IAT-II Solutions Oct 2019

CMR

1) for a complex valued sequence $x(n)$ of N points, the DFT may be expressed as

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos\left(\frac{2\pi}{N}kn\right) + x_I(n) \sin\left(\frac{2\pi}{N}kn\right) \right] \longrightarrow (1)$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin\left(\frac{2\pi}{N}kn\right) - x_I(n) \cos\left(\frac{2\pi}{N}kn\right) \right] \longrightarrow (2)$$

The direct computation of equations (1) & (2) requires

1) $2N^2$ evaluation of trigonometric functions
2) $4N^2$ real multiplications
3) $4N(N-1)$ real additions
4) N^2 complex multiplications
5) $N(N-1)$ complex additions.

2) Obtain the circular convolution of the sequences $x_1(n) = (1,2,3,4)$ & $x_2(n) = (2,1,2,1)$ using FFT algorithms.

Sol: Using the circular convolution property

$$x_3(n) = x_1(n) \circledN x_2(n) \xleftrightarrow[N]{DFT} X_3(k) = X_1(k) \cdot X_2(k)$$

$$\text{ie } x_3(n) = IDFT[X_3(k)]$$

let us find the DFT's of $x_1(n)$ & $x_2(n)$ using DIF-FFT algorithms.

$$X_1(k) = DFT[x_1(n)]$$

$$X_1(k) = \sum_{n=0}^{3} x_1(n) W_4^{kn} \quad , \quad k = 0,1,2,3$$

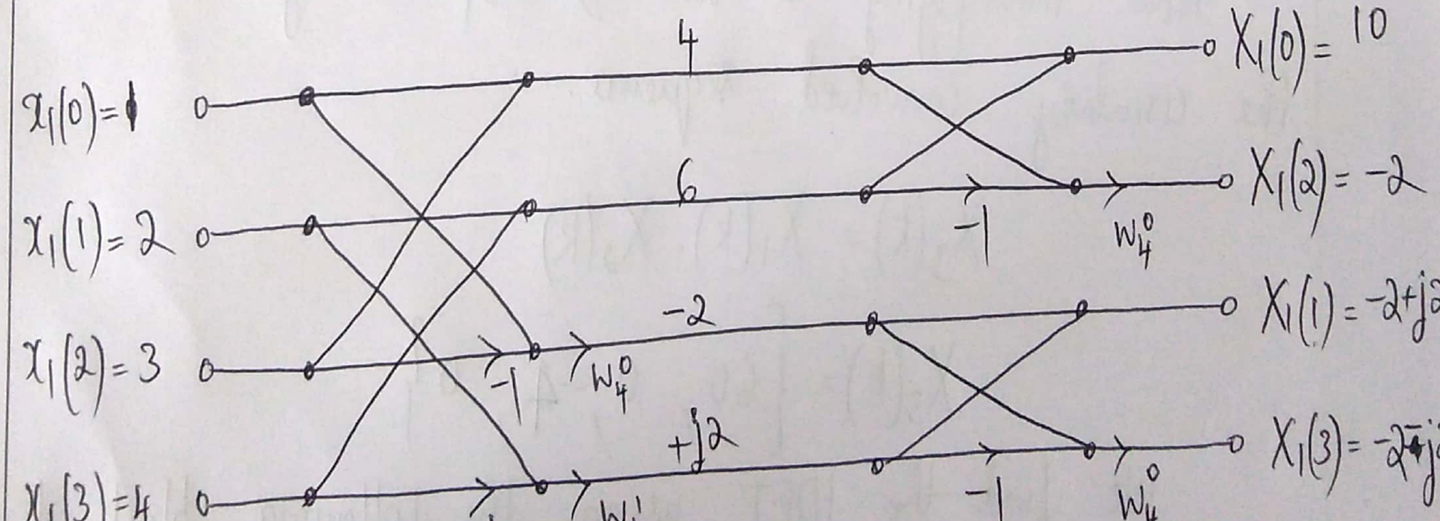

fig: DIF-FFT flowchart for N=4

$$\therefore X_1(k) = \{10, (-2+j2), -2, (-2-j2)\}$$

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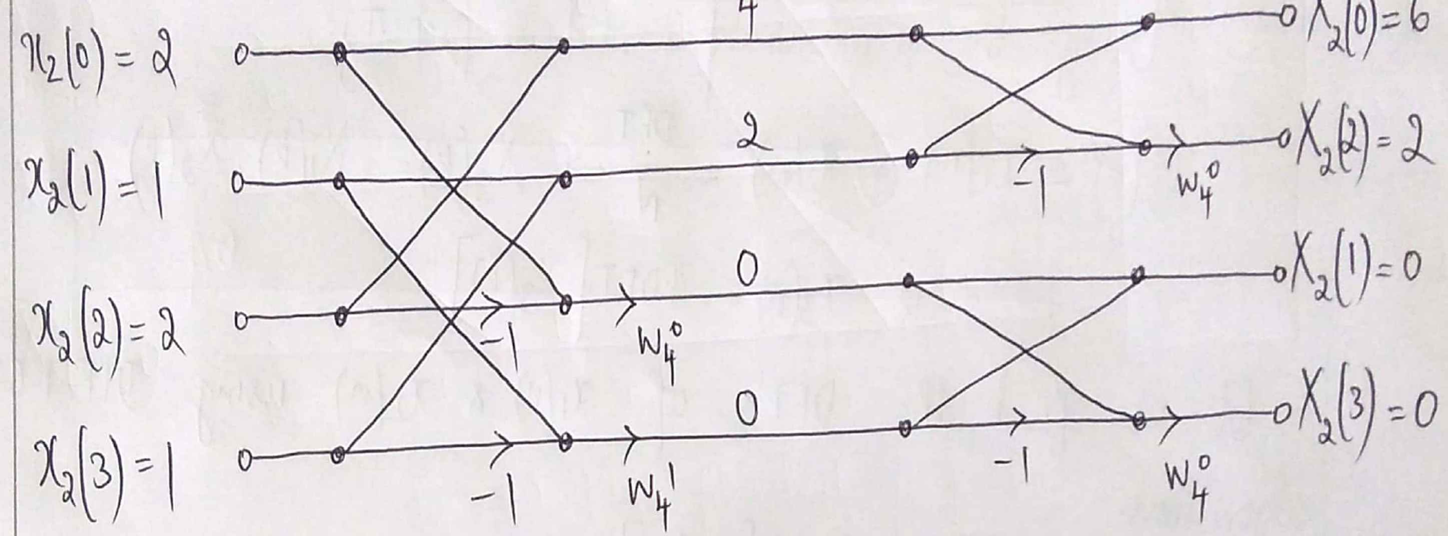
Let us find the DFT of $x_2(n)$ using DIF-FFT algorithm



