(iii)



$\vec{R}_2 = -0.5\vec{a}_x + \vec{a}_y - \vec{a}_z$

$= \sqrt{(0.5)^2 + 1^2 + 1^2}$

$|\vec{R}_2| = 1.5$

$V_{p2} = V_p - V_2$

$= \dfrac{Q}{4\pi\epsilon_0 |\vec{R}_p|} - \dfrac{Q}{4\pi\epsilon_0 |\vec{R}_2|}$

$= 6\times10^{-9} \times 9\times10^{9} \times \left[\dfrac{1}{0.6} - \dfrac{1}{1.5}\right]$

$= 6\times9\times\left[\dfrac{10}{6} - \dfrac{10}{15}\right] = 6\times9\times10\times\left[\dfrac{5-2}{30}\right] = 6\times9\times10\left[\dfrac{3}{30}\right]$

$\boxed{V_{p2} = 54\ V}$

$V_2 = 20\,V$

$\therefore \quad V_p = V_{p2} + V_2$

$\boxed{V_p = 74\,V}$

4. (c)

soln.

Current through a closed surface,

$$I = \oint_s \vec{J} \cdot \vec{ds}.$$

Outward flow of +ve charge is balance by a decrease of +ve charge within the closed surface.

let, Q_i be the charge inside the closed surface

$$\therefore I = \oint_s \vec{J} \cdot \vec{ds} = -\frac{dQ_i}{dt}$$ → reduction in charge, giving -ve sign.

Using divergence theorem,

$$\oint_s \vec{J} \cdot \vec{ds} = \int_{vol} (\vec{\nabla} \cdot \vec{J})\, dv.$$

Now, $Q = \int_{vol} \rho_v \, dv$

$$\therefore \int_{vol} (\vec{\nabla} \cdot \vec{J})\, dv = -\frac{d}{dt} \int_{vol} \rho_v \, dv$$

If the surface is constant derivative becomes a partial derivative.

$$\therefore \int_{vol} (\vec{\nabla} \cdot \vec{J})\, dv = \int_{vol} -\frac{\partial \rho_v}{\partial t}\, dv$$

This is true for any value however small.

This is true for an incremental volume.

$$\therefore (\vec{\nabla} \cdot \vec{J})\, \Delta v = -\frac{\partial \rho_v}{\partial t} \Delta v.$$

$\therefore$ Point form of continuity equation,

$$\boxed{(\vec{\nabla} \cdot \vec{J}) = -\frac{\partial \rho_v}{\partial t}}$$

5.(a) Derive the expression for Poisson's and Laplace's equation.

Soln. From Gauss's law,

$\vec{\nabla}\cdot\vec{D} = \rho_v$

$\vec{D} = \epsilon\vec{E}$

and $\vec{E} = -\vec{\nabla}V$

$\therefore \vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot(\epsilon\vec{E}) = -\vec{\nabla}(\epsilon\vec{\nabla}V) = \rho_v$

$\Rightarrow -\vec{\nabla}\cdot(\vec{\nabla}V) = \frac{\rho_v}{\epsilon}$

$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}V) = -\frac{\rho_v}{\epsilon}$

$\Rightarrow \nabla^2 V = -\frac{\rho_v}{\epsilon}$

$\boxed{\nabla^2 V = \frac{-\rho_v}{\epsilon}}$ → Poisson's equation

Now, $\vec{\nabla}\cdot\vec{A} = \left(\hat{a}_x\frac{\partial}{\partial x} + \hat{a}_y\frac{\partial}{\partial y} + \hat{a}_z\frac{\partial}{\partial z}\right)\cdot(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z)$

$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$

$\vec{\nabla}\cdot\vec{\nabla}V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$

$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$ → rectangular co-ordina

If $\rho_v = 0$ then $\nabla^2 V = 0$

In cylindrical co-ordinates,

$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial\phi^2}\right) + \frac{\partial^2 V}{\partial z^2}$

$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$

5.(b)

$$\frac{\partial^2 V}{\partial x^2} = 0$$

$$\frac{d^2 V}{dx^2} = 0$$

$$\Rightarrow \frac{dV}{dx} = A$$

$$\Rightarrow V = Ax + B$$

$V = V_1$ at $x = x_1$

$V = V_2$ at $x = x_2$

$$A = \frac{V_1 - V_2}{x_1 - x_2} \qquad B = \frac{V_2 x_1 - V_1 x_2}{x_1 - x_2}$$

