USN

CMRIT
CMR INSTITUTE OF TECHNOLOGY, BENGALURU.
ACCREDITED WITH A+ GRADE BY NAAC

Internal Assessment Test III – December 2019

Sub:	Calculus and Linear Algebra					Sub Code:	18MAT11	
Date:	09/12/2019	Duration:	90 mins	Max Marks:	50	Sem / Sec:	I / A-G (PHY CYCLE)	OBE

Question 1 is compulsory and answer any SIX questions from the rest.

		MARKS	CO	RBT
1.	(a) Solve $yp^2+(x-y)p-x=0$.	[04]	CO5	L3
	(b) If the temperature of the air is $30°C$ and a metal ball cools from $100°C$ to $70°C$ in 15 minutes. Find how long will it take for the metal ball to reach a temperature of $40°C$.	[04]	CO4	L3
2.	Find the orthogonal trajectory of $r^n \sin n\theta = a^n$, where 'a' is the parameter.	[07]	CO4	L3
3.	Find by double integration, the smaller area between circle $x^2+y^2=9$ and the straight line $x+y=3$.	[07]	CO3	L3
4.	Solve: $\dfrac{dy}{dx}+y\tan x = y^3 \sec x$.	[07]	CO4	L3

5.	Find the general solution of $(px-y)(py+x)=0$ by reducing to Clairaut's form using $u=x^2$ and $v=y^2$.	[07]	CO5	L3
6.	Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dy\, dx$ by changing the order of integration.	[07]	CO3	L3
7.	Prove that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \cdot \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} d\theta = \pi$.	[07]	CO3	L3
8.	Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dy\, dx$ by changing into polar coordinate system.	[07]	CO3	L3

Internal Assessment Test - III Solutions
(PHY Cycle).

1. (a) $yp^2+(x-y)p-x=0$, solvable for p.

$$p = \frac{-(x-y) \pm \sqrt{(x-y)^2-4y(-x)}}{2y}$$

$$= \frac{-x+y \pm \sqrt{x^2+y^2-2xy+4xy}}{2y}$$

$$= \frac{-x+y \pm \sqrt{(x+y)^2}}{2y} = \frac{-x+y \pm (x+y)}{2y}.$$

$$p = \frac{-x+y+x+y}{2y}$$

$$\frac{dy}{dx} = 1$$

$\Rightarrow dy = dx$

Integrating both sides,

$y = x + c$

$\Rightarrow y - x - c = 0.$

$$p = \frac{-x+y-x-y}{2y}$$

$$\frac{dy}{dx} = \frac{-2x}{2y}.$$

$y\,dy = -x\,dx$

Integrating both sides,

$y^2/2 = -x^2/2 + c$

$$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} - c = 0.$$

$\therefore$ General soln is, $(y-x-c)\left(\frac{y^2}{2}+\frac{x^2}{2}-c\right)=0.$

(b) Initial temperature, $t_1 = 100°C$

Temperature of surrounding media, $t_2 = 30°C$.

$T = 70°C$ at $t = 15$ mins.

By Newton's Law of cooling we have,

$$T = t_2 + (t_1 - t_2)e^{-kt}$$

$$\Rightarrow 70 = 30 + (100-30)e^{-k(15)}$$

$$\Rightarrow \frac{70-30}{70} = e^{-15k}$$

$$\Rightarrow \frac{4}{7} = e^{-15k} \Rightarrow k = -\frac{1}{15}\ln\left(\frac{4}{7}\right) = 0.0373077$$

To find :- t when $T = 40°C$.

$$T = t_2 + (t_1 - t_2)e^{-kt}$$

$$40 = 30 + (100-30)e^{-0.0373077t}$$

$$\Rightarrow e^{-0.0373077t} = \frac{10}{70}.$$

$$\Rightarrow t = \frac{-1}{0.0373077}\ln\left(\frac{1}{7}\right).$$

$= 52.15 \simeq 52$ minutes.

2. $r^n \sin n\theta = a^n$. ——— (1)

Applying log both sides,

$n \log r + \log(\sin n\theta) = n \log a$.