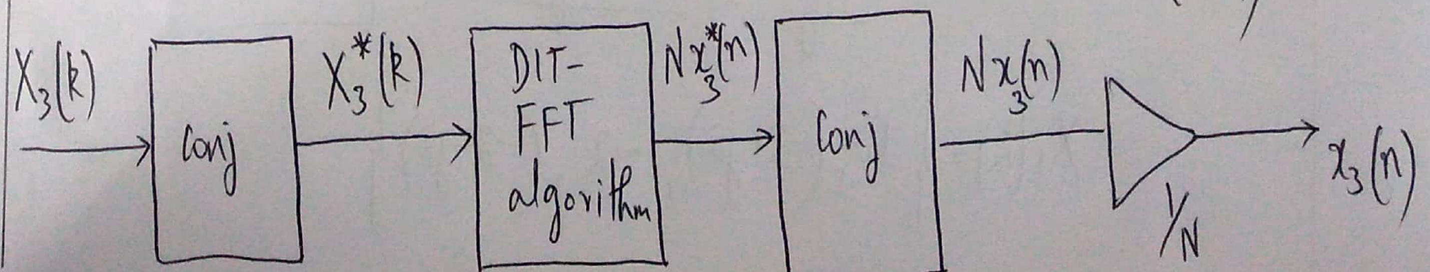
$$\therefore X_2(k) = \{6, 0, 2, 0\}$$

Now multiplying the two DFT's, we get the DFT of the circularly convolved sequence.

$$X_3(k) = X_1(k) \cdot X_2(k)$$

$$X_3(k) = \{60, 0, -4, 0\}$$

We find the IDFT using the following block diagram (N=4)



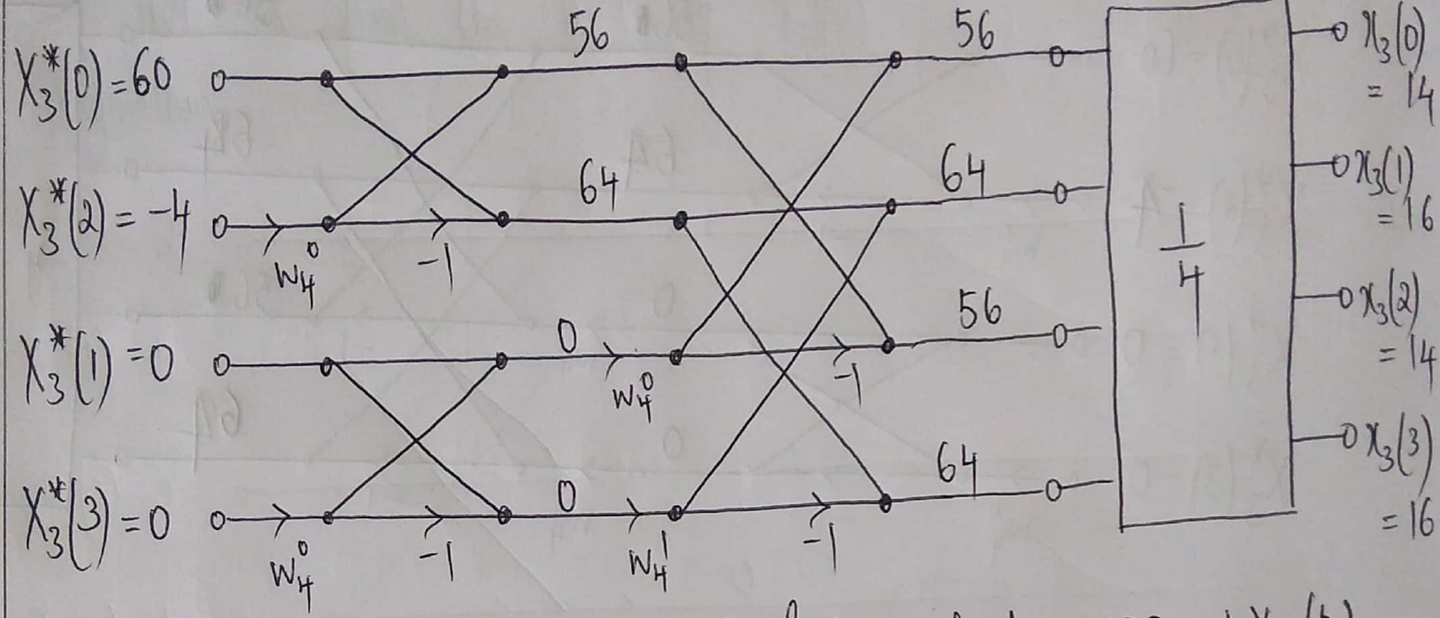


fig:- DIT-FFT flowchart for finding IDFT of $X_3(k)$

∴ The circularly convolved sequence is

$$x_3(n) = \{14, 16, 14, 16\}$$

$$W_4^1 = e^{-j\frac{2\pi}{4}} = e^{-j\pi/2} = -j$$

Natural order

3) Decimation in freq (DIF) FFT algorithm

$$X(k) = \sum_{n=0}^{N/2-1} x(n)\, W_N^{kn} + \sum_{n=N/2}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} x(n)\, W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{kn}$$

Since $W_N^{kN/2} = (-1)^k$

$$X(k) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^k x\left(n+\frac{N}{2}\right)\right] W_N^{kn} \longrightarrow ①$$

let us split (decimate) $X(k)$ into even & odd numbered sam

$$X(2k) = \sum_{n=0}^{N/2-1} \left[x(n) + x\left(n+\frac{N}{2}\right)\right] W_{N/2}^{kn}, \quad k = 0, 1 \ldots N/2 - 1$$

and

$$X(2k+1) = \sum_{n=0}^{N/2-1} \left\{\left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{n}\right\} W_{N/2}^{kn}$$

Put ~~n = 2k+1~~ $k = 2m+1$

$W_N^{(2m+1)n}$

let $g_1(n) = x(n) + x\left(n+\frac{N}{2}\right)$, $n = 0, 1 \ldots \frac{N}{2}-1$