$V = 0$ at $x = 0$

$V = V_0$ at $x = d$

$\therefore$ $B = 0$, $A = \frac{V_0}{d}$.

$$\therefore V = \frac{V_0 x}{d}$$

$$\therefore \vec{E} = -\frac{V_0}{d}\hat{a}_x$$

$$\vec{D} = \epsilon\vec{E} = -\frac{\epsilon V_0}{d}\hat{a}_x$$

$$D_s = D|_{x=0} = -\frac{\epsilon V_0}{d}\hat{a}_x = \rho_s = D_N$$

$$Q = -\int_s \frac{\epsilon V_0}{d}\, d\vec{s} = -\frac{\epsilon V_0 S}{d}$$

$$\Rightarrow C = \frac{|Q|}{V} = \frac{\epsilon S}{d}.$$

5.(c) State and prove uniqueness theorem.

$\therefore \; V_{1b} = V_{2b} = V_b$

$\Rightarrow V_{1b} - V_{2b} = 0$

considering the vector identity

$$\vec{\nabla}\cdot(V\vec{D}) = V(\vec{\nabla}\cdot\vec{D}) + \vec{D}\cdot(\vec{\nabla}V)$$

considering a scalar to be $(V_1 - V_2)$

considering the vector to be $\vec{\nabla}(V_1 - V_2)$

$$\vec{\nabla}\cdot\left[(V_1 - V_2)\vec{\nabla}(V_1 - V_2)\right] = (V_1 - V_2)\left[\vec{\nabla}\cdot\vec{\nabla}(V_1 - V_2)\right] + \left[\vec{\nabla}(V_1 - V_2)\cdot\vec{\nabla}(V_1 - V_2)\right]$$

taking volume integral on both sides

$$\int_{vol}\vec{\nabla}\cdot\left[(V_1 - V_2)\vec{\nabla}(V_1 - V_2)\right]dv = \int_{vol}\underbrace{(V_1 - V_2)\nabla^2(V_1 - V_2)}_{0\text{ [by hypothesis]}}dv + \int_{vol}\left[\vec{\nabla}(V_1 - V_2)\right]^2 dv \quad (1)$$

$$LHS = \oint_S\left[(V_1 - V_2)\vec{\nabla}(V_1 - V_2)\right]\cdot d\vec{S}$$

$$= \oint_S\left[(V_{1b} - V_{2b})\vec{\nabla}(V_{1b} - V_{2b})\right]\cdot d\vec{S}$$

$$= 0$$

$$\therefore \int_{vol}\left[\vec{\nabla}(V_1 - V_2)\right]^2 dv = 0$$

[Note: an integral may be zero if either the integrand is everywhere zero, or the integrand is +ve in some regions and -ve in other regions, and the contributions cancel algebraically. In this case the 1st reason holds good as $[\vec{\nabla}(V_1 - V_2)]^2$ can not be -ve.

$\Rightarrow \vec{\nabla}(V_1 - V_2) = 0$

$V_1 - V_2 = \text{Constant}$

on the boundary

$V_1 - V_2 = V_{1b} - V_{2b} = 0$

$\Rightarrow V_1 - V_2 = 0$

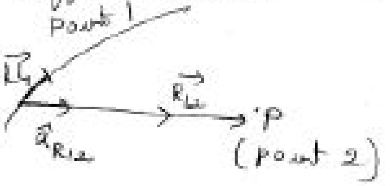
$$\boxed{V_1 = V_2}$$

6.(a) State and explain Biot-Savart's law.

(*) Biot-Savart's law.-

Let's consider a differential current element as a vanishingly small section of a current-carrying filamentary conductor, where a filamentary conductor is the limiting case of a cylindrical conductor of circular cross-section as the radius approaches zero.

