CBCS SCHEME

17MAT41

USN

Fourth Semester B.E. Degree Examination, Jan./Feb.2021
Engineering Mathematics – IV

Time: 3 hrs. Max. Marks: 100

Note: *Answer any FIVE full questions, choosing ONE full question from each module.*

Module-1

1 a. Using Taylor's series method, compute the solution of $\frac{dy}{dx} = x - y^2$ with $y(0) = 1$ at $x = 0.1$, correct to fourth decimal place. (06 Marks)

b. Using modified Euler's formula, solve the $\frac{dy}{dx} = x + \left|\sqrt{y}\right|$ with $y(0.2) = 1.23$ at $x = 0.4$ by taking $h = 0.2$. (07 Marks)

c. The following table gives the solution of $\frac{dy}{dx} = x^2 + \frac{y}{2}$. Find the value of y at $x = 1.4$ by using Milne's Predictor-Corrector method.

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514

(07 Marks)

OR

2 a. Using modified Euler's method, solve $\frac{dy}{dx} = \log_{10}\left(\frac{x}{y}\right)$ with y(20) = 5 at x = 20.2 by taking h = 0.2. (06 Marks)

b. Employ the Range-Kutta method of fourth order to solve $\frac{dy}{dx} = 3x + \frac{y}{2}$, with $y(0) = 1$ at $x = 0.1$ by taking $h = 0.1$. (07 Marks)

c. Using Adams-Bashforth method, find y when $x = 1.4$ given $\frac{dy}{dx} = x^2(1 + y)$, with y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979 (07 Marks)

Module-2

3 a. Using Runge-Kutta method of fourth order solve the differential equation, $\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right)$ for $x = 0.1$. Correct to four decimal places with initial conditions $y(0) = 1$, $y'(0) = 0.5$. (06 Marks)

b. Obtain the series solution of Legendre Differential equation leading to Pn(x). (07 Marks)

c. With usual notation, show that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$. (07 Marks)

OR

4 a. Apply Milne's method to compute y(1.4) given that $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ and

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y′	2	2.3178	2.6725	3.0657

(06 Marks)

b. State and prove Rodrigue's formula. (07 Marks)
c. Express $f(x) = 3x^3 - x^2 + 5x - 2$ in terms of Legendre's polynomials. (07 Marks)

Module-3

5 a. State and prove Cauchy-Riemann equations in polar form. (06 Marks)
b. If $V = e^{-2y} \sin 2x$, find the analytic function f(z). (07 Marks)
c. Find the bilinear transformation that maps the points 0, i, ∞ onto the points 1, $-i$, -1. (07 Marks)

OR

6 a. State and prove Cauchy's theorem on complex integration. (06 Marks)
b. Evaluate $\oint_C \frac{z^2+5}{(z-2)(z-3)} dz$, where $C: |z| = \frac{5}{2}$. (07 Marks)
c. Discuss the transformation $W = Z + \frac{1}{Z}$. (07 Marks)

Module-4

7 a. A box contains 100 transistors, 20 of which are defective and 10 are selected at random, find the probability that (i) all are defective (ii) at least one is defective (iii) all are good (iv) at most three are defective. (06 Marks)
b. Show that mean and standard deviation of exponential distribution are equal. (07 Marks)
c. The joint probability is,

X \ Y	0	1	2	3
0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0

(i) Find marginal distributions of X and Y.
(ii) Also find E(X), E(Y) and E(XY). (07 Marks)

OR

8 a. Find the mean and variance of binomial distribution. (06 Marks)
b. In an examination taken by 500 candidates the average and the standard deviation of marks obtained (normally distributed) are 40% and 10%. Find approximately,
(i) How many will pass, if 50% is fixed as a minimum?
(ii) What should be the minimum if 350 candidates are to pass?
(iii) How many have scored marks above 60%? (07 Marks)
c. Suppose X and Y are independent random variables with the following distributions:

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Find the joint distribution of X and Y. Also find the expectations of X and Y and covariance of X and Y. (07 Marks)