$g_2(n) = \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^n$

then
$$X(2k) = \sum_{n=0}^{N/2-1} g_1(n)\, W_{N/2}^{kn}$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} g_2(n)\, W_{N/2}^{kn}$$

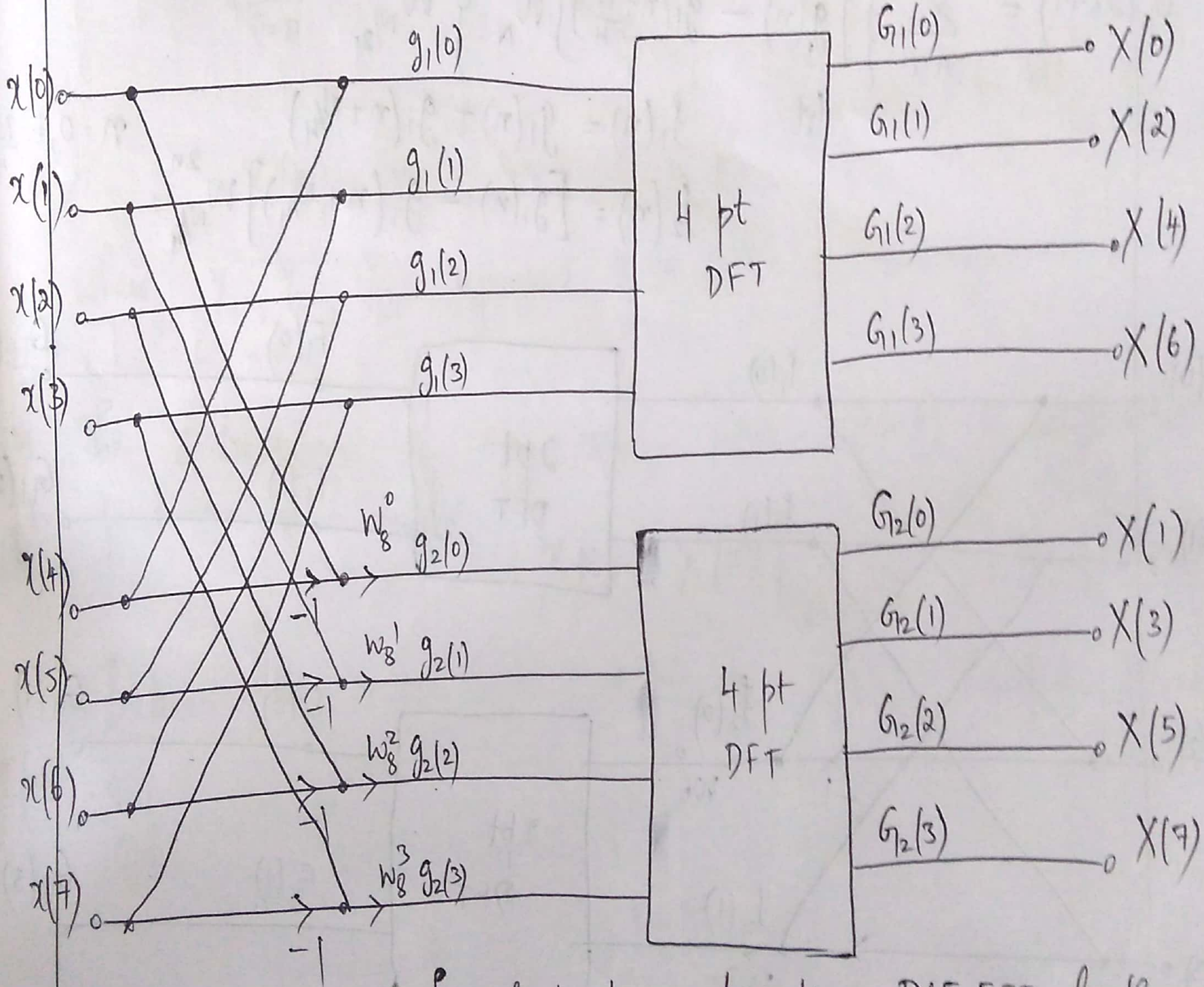


fig: first stage of decimation DIF-FFT algorithm

$$G_{11}(k) = \sum_{n=0}^{N/2-1} g_1(n)\, W_{N/2}^{kn}$$

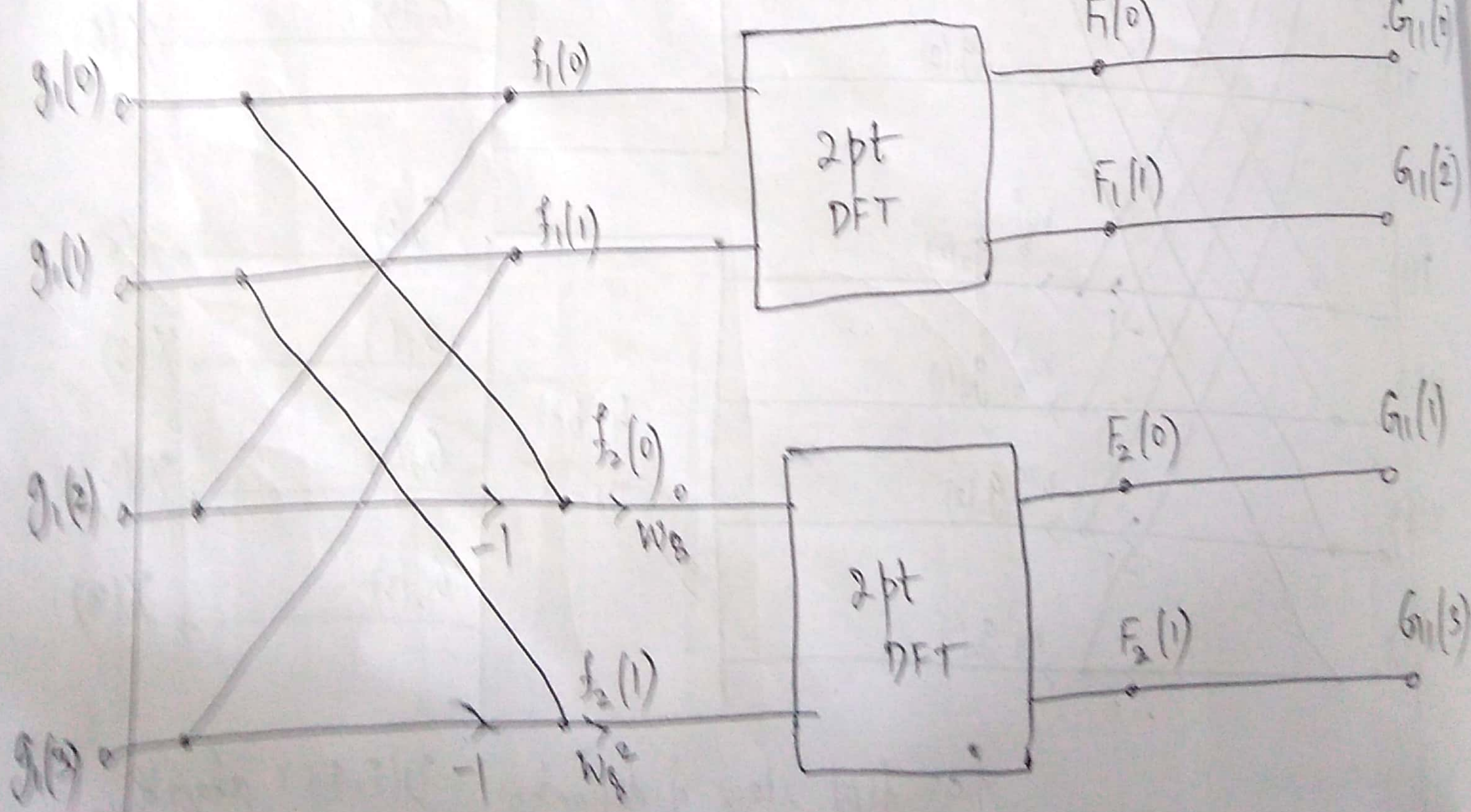
$$G_{11}(k) = \sum_{n=0}^{N/4-1} \left[g_1(n) + \underset{(-1)^k}{g_1\left(n+\frac{N}{4}\right)} \right] W_{N/2}^{kn}$$