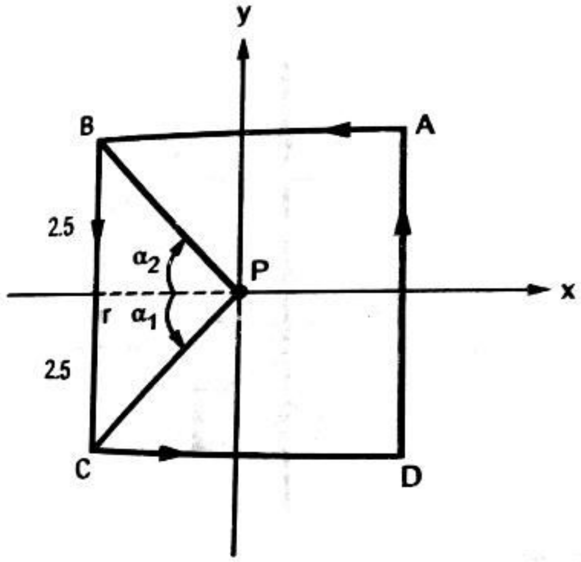
We assume a current I flowing in a different vector length of the filament $d\vec{L}$. The Biot-Savart's law then states that, at any point P the mag. of the mag. field intensity produced by the differential element is proportional to the product of the current, the mag. of the differential length and the sine of the angle lying b/w the filament and a line connecting the filament to the point P at which the field is desired. Also, the magnitude of the mag. field intensity is inversely proportional to the square of the distance from the differential element to point P.



$$d\vec{H_2} = \frac{I_1 d\vec{L_1} \times \hat{a}_{R_{12}}}{4\pi R_{12}^2}$$

where $d\vec{H_2}$ → mag. field intensity produced by a differential current element $I_1 d\vec{L_1}$. The direction of $d\vec{H_2}$ is into the page.

6.(b)



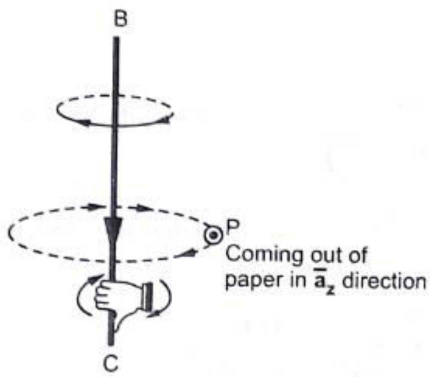
Alternative method : Consider one side of a square as shown in the Fig. 7.20, in xy plane. Consider segment BC, which is finite length of the conductor. As B is above P, α_1 is negative and α_2 is positive.

$$\alpha_1 = \tan^{-1}\frac{2.5}{2.5} = 45°, \text{ but}$$

$$\alpha_1 = -45°$$

$$\alpha_2 = +45°$$

$$\therefore \quad |\bar{H}| = \frac{I}{4\pi r}\sin\alpha_2 - \sin\alpha_1 = \frac{10}{4\pi \times 2.5}[\sin(45°) - \sin(-45°)] = 0.4501 \text{ A/m}$$



Important note : As BC segment is not along z axis while using formula derived earlier do not use direction as $\bar{a}_\phi$. **Remember that $\bar{H}$ direction is normal to the plane containing the source. In this case, square is in xy plane normal to which is $\bar{a}_z$ hence direction of $\bar{H}$ is $\bar{a}_z$, as shown in the Fig. 7.21 by right handed screw rule.**

$$\therefore \quad \bar{H} = 0.4501\ \bar{a}_z \ \text{A/m}$$

$$\therefore \bar{H}_{total} = 4\bar{H} = 1.8\ \bar{a}_z \ \text{A/m}$$

6.(c)

soln.

$$A_x = 4x + 3y + 2z$$
$$A_y = 5x + 6y + 3z$$
$$A_z = 2x + 3y + 5z.$$

$$\therefore \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (4x+3y+2z) & (5x+6y+3z) & (2x+3y+5z) \end{vmatrix}$$

$$= \hat{a}_x (3-3) + \hat{a}_y (2-2) + \hat{a}_z (5-3)$$

$$= 2\,\hat{a}_z \ \mathrm{Wb/m^2} = \vec{B}.$$

7.(a)