Module-5

9 a. The average income of persons was Rs.210 with a standard deviation of Rs.10 in sample of 100 people of a city. For another sample of 150 persons, the average income was Rs.220 with standard deviation of Rs.12. The standard deviation of the incomes of the people of the city was Rs.11. Test whether there is any significant difference between the average incomes of the localities. (Use $Z_{0.05} = 1.96$) **(06 Marks)**

b. A certain stimulus administered to each of the 12 patients resulted in the following change in blood pressure : 5, 2, 8, −1, 3, 0, 6, −2, 1, 5, 0, 4. Can it be concluded that the stimulus will increase the blood pressure? ($t_{0.05}$ for 11 d.f = 2.201). **(07 Marks)**

c. Define stochastic matrix. Find a unique fixed probability vector for the matrix
$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$
(07 Marks)

OR

10 a. Explain the following terms:
(i) Type I and Type II errors.
(ii) Null hypothesis.
(iii) Level of significance.
(iv) Confidence limits. **(06 Marks)**

b. Eleven school boys were given a test in mathematics carrying a maximum of 25 marks. They were given a month's extra coaching and a second test of equal difficulty was held thereafter. The following table gives the marks in two tests.

Boy	1	2	3	4	5	6	7	8	9	10	11
Marks (I test)	23	20	21	18	18	20	18	17	23	16	19
Marks (II test)	24	19	18	20	20	22	20	20	23	20	17

Do the marks give evidence that the students have benefitted by extra coaching? (Given $t_{0.05} = 2.228$ for 10 d.f) **(07 Marks)**

c. Three boys A, B and C are throwing ball to each other. A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball, find the probabilities that after three throws (i) A has the ball, (ii) B has the ball, (iii) C has the ball. **(07 Marks)**

* * * * *

1a Use Taylor's Series method to compute solution of $x - y^2 = \frac{dy}{dx}$ given $y(0)=1$ at $x=0.1$ correct to 4 decimals.

Soln:- Taylor's Series expansion is given by.

$$y(x) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}\,y''(x_0) + \frac{(x-x_0)^3}{3!}\,y'''(x_0) + \cdots$$

Given $x_0 = 0$; $y_0 = 1$; $x = 0.1$

$y' = x - y^2 \Rightarrow y'(0) = -1$

$\Rightarrow y'' = 1 - 2yy' \Rightarrow y''(0) = 1 - 2(1)(-1) = 3$

$\Rightarrow y''' = 0 - 2(yy'' + y'^2) \Rightarrow y'''(0) = -8$

$\Rightarrow y^{iv} = -2[(yy''' + y'y'') + 2y'y'']$

$= -2[yy''' + 3y'y''] \Rightarrow y^{iv}(0) = 34.$

$$\therefore y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(-8) + \frac{(0.1)^4}{4!}\,34 + \cdots$$

$y(0.1) = 0.9138$

1b Use modified Euler's method solve $\frac{dy}{dx} = x + |\sqrt{y}|$, $y(0.2) = 1.23$ at $x = 0.4$, taking $h = 0.2$

Soln:- Given $x_0 = 0.2$; $y_0 = 1.23$; $h = 0.2$; $y' = f(x,y) = x + |\sqrt{y}|$.

By Euler's formula; $y_1^{(0)} = y_0 + h f(x_0, y_0)$ & $f(x_0, y_0) = 0.2 + |\sqrt{1.23}|$ $= 1.3095$

$= 1.23 + 0.2(1.3095)$

$= 1.4928$

$y_1^{(1)} = 1.23 + \frac{0.2}{2}\left[1.3095 + x_1 + |\sqrt{y_1^{(0)}}|\right] = 1.23 + 0.1\left[1.3095 + 0.4 + \sqrt{1.4928}\right] = 1.524$

$y_1^{(2)} = 1.23 + \frac{0.2}{2}\left[1.3095 + x_1 + \sqrt{y_1^{(1)}}\right] = 1.23 + 0.1\left[1.3095 + 0.4 + \sqrt{1.524}\right] = 1.5253$

$y_1^{(3)} = 1.23 + \frac{0.2}{2}\left[1.3095 + x_1 + \sqrt{y_1^{(2)}}\right] = 1.23 + 0.1\left[1.3095 + 0.4 + \sqrt{1.5253}\right] = 1.5254$