$$G_{11}(2k) = \sum_{n=0}^{N/4-1} \left[g_1(n) + g_1(n+N/4) \right] W_{N/4}^{kn} \qquad k=0,1\cdots\frac{N}{4}-1$$

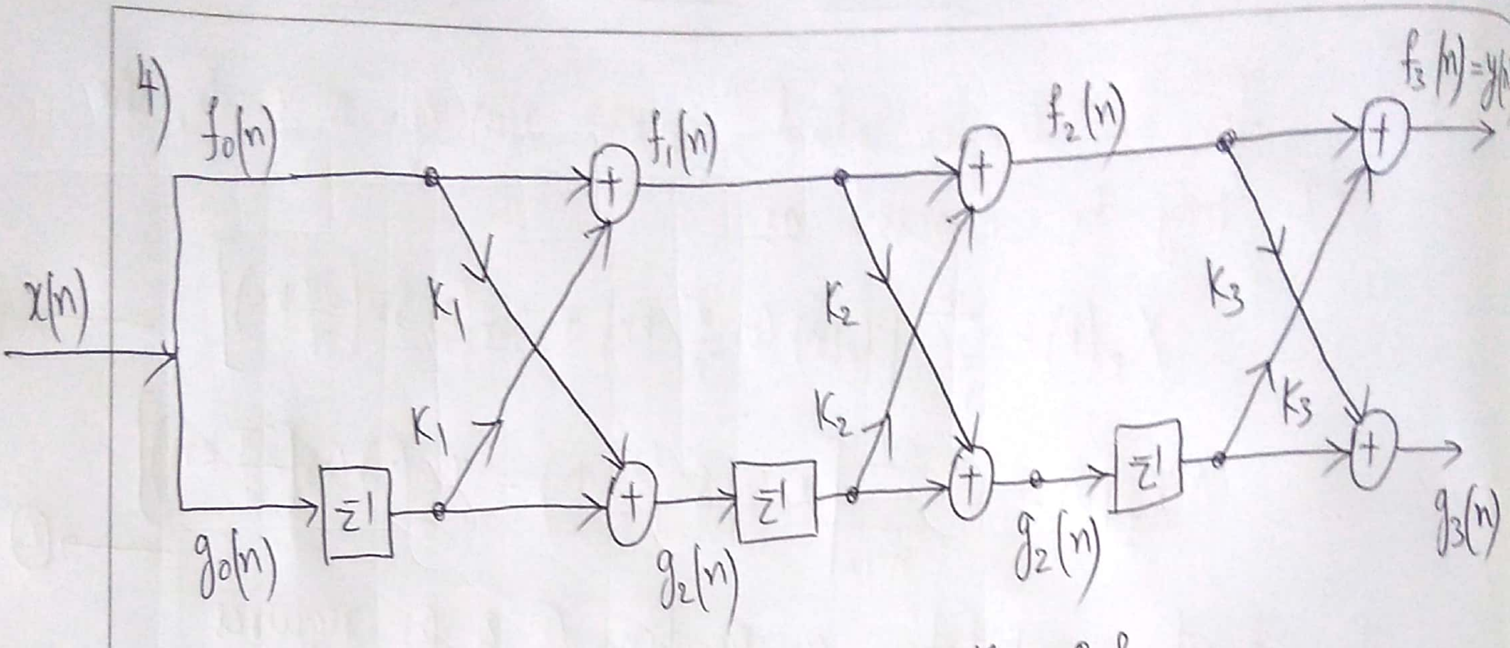
$$G_{11}(2k+1) = \sum_{n=0}^{N/4-1} \left\{ \left[g_1(n) - g_1\left(n+\frac{N}{4}\right) \right] W_N^{2n} \right\} W_{N/4}^{kn}$$

let $f_1(n) = g_1(n) + g_1(n+N/4)$ $\quad n=0,1\cdots\frac{N}{4}-1$

$f_2(n) = \left[g_1(n) - g_1(n+N/4) \right] W_N^{2n}$



4)