Consider the case of current carrying conductors

convection current density,

$$\boxed{\vec{J} = \rho_v \vec{v}}$$

$$dQ = \rho_v\, dv.$$

$$\therefore \vec{dF} = \rho_v\, dv\, \vec{v} \times \vec{B}$$

$$\Rightarrow \boxed{\vec{dF} = \vec{J} \times \vec{B}\, dv} \quad [\because \vec{J} = \rho_v \vec{v}]$$

$$\vec{J}\, dv = \vec{K}\, ds = I\, \vec{dL}$$

$$\therefore \vec{dF} = \sigma\, \vec{K} \times \vec{B}\, ds$$

$$d\vec{F} = I\, \vec{dL} \times \vec{B}.$$

$\vec{J} = \rho_v \vec{v}$

$d\vec{F} = \rho_v\, dv\, \vec{v} \times \vec{B}$

$= \vec{J} \times \vec{B}\, dv$

$= \sigma \vec{K} \times \vec{B}$

$= \vec{dL} \times \vec{B}$

Integrating,

$$\vec{F} = \int_{vol} \vec{J} \times \vec{B}\, dv$$

$$\vec{F} = \oint_s \vec{K} \times \vec{B}\, ds$$

$$\vec{F} = \oint I\, \vec{dL} \times \vec{B} = -I \oint \vec{B} \times \vec{dL}$$

$$\vec{F} = Q\vec{E}.$$

direction same as electric field intensity.

A charge Q in motion in a magnetic field with flux-density $\vec{B}$.

velocity of charge $= \vec{v}$.

$$\boxed{\vec{F} = Q\,\vec{v} \times \vec{B}}$$

For a force which is applied in a direction $\perp^r$ to the direction of motion can't change magnitude of particle velocity.

i.e. steady magnetic field can not do any energy transfer to the moving charge

Force on a moving particle due to combined effe of electric and magnetic fields.

$$\vec{F} = Q\,(\vec{E} + \vec{v} \times \vec{B}).$$

7.(b) Derive an expression for force on a differential current element placed in a magnetic field.

Force b/w differential current elements

Consider

Using Biot-Savart's law, the field at point 2 due to a current element at point 1 is,

$$d\vec{H_2} = \frac{I_1 \, d\vec{L_1} \times \hat{a}_{R_{12}}}{4\pi R_{12}^2}$$

Differential force on a differential current element is, $d\vec{F} = (I d\vec{L} \times \vec{B})$.

Here, in place of $\vec{B}$, use $d\vec{B_2}$. $d\vec{B_2}$ is the differential flux density at point 2 caused by current element 1.

One more current element $I_2 \, d\vec{L_2}$.

~~Force~~ differential force on element 2 $d(d\vec{F_2})$.

$$\therefore d(d\vec{F_2}) = I_2 \, d\vec{L_2} \times d\vec{B_2}$$

$$= I_2 \, d\vec{L_2} \times \mu_0 \, d\vec{H_2}$$

$$= \mu_0 I_2 \, d\vec{L_2} \times \frac{I_1 \, d\vec{L_1} \times \hat{a}_{R_{12}}}{4\pi R_{12}^2}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi R_{12}^2} \, d\vec{L_2} \times (d\vec{L_1} \times \hat{a}_{R_{12}})$$

$\therefore$ The total force b/w two filamentary circuits is obtained by integrating twice.

$$\vec{F_2} = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[d\vec{L_2} \times \oint \frac{d\vec{L_1} \times \hat{a}_{R_{12}}}{R_{12}^2} \right]$$

$$= \frac{\mu_0 I_1 I_2}{4\pi} \oint \left[\oint \frac{\hat{a}_{R_{12}} \times d\vec{L_1}}{R_{12}^2} \right].$$

7.(c)

A conductor 4 m long lies along the y-axis with a current of 10A in $\vec{a_y}$ direction. Find the force on the conductor if the field in the region is $\vec{B} = 0.005 \, \vec{a_x}$ T.

$$\vec{F} = I d\vec{L} \times \vec{B}$$

$$d\vec{L} = 4\vec{a_y}$$

$$\vec{B} = 0.005 \, \vec{a_x}$$

$$I = 10A$$

$$\vec{F} = 10 \times 4\vec{a_y} \times 0.005 \, \vec{a_x}$$

$$\boxed{\vec{F} = -0.2 \, \vec{a_z}} \text{ N}$$

8.(a)

$\therefore$ Magnetization M is the magnetic dipole moment per unit volume,

$$M = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n \Delta v} m_i \quad A/m.$$

Permeability: A quantity measuring the influence of a substance on the magnetic flux in the region it occupies.