Diff. wrt θ,

$$\frac{n}{r}\frac{dr}{d\theta} + \frac{n \cos n\theta}{\sin n\theta} = 0.$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\cot n\theta \longrightarrow \text{diff. eq}^n \text{ of (1).}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$,

$$\frac{1}{r} \times -r^2 \frac{d\theta}{dr} = -\cot n\theta \longrightarrow \text{diff eq}^n \text{ of OT.}$$

$$r\frac{d\theta}{dr} = \cot n\theta \Rightarrow \frac{1}{r}dr = \tan n\theta \cdot d\theta.$$

Integrating, $\log r = \frac{1}{n}\log(\sec n\theta) + \log c$.

$$n \log r - \log(\sec n\theta) = n \log c.$$

$$\Rightarrow \frac{r^n}{\sec n\theta} = c^n$$

$$\Rightarrow r^n \cos n\theta = c^n, \text{ required OT.}$$

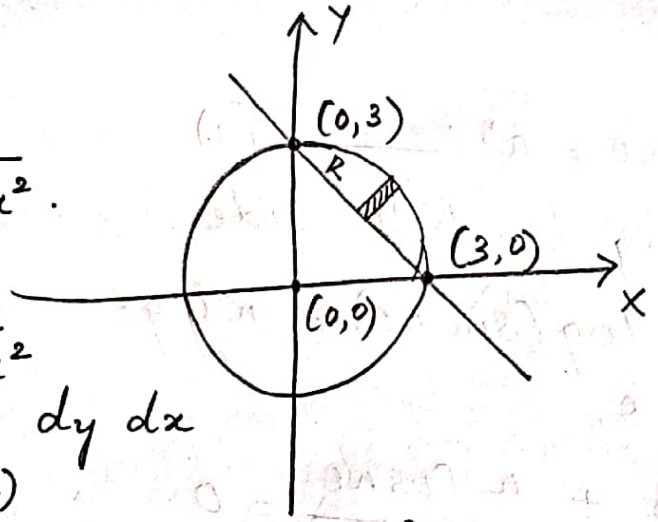
3. By double integration,

Area $= \iint_R dx\,dy$, where R is the area of the smaller region bounded by $x^2+y^2=9$ and $x+y=3$.

Scanned by CamScanner

$x : 0 \to 3$

$y : (3-x) \to \sqrt{9-x^2}$.



$$\therefore A = \int_{x=0}^{3} \int_{y=(3-x)}^{\sqrt{9-x^2}} dy\, dx$$

$$= \int_{x=0}^{3} y \Big|_{y=3-x}^{\sqrt{9-x^2}} dx \quad = \int_{x=0}^{3} \left[\sqrt{9-x^2} - (3-x)\right] dx$$

$$= \left[\frac{x}{3}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - 3x + \frac{x^2}{2}\right]_{x=0}^{3}$$

$$= \left[\left\{0 + \frac{9}{2}\cdot\frac{\pi}{2} - 9 + \frac{9}{2}\right\} - \left\{0\right\}\right]$$

$$= \frac{9\pi}{4} - \frac{9}{2} = \frac{9}{4}(\pi - 2) \text{ Sq. units.}$$

4. $\frac{dy}{dx} + y\tan x = y^3 \sec x$, Bernoulli in y.

$P = \tan x$, $Q = \sec x$, $y^n = y^3$.

$\div y^3$, $\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\tan x = \sec x \quad -(1)$

Let $\frac{1}{y^2} = v \Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$

$(1) \Rightarrow -\frac{1}{2}\frac{dv}{dx} + v\tan x = \sec x$.

$\times(-2)$

$$\Rightarrow \frac{dv}{dx} - 2v\tan x = -2\sec x, \text{ linear in } v.$$

$P_1 = -2\tan x;\ Q_1 = -2\sec x.$

$$I.F = e^{\int P_1\,dx}$$

$$= e^{-2\int \tan x\,dx} = e^{-2\log(\sec x)} = \cos^2 x.$$

$\therefore$ The soln. is

$$v\cos^2 x = \int -2\sec x.\cos^2 x.\,dx + c$$

$$\frac{\cos^2 x}{y^2} = -2\int \cos x.\,dx + c$$

$$\frac{\cos^2 x}{y^2} = -2\sin x + c.$$

5. $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$v = y^2 \Rightarrow \frac{dv}{dy} = 2y$

$$p = \frac{dy}{dx} = \frac{dy}{dv}\,\frac{dv}{du}\,.\,\frac{du}{dx} = \frac{1}{2y}P.2x = \frac{x}{y}P$$