$K_1 = 0.65$, $K_2 = -0.34$, $K_3 = 0.8$

$x(n) = \delta(n)$

hence $f_0(n) = g_0(n) = \delta(n)$

1^{st} stage O/P, $f_1(n) = \delta(n) + 0.65\,\delta(n-1)$

$g_1(n) = 0.65\,\delta(n) + \delta(n-1)$

2^{nd} stage O/P, $f_2(n) = f_1(n) - 0.34\,g_1(n-1)$

$f_2(n) = \delta(n) + 0.429\,\delta(n-1) - 0.34\,\delta(n-2)$

$g_2(n) = -0.34\,\delta(n) + 0.429\,\delta(n-1) + \delta(n-2)$

Impulse response, $h(n) = f_3(n) = f_2(n) + 0.8\,g_2(n-1)$

$h(n) = \delta(n) + 0.157\,\delta(n-1) + 0.0032\,\delta(n-2) + 0.8\,\delta(n-3)$

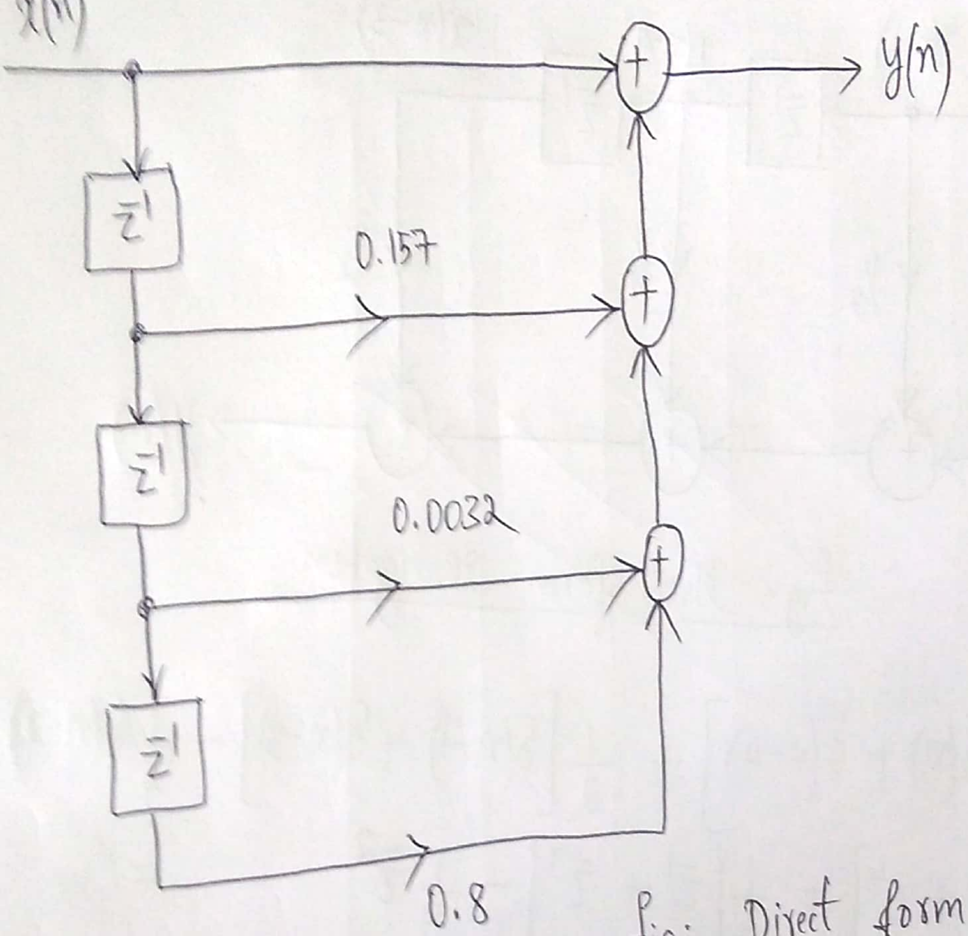


fig: Direct form structure

5) i) $h(n) = \left(\frac{1}{2}\right)^n \left[u(n) - u(n-4)\right]$

$$h(n) = \begin{cases} \left(\frac{1}{2}\right)^n & \text{for } 0 \le n \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$h(n) = \frac{1}{2}\delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{2}\delta(n-2) + \frac{1}{2}\delta(n-3)$$

$$H(z) = \frac{1}{2}\left[1 + z^{-1} + z^{-2} + z^{-3}\right]$$

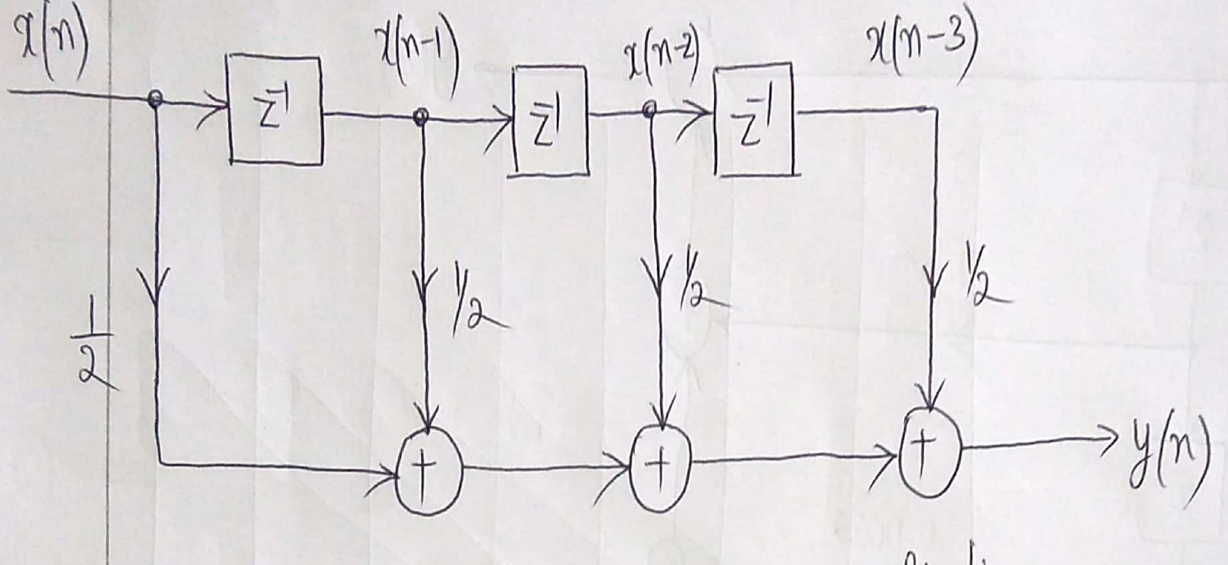


fig:- Direct form realization

ii) $$h(n) = \left[\delta(n) + \delta(n-4)\right] + \frac{1}{2}\left[\delta(n-1) + \delta(n-3)\right] - \frac{1}{4}\delta(n-2)$$

$$H(z) = \left[1 + z^{-4}\right] + \frac{1}{2}\left[z^{-1} + z^{-3}\right] - \frac{1}{4}z^{-2}$$

or $$y(n) = \left[x(n) + x(n-4)\right] + \frac{1}{2}\left[x(n-1) + x(n-3)\right] - \frac{1}{4}x(n-2)$$