8.(b)

(i) $\mu = 1.8 \times 10^{-5}$ H/m $= \mu_0 \mu_r$ & $\vec{H} = 120$ A/m.

$\mu_0 = 4\pi \times 10^{-7}$

$\mu_r = \frac{\mu}{\mu_0} = \frac{1.8 \times 10^{-5}}{4\pi \times 10^{-7}} = 14.3239$

$\chi_m = 13.3239$, $\vec{M} = \chi_m \vec{H}$.

$\boxed{\vec{M} = 1598.873 \text{ A/m}}$

(iii) $B = 300\ \mu T$

$\vec{H} = \frac{\vec{B}}{\mu}$. $\mu_r = 1 + \chi_m$

$\mu = \mu_0 \mu_r = \mu_0 \times 16$

$\vec{M} = \chi_m \cdot \vec{H}$

$= 15 \times \frac{300 \times 10^{-6}}{\mu_0 \times 16}$

$\boxed{\vec{M} = 223.811}$ A/m

(ii) Magnetization is total magnetic dipole moment per unit volume

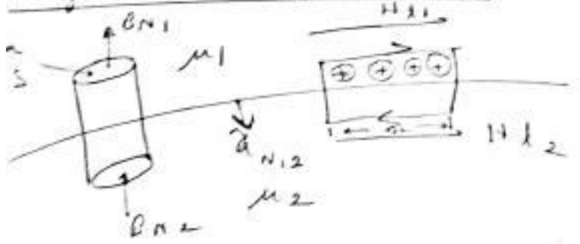
$m_i = 4.5 \times 10^{-27}$ A m^2 (dipole moment for each atom)

$n = 8.3 \times 10^{28}$ atoms/m^3 (total no of atoms)

$\vec{M} = n \times \vec{m_i} = 373.5 \ \frac{A\,m^2}{m^3}$ $\boxed{M = 373.5 \text{ A/m}}$

8.(c) Discuss the boundary conditions at the interface between two media of different permeabilities.

Magnetic boundary conditions



Gauss's law of mag.

$$\oint_S \vec{B}\cdot d\vec{s} = 0$$

$$B_{N1}\,\Delta S - B_{N2}\,\Delta S = 0$$

$$\Rightarrow \boxed{B_{N1} = B_{N2}}$$

$$\therefore H_{N2} = \frac{\mu_1}{\mu_2} H_{N1}$$

Normal comp. of B is continuous, of H is discontinuous by ratio $\left(\frac{\mu_1}{\mu_2}\right)$.

$$M_{N2} = \chi_{m2}\left(\frac{\mu_1}{\mu_2} H_{N1}\right) \quad \to H_{N2}$$

$$= \frac{\chi_{m2}\mu_1}{\chi_{m1}\mu_2} M_{N1} \qquad \left[M = \chi_m H\right] \quad H_{N1} = \frac{M_{N1}}{\chi_{m1}}$$

$\therefore$ Now, $\oint \vec{H}\cdot d\vec{l} = I$.

small closed path in a plane normal the boundary surface.

$$H_{t1}\,\Delta L - H_{t2}\,\Delta L = K\,\Delta L$$

Surface current $\vec{K}$ $\to$ comp. normal to the plane of closed path is K.