$$= \frac{\sqrt{u}}{\sqrt{v}}P.$$

Given $(px-y)(py+x)=0.$

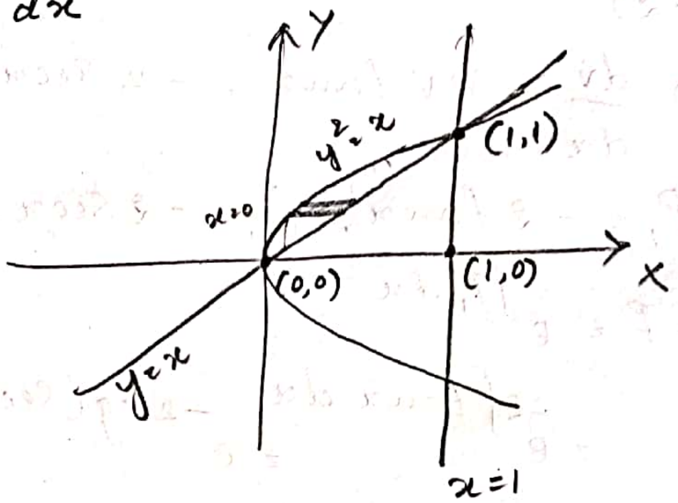
$$\left[\frac{\sqrt{u}}{\sqrt{v}}P.\sqrt{u}-\sqrt{v}\right]\left[\frac{\sqrt{u}P}{\sqrt{v}}\sqrt{v}+\sqrt{u}\right]=0.$$

$$\frac{(Pu-v)}{\sqrt{v}}\times\sqrt{u}(P+1) = 0. \Rightarrow Pu - v = 0$$

$\Rightarrow v = Pu + 0$, Clairaut's eqn.

$\therefore$ The G.S is $v = cu \Rightarrow y^2 = cx^2.$

6. $$I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy \, dy \, dx$$



$x=0, \ x=1$

$y=x, \ y=\sqrt{x}$

$y^2 = x.$

$x: y^2 \to y$

$y: 0 \to 1$

$$I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy \, dx \, dy = \int_{y=0}^{1} y \left(\frac{x^2}{2}\right)_{x=y^2}^{y} dy$$

$$= \frac{1}{2} \int_{y=0}^{1} y \left(y^2 - y^4\right) dy$$

$$= \frac{1}{2} \int_{y=0}^{1} \left(y^3 - y^5\right) dy$$

$$= \frac{1}{2} \left(\frac{y^4}{4} - \frac{y^6}{6}\right)_0^1$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{4}\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{4}\left(\frac{1}{6}\right) = \frac{1}{24}.$$

7. $$I_1 = \int_0^{\pi/2} \sqrt{\sin\theta} \, d\theta.$$

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^0\theta \cdot d\theta.$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

$2m-1 = \frac{1}{2}$

$\Rightarrow m = \frac{3}{4}.$

$2n-1 = 0$

$\Rightarrow n = \frac{1}{2}$

$$\therefore I_1 = \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{2}\right).$$

$$I_2 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} \cdot d\theta$$

$$= \frac{1}{2} \times 2\int_0^{\pi/2} \sin^{-1/2}\theta \; \cos^0\theta \cdot d\theta$$

$$= \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right).$$

$2m-1 = -\frac{1}{2}$

$2m = \frac{1}{2}$

$m = \frac{1}{4}$

and $2n-1 = 0$

$n = 1/2$.

$$\therefore I_1 \times I_2 = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right) \cdot \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4}\, \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}.$$

$$= \frac{1}{4}\, \frac{\sqrt{\pi} \times \sqrt{\pi} \times \pi\sqrt{2}}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{\cancel{4}}\, \frac{\pi \times \pi\sqrt{2}}{\frac{1}{\cancel{4}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\pi \times \pi\sqrt{2}}{\pi\sqrt{2}} = \pi.$$

8. $$I = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\; dy\, dx.$$

In polar coordinate system, $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

$$\therefore I = \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} r^2\sin^2\theta \times r^2\, d\theta\, dr$$

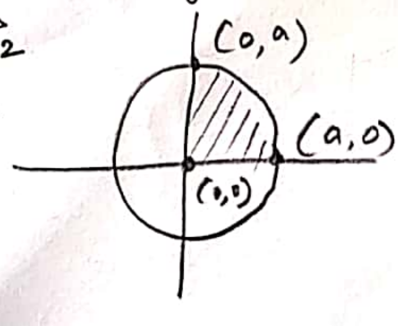
$x=0$, $x=a$

$y=0$, $x^2+y^2=a^2$.

$\theta : 0 \to \pi/2$

$r : 0 \to a$

$$= \int_{r=0}^{a} r^4 \int_{\theta=0}^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right) d\theta\, dr.$$



$$= \frac{1}{2}\int_{r=0}^{a} r^4 \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} dr$$

$$= \frac{1}{2}\int_{r=0}^{a} r^4 \left[\frac{\pi}{2}\right] dr$$

$$= \frac{\pi}{4}\left(\frac{r^5}{5}\right)_{r=0}^{a} = \frac{\pi a^5}{20}.$$