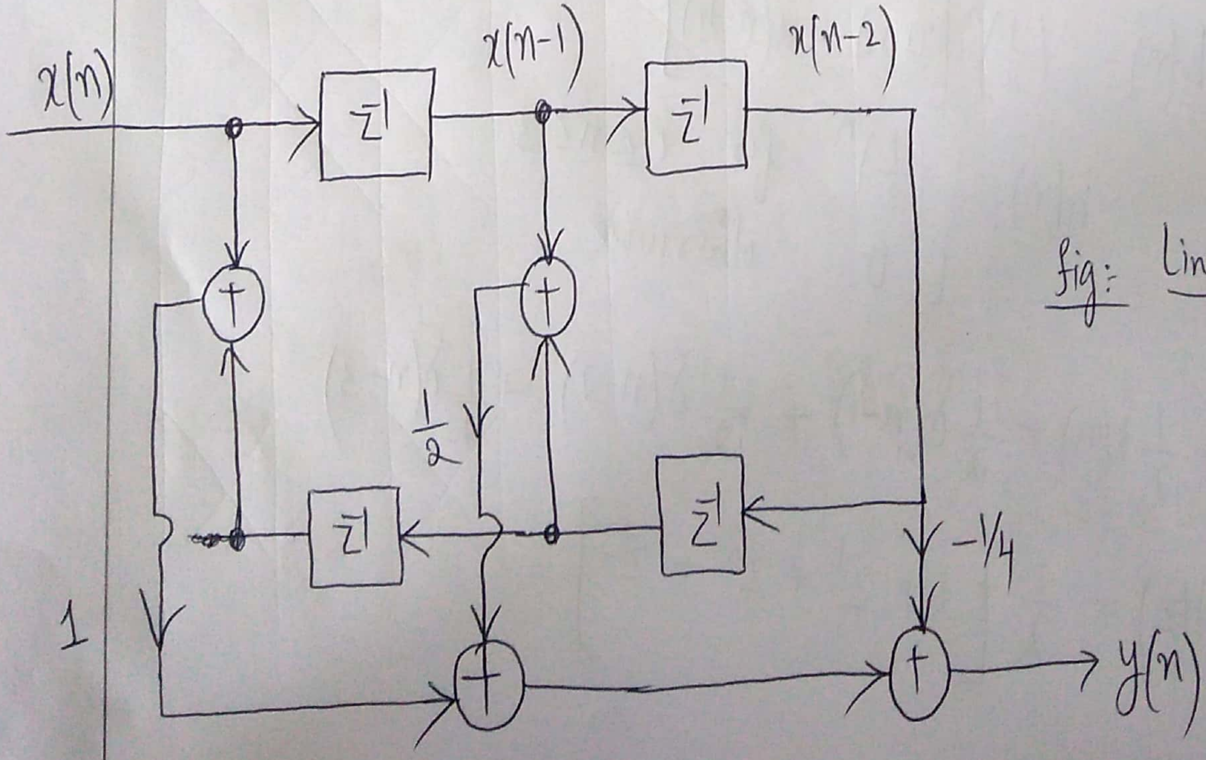


fig:- Linear phase realization

6) **Goertzel Algorithm** : exploits the periodicity of the phase factors $\{W_N^{k}\}$ & allows to express the computation of DFT as linear filtering operation. Since $W_N^{-kN} = 1$

$$X(k) = W_N^{-kN} \sum_{m=0}^{N-1} x(m) W_N^{km} = \sum_{m=0}^{N-1} x(m) W_N^{-k(N-m)} \longrightarrow (1)$$

This is in form of ~~...~~ convolution. If we define seq $y_k(n)$ as

$$y_k(n) = \sum_{m=0}^{N-1} x(m) W_N^{-k(n-m)} \longrightarrow (2)$$

It is clear that $y_k(n)$ is the convolution of finite duration seq $x(n)$ of length N with the filter of impulse response

$$h_k(n) = W_N^{-kn} u(n) \longrightarrow (3)$$

The o/p of this filter at $n=N$ yields the DFT at freq $\omega_k = \frac{2\pi k}{N}$, ie

$$X(k) = y_k(n)\Big|_{n=N} \longrightarrow (4)$$

The filter with impulse response $h_k(n)$ has system fn

$$H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}} \longrightarrow (5)$$

Pole on the unit circle at freq $\omega_k = \frac{2\pi k}{N}$

$$z = W_N^{-k} = e^{+j\frac{2\pi k}{N}}$$

Instead of performing DFT via convolution, we can use difference eqn to compute $y_k(n)$ recursively.

$$y_k(n) = W_N^{-k}\, y_k(n-1) + x(n), \quad y_k(-1) = 0 \longrightarrow (6)$$

The desired o/p is $X(k) = y_k(N)$ for $k = 0, 1, \dots, N-1$. To perform this computation, we can compute once & store phase factor W_N^{-k}.

$$H_k(z) = \frac{1 - W_N^k z^{-1}}{1 - 2\cos\left(\frac{2\pi}{N}k\right) z^{-1} + z^{-2}} = \frac{Y_k(z)}{V_k(z)} \cdot \frac{V_k(z)}{X(z)}$$

Let $\frac{V_k(z)}{X(z)} = \frac{1}{1 - 2\cos\left(\frac{2\pi}{N}k\right) z^{-1} + z^{-2}} \Rightarrow v_k(n) = 2\cos\left(\frac{2\pi}{N}k\right) v_k(n-1) - v_k(n-2) + x(n)$