$$\therefore \boxed{H_{t1} - H_{t2} = K}$$

$$\Rightarrow (\vec{H}_1 - \vec{H}_2) \times \hat{a}_{N12} = \vec{K}$$

$\hat{a}_{N12}$ $\to$ unit normal at the boundary directed from region 1 to region 2

$$(\vec{H}_{t1} - \vec{H}_{t2}) = \hat{a}_{N12} \times \vec{K}$$

$$\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K$$

$$\left[M_{t2} = \frac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m2} K\right]$$

$$H_{t_1} - H_{t_2} = k$$

$$\Rightarrow \frac{M_{t_1}}{\chi_{m_1}} - \frac{M_{t_2}}{\chi_{m_2}} = k$$

$$\Rightarrow M_{t_1}\chi_{m_2} - M_{t_2}\chi_{m_1} = k\,\chi_{m_1}\chi_{m_2}$$

$$\Rightarrow M_{t_2}\chi_{m_1} = -k\,\chi_{m_1}\chi_{m_2} + M_{t_1}\chi_{m_2}$$

$$\Rightarrow \boxed{M_{t_2} = \frac{\chi_{m_2}}{\chi_{m_1}} M_{t_1} - k\,\chi_{m_2}}$$

9. (a)

– Two separate windings on an iron toroid and placed a galvanometer in one ckt. and battery on the other.

— closing the battery ckt, momentary deflection of the galvanometer.

– A similar deflection in the opposite direction occurred when the battery was disconnected

– either a moving mag. field or a moving coil could also produce a galvanometer deflection

– Time varying mag. field produces an emf which may establish a current in a suitable closed ckt.

Faraday's law: $e \propto -\frac{d\phi}{dt} \Rightarrow e = -k\frac{d\phi}{dt}$ in SI $k=1$

$$emf = -\frac{d\phi}{dt}$$

$\therefore e = -\frac{d\phi}{dt}$ conducting line or not

~~If the~~ we consider a closed path, the mag. flux is that flux that passes through any and every surface whose perimeter is the closed path.

and $\frac{d\phi}{dt} \rightarrow$ time rate of change of flux

A non-zero value of $\frac{d\phi}{dt}$ occurs when:

i) Time-changing flux linking a closed path.

ii) Relative motion b/w a steady flux and a closed path.

iii) A combination of the two.

The minus sign is an indication that the emf is in such a direction as to produce a current whose

- If the closed path is that taken N-turn filamentary conductor, it is sufficiently accurate to consider the turns as coincident and let,

$$emf = -N\frac{d\phi}{dt}$$

where, $\phi \rightarrow$ flux passing through any one of the N coincident paths.

e.m.f is defined as,

$emf = \oint \vec{E}.d\vec{l} \rightarrow$ voltage about a specific closed path

for electrostatic $\oint \vec{E}.d\vec{l} = 0$

$$e.m.f = \oint \vec{E}.d\vec{l} = -\frac{d\phi}{dt} = -\frac{d}{dt}\int \vec{B}.d\vec{s}$$

fingers of our right hand indicate the direction of the closed path, thumb indicates the direction of $d\vec{s}$.

- A flux-density $\vec{B}$ in the direction of $d\vec{s}$ and increasing with time thus produces an average value of $\vec{E}$ which is opposite to the +ve direction about the path.

Stationary path

magnetic flux $\rightarrow$ time varying quantity.

$$\therefore emf = \oint \vec{E}.d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}.d\vec{s}$$

Applying stokes's theorem,

$$\int_s (\vec{\nabla}\times\vec{E}).d\vec{s} = -\int_s \frac{\partial \vec{B}}{\partial t}.d\vec{s}$$

Surface integrals taken over identical general surfaces.

$$\therefore (\vec{\nabla}\times\vec{E}).d\vec{s} = -\frac{\partial \vec{B}}{\partial t}.d\vec{s}$$

$$\Rightarrow \boxed{(\vec{\nabla}\times\vec{E}) = -\frac{\partial \vec{B}}{\partial t}} \rightarrow \text{differential or point form}$$

$$\boxed{\oint \vec{E}.d\vec{l} = -\int \frac{\partial B}{\partial t}.d\vec{s}} \rightarrow \text{integral}$$

9.(b)

$$|\sigma \vec{E}| = |j\omega\epsilon \vec{E}|$$

$$\frac{\sigma}{\omega\epsilon} = 1$$

or $\sigma = \omega\epsilon$

$$\omega = \frac{\sigma}{\epsilon} = \frac{2\times10^{-4}}{8.854\times10^{-12}\times81} = \frac{2\times10^{-4}\times 36\pi\times10^{9}}{81}$$

$$= \frac{8\pi}{9}\times10^{5}$$

$$\boxed{\omega = 2.679\times10^{8}\ \text{rad/s}}$$

$$f = \frac{\omega}{2\pi} = \frac{8\pi}{2\pi\times9}\times10^{5}$$

$$f = 4.44\times10^{6}\ \text{Hz}$$

$$\boxed{f = 4.44\ \text{MHz}}$$

9.(c)

- Maxwell's eqn. in point form,

$$\vec{\nabla}\cdot\vec{D} = \rho_v$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

Maxwell's eqn. in Integral form,

$$\oint \vec{D}\cdot\vec{ds} = Q$$

$$\oint \vec{B}\cdot\vec{ds} = 0$$

$$\oint \vec{E}\cdot\vec{dl} = -\frac{\partial}{\partial t}\int_s \vec{B}\cdot\vec{ds}$$

$$\oint \vec{H}\cdot\vec{dl} = I + \int_s \frac{\partial\vec{D}}{\partial t}\cdot\vec{ds}$$

10.(a)

Solution of wave equation (Uniform Plane Waves). (iii)

$$\vec{\nabla}^2 \vec{E_s} = -\gamma^2 \vec{E_s}$$

$$\boxed{\vec{E_x}(z,t) = \vec{E_{xs}}(z) e^{j\omega t}}$$
$$\boxed{\vec{H_y}(z,t) = \vec{H_{ys}}(z) e^{j\omega t}}$$

$$\boxed{\nabla^2 E_{xs} = -\gamma^2 E_{xs}}$$

$$\boxed{\vec{E_{xs}} = E_{xs} \vec{a_x}}$$
$$\boxed{\vec{H_{ys}} = H_{ys} \vec{a_y}}$$

$$\underbrace{\frac{\partial^2 E_{xs}}{\partial x^2}}_{0} + \underbrace{\frac{\partial^2 E_{xs}}{\partial y^2}}_{0} + \frac{\partial^2 E_{xs}}{\partial z^2} = -\gamma^2 E_{xs}$$

$$\boxed{\frac{d^2 E_{xs}}{dz^2} = -\gamma^2 E_{xs}}$$

Solution of the above equation is

$$\boxed{E_{xs}(z) = \underbrace{E_{xo} e^{-j\gamma z}}_{\text{Fwd travelling wave}} + \overbrace{E_{xo}' e^{j\gamma z}}^{\text{Backward travelling wave}}}$$

$$\vec{E_x}(z,t) = \vec{E_{xs}}(z) \cdot e^{j\omega t}$$

$$\vec{E_x}(z,t) = \left(E_{xo} e^{-j\gamma z} + E_{xo}' e^{j\gamma z}\right) e^{j\omega t} \vec{a_x}$$

$$\boxed{\vec{E_x}(z,t) = \left[E_{xo} \underbrace{e^{-\alpha z} \cdot e^{-j\beta z} \cdot e^{j\omega t}}_{\text{Fwd travelling wave}} + E_{xo}' \underbrace{e^{\alpha z} \cdot e^{j\beta z} \cdot e^{j\omega t}}_{\text{Backward travelling wave}}\right] \vec{a_x}}$$

From the Maxwell's equation

$$\vec{\nabla} \times \vec{E_s} = -j\omega\mu \vec{H_s}$$

$$\vec{\nabla} \times \vec{E_s} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_{xs} & 0 & 0 \end{vmatrix} = \vec{a_x}(0-0) - \vec{a_y}\left(0 - \frac{\partial E_{xs}}{\partial z}\right) + \vec{a_z}(0-0)$$

$$\boxed{\vec{\nabla} \times \vec{E_s} = \frac{\partial E_{xs}}{\partial z} \vec{a_y}}$$

$$\Rightarrow \frac{\partial E_{xs}}{\partial z}\vec{a_y} = -j\omega\mu H_{ys}\vec{a_y}$$

$$H_{ys} = -\frac{1}{j\omega\mu}\cdot\frac{\partial}{\partial z}\left[E_{xo}e^{-j\gamma z} + E_{xo}'e^{j\gamma z}\right]$$

$$H_{ys} = E_{xo}\cdot\frac{e^{-j\gamma z}(-j\gamma)}{(-j\omega\mu)} + E_{xo}'\frac{e^{j\gamma z}(j\gamma)}{(-j\omega\mu)}$$

$$H_{ys} = \frac{E_{xo}\cdot e^{-j\gamma z}}{\left(\frac{j\omega\mu}{j\gamma}\right)} + \frac{E_{xo}'e^{j\gamma z}}{-\left(\frac{j\omega\mu}{j\gamma}\right)}$$