$\frac{Y_k(z)}{V_k(z)} = 1 - W_N^k z^{-1} \Rightarrow y_k(n) = v_k(n) - W_N^k v_k(n-1)$

with $v_k(-1) = v_k(-2) = 0$

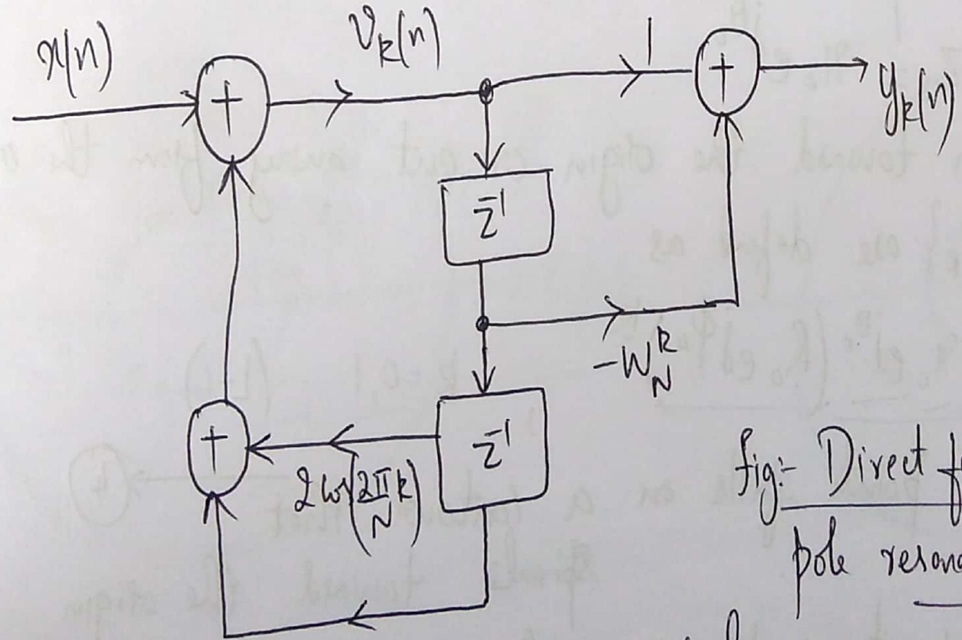


fig:- Direct form II realization of two pole resonator for computation of DFT

Each iteration requires 1 ~~complex~~ real multiplication & 2 addition.

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7) $g(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

8-point DFT is

$$G(k) = \sum_{n=0}^{7} g(n)\, W_8^{kn} \quad , \quad k = 0, 1, 2 \cdots 7$$

$W_8^0 = 1$

$W_8^1 = 0.707 - j0.707$

$W_8^2 = -j$

$W_8^3 = -0.707 - j0.707$

$W_8^4 = -1$

$W_8^5 = -0.707 + j0.707$

$W_8^6 = j$

$W_8^7 = 0.707 + j0.707$

Twiddle factors.

The DIT-FFT flowchart for computing the DFT of the sequence $g(n)$ is shown in the figure along with the intermediate stage results.

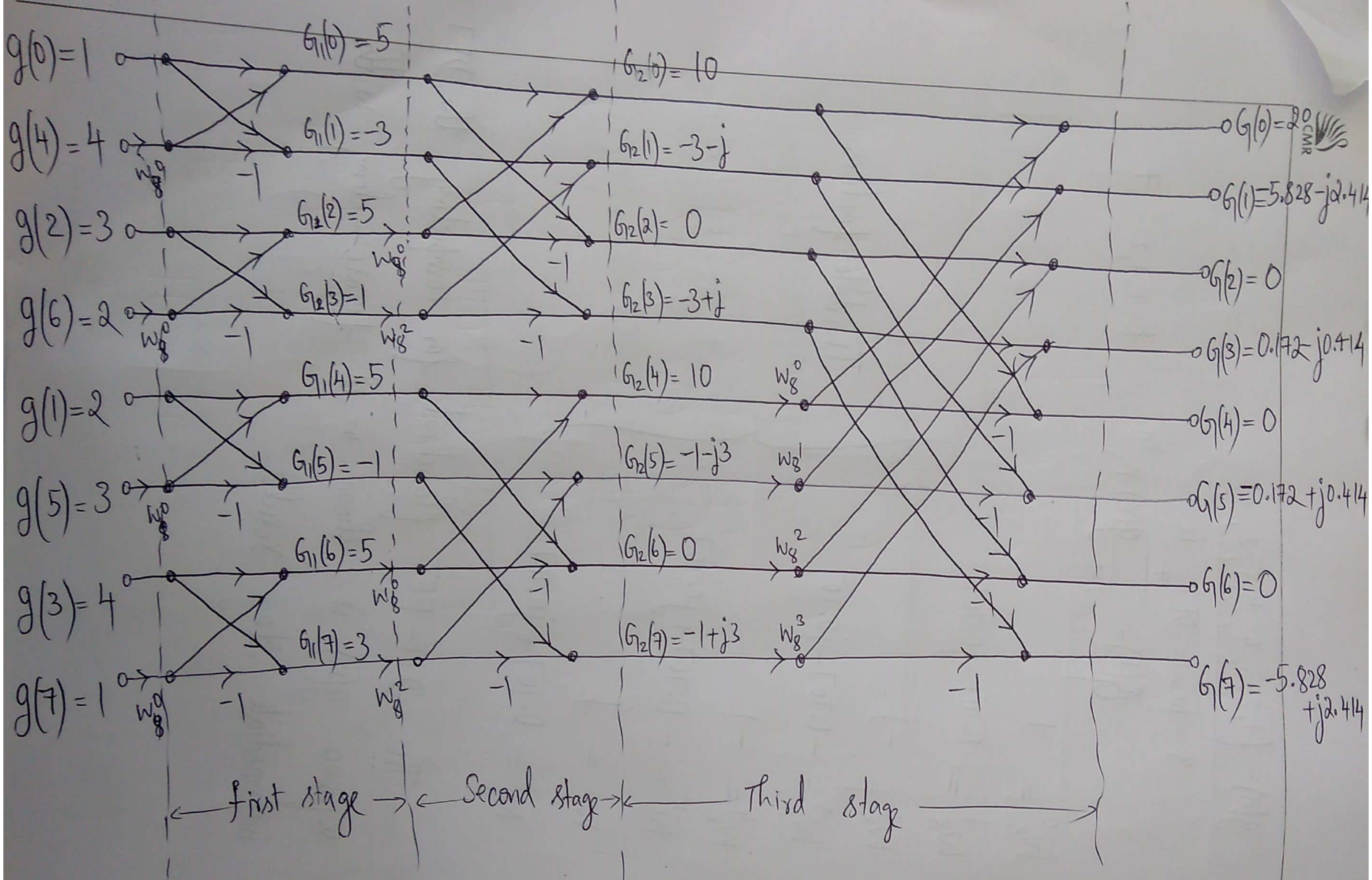
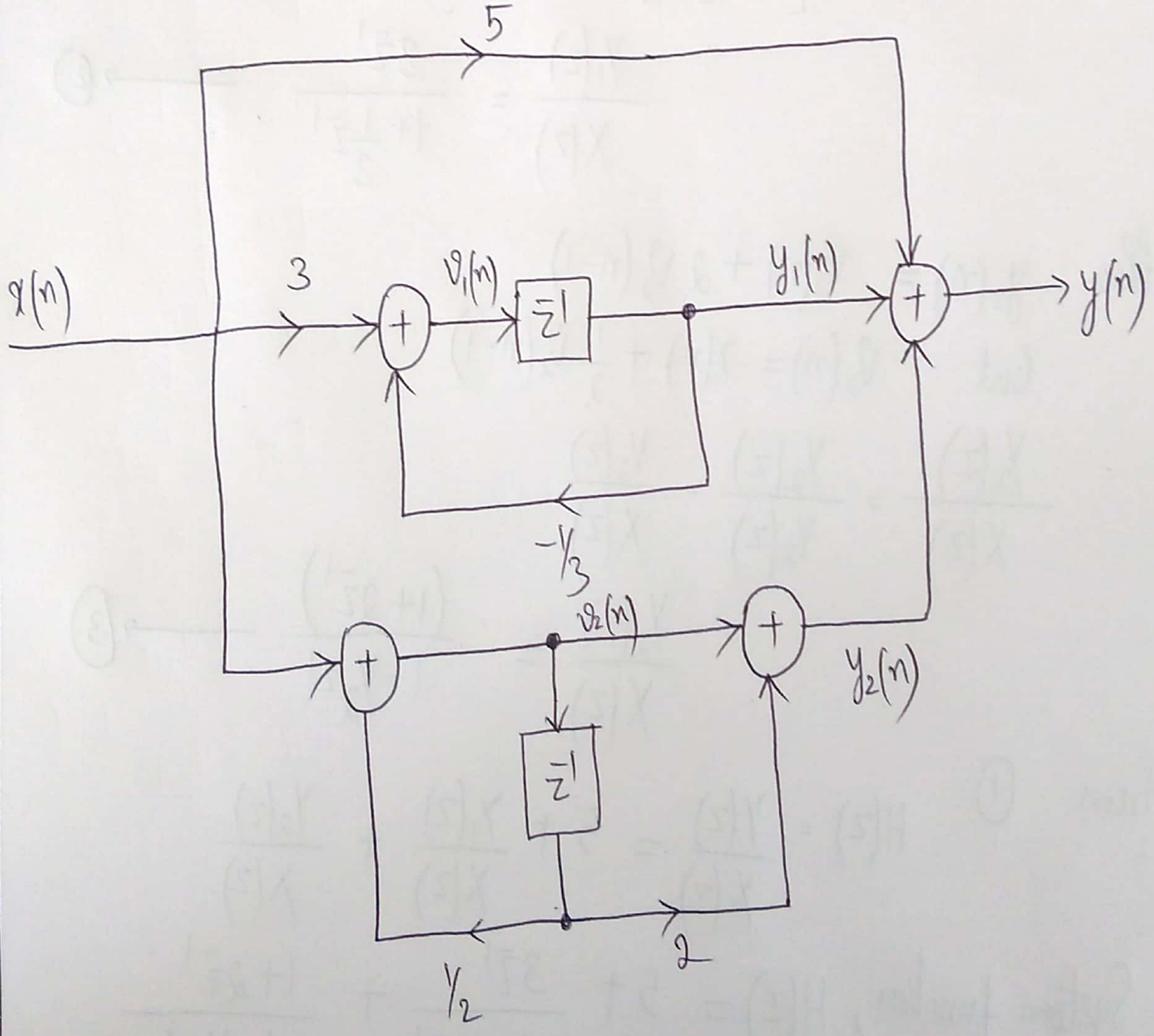


fig: DIT-FFT flowchart for computation of DFT of g(n)

$\therefore \; G(k) = \{20, (-5.828 - j2.414), 0, (-0.172 - j0.414), 0, (-0.172 + j0.414), 0, (-5.828 + j2.414)\}$

8)



from figure $\quad y(n) = 5\,x(n) + y_1(n) + y_2(n) \longrightarrow (1)$

$$y_1(n) = v_1(n-1)$$

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but $v_1(n) = 3x(n) - \frac{1}{3}y_1(n)$

$$\therefore \quad y_1(n) = 3x(n-1) - \frac{1}{3}y_1(n-1)$$

$$Y_1(z)\left[1 + \frac{1}{3}z^{-1}\right] = 3z^{-1}X(z)$$

$$\therefore \quad \frac{Y_1(z)}{X(z)} = \frac{3z^{-1}}{1+\frac{1}{3}z^{-1}} \longrightarrow (2)$$

Also $y_2(n) = v_2(n) + 2v_2(n-1)$

but $v_2(n) = x(n) + \frac{1}{2}v_2(n-1)$

$$\frac{Y_2(z)}{X(z)} = \frac{Y_2(z)}{V_2(z)} \cdot \frac{V_2(z)}{X(z)}$$

$$\frac{Y_2(z)}{X(z)} = \frac{(1+2z^{-1})}{1-\frac{1}{2}z^{-1}} \longrightarrow (3)$$

from (1) $H(z) = \frac{Y(z)}{X(z)} = 5 + \frac{Y_1(z)}{X(z)} + \frac{Y_2(z)}{X(z)}$

System function, $H(z) = 5 + \frac{3z^{-1}}{1+\frac{1}{3}z^{-1}} + \frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}$

Impulse response, $h(n) = 5\delta(n) + 3\left(\frac{-1}{3}\right)^{n-1}u(n-1) + \left(\frac{1}{2}\right)^{n}u(n) + 2\left(\frac{1}{2}\right)^{n-1}u(n-1)$