η = Intrinsic impedance

$$\boxed{\frac{E_{xo}}{\eta} = H_{yo}} \quad \& \quad \boxed{\frac{E_{xo}'}{-\eta} = H_{yo}'}$$

$$\boxed{H_{ys} = H_{yo}e^{-j\gamma z} + H_{yo}'e^{j\gamma z}} \quad \& \quad \eta = \frac{j\omega\mu}{j\gamma}$$

$$j\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

Intrinsic Impedance,

$$\boxed{\eta = \frac{j\omega\mu}{j\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma+j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\varepsilon}}}$$

$$\boxed{\vec{H_y}(z,t) = \left[\frac{E_{xo}}{\eta}e^{-\alpha z}e^{-j\beta z}e^{j\omega t} - \frac{E_{xo}'}{\eta}e^{\alpha z}e^{j\beta z}e^{j\omega t}\right]\vec{a_y}}$$

10.(b)

Poynting's theorem :-

Poynting's theorem states that the net power flowing out of a given volume v is equal to the time-rate of decrease in the energy stored within v minus the ohmic losses

If the medium is conductive, Maxwell's equation becomes,

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{--- (1)}$$

Taking scalar product of both sides with $\vec{E}$,

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad \text{--- (2)}$$

consider the vector identity

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{E} \cdot (\vec{\nabla} \times \vec{H}) + \vec{H} \cdot (\vec{\nabla} \times \vec{E})$$

$$\Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) \quad \text{--- (3)}$$

Using this result in eqn. (2),

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \vec{H} \cdot \left(-\frac{\partial \vec{B}}{\partial t}\right) - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{J} \cdot \vec{E} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow -\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{J} \cdot \vec{E} + \epsilon \vec{E} \frac{\partial \vec{E}}{\partial t} + \mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} \quad \text{--- (4)}$$

$$\text{Now,} \quad \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t} \frac{1}{2} (\vec{D} \cdot \vec{E})$$

$$\text{and} \quad \mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{1}{2} \vec{B} \cdot \vec{H}\right) \quad \text{--- 4}$$

∴ Eqn. ⑤ becomes,

$$-\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{J}\cdot\vec{E} + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}\cdot\vec{E}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}\cdot\vec{H}\right)$$

Integrating over a volume,

$$-\int_{vol}\vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dv = \int_{vol}\vec{J}\cdot\vec{E}\,dv + \int_{vol}\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}\cdot\vec{E}\right)dv + \int_{vol}\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}\cdot\vec{H}\right)dv$$

$$\Rightarrow -\oint_{area}(\vec{E}\times\vec{H})\cdot d\vec{s} = R.H.S$$

→ ~~Poynting's theorem~~

On R.H.S,

1st term = ohmic power dissipated within the volu

2nd term = total energy stored in the electric field

3rd " = " " " " " magnetic field

R.H.S = total power flowing into the volume.

∴ The total power flowing out of the volume is,

$$\oint_{area}(\vec{E}\times\vec{H})\cdot d\vec{s} \quad W$$

The cross-product, $(\vec{E}\times\vec{H}) = \vec{S}\ W/m^2 \rightarrow$ Poynting's vector.

Direction of power flow normal to both $\vec{E}$ & $\vec{H}$ vectors

10.(c)

Given: $\delta = 0.1\,m$ (depth of penetration).

$f = 1\,MHz$

To find: $\sigma = ?$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \Rightarrow \delta^2 = \frac{1}{\pi f \mu \sigma}$$

$$\sigma = \frac{1}{\pi f \mu \delta^2} = \frac{1}{\pi \times 1\times10^6 \times \mu_0 \times (0.1)^2}$$

$$\sigma = \frac{1}{\pi\times10^6\times 4\pi\times10^{-7}\times(0.1)^2} = 25.33$$

$$\boxed{\sigma = 25.33\ S/m}$$