Likewise $y_1^{(4)} = 1.5254$ $\therefore$ $y(0.4) = 1.5254.$

CMR

1c Using Milnes Predictor Corrector method find y at $x = 1.4$

given

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514

given $y' = x^2 + \frac{y}{2}$

Soln:- Let us form the table of required Values

$h = 0.1$

x	y	$y' = x^2 + \frac{y}{2}$
$x_0 = 1$	$y_0 = 2$	$y_0' = 1^2 + \frac{2}{2} = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y_1' = (1.1)^2 + \frac{2.2156}{2} = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4649$	$y_2' = (1.2)^2 + \frac{2.4649}{2} = 2.67245$
$x_3 = 1.3$	$y_3 = 2.7514$	$y_3' = (1.3)^2 + \frac{2.7514}{2} = 3.0657.$

By Milne's method, predictor $y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$$\Rightarrow y_4^{(P)} = 2 + \frac{4(0.1)}{3}\left(2(2.3178) - 2.67245 + 2(3.0657)\right)$$

$$= 3.0793$$

and $y_4' = x_4^2 + \frac{y_4}{2} = (1.4)^2 + \frac{3.0793}{2} = 3.49965$

now the Corrector $y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$

$$\Rightarrow y_4^{(C)} = 2.4649 + \frac{0.1}{3}\left(2.67245 + 4(3.0657) + 3.49965\right)$$

$$= 3.0794$$

$\therefore y_4 = y(1.4) = 3.0794.$

Use Modified Euler's method, to solve $\frac{dy}{dx} = \log_{10}\left(\frac{x}{y}\right)$ with $y(20) = 5$ at $x = 20.2$ taking $h = 0.2$.

Soln:- Given $x_0 = 20$; $y_0 = 5$; $h = 0.2$; $\frac{dy}{dx} = f(x,y) = \log_{10}\left(\frac{x}{y}\right)$

$\Rightarrow f(x_0, y_0) = \log_{10}\left(\frac{20}{5}\right) = \log_{10} 4 = 0.6021$.

By Euler's formula $y_1^{(0)} = y_0 + h f(x_0, y_0)$.

$y_1^{(0)} = 5 + 0.2(0.6021) = 5.1204$

By Modified Euler's formula, $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$\Rightarrow y_1^{(1)} = 5 + \frac{0.2}{2}\left[0.6021 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right]$

$= 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1204}\right)\right] = 5.1198$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + \log_{10}\left(\frac{x_1}{y_1^{(1)}}\right)\right] = 5 + \frac{0.2}{2}\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1198}\right)\right]$

$= 5.1198$.

$\therefore y_1 = y(20.2) = 5.1198$.

Using Runge-Kutta Method of order 4 to solve $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$ at $x = 0.1$, taking $h = 0.1$

Soln:- Given $x_0 = 0$; $y_0 = 1$; $h = 0.1$; $\frac{dy}{dx} = f(x,y) = 3x + \frac{y}{2}$

we have $y_1 = y(0.1) = y_0 + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$.

where $k_1 = h\, f(x_0, y_0) = 0.1\, f(0,1) = 0.1 \left(3(0) + \frac{1}{2}\right) = 0.1\left(\frac{1}{2}\right) = 0.05$

$K_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\left[3\left(0 + \frac{0.1}{2}\right) + \frac{1.025}{2}\right] = 0.1\left[0.15 + 0.5125\right]$

$= 0.06625$

$K_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\left[3\left(0 + \frac{0.1}{2}\right) + \frac{\left(1 + \frac{0.06625}{2}\right)}{2}\right].$

$= 0.1\left[0.15 + 0.565625\right] = 0.06665$

$k_4 = h\, f(x_0 + h, y_0 + k_3) = 0.1\left[3(0 + 0.1) + \frac{1 + 0.06665}{2}\right].$

$= 0.1\left[0.3 + 0.51666\right] = 0.081667$

$\therefore y_1 = y_0 + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$= 1 + \frac{0.05 + 2(0.06625) + 2(0.06665) + 0.081667}{6}.$

$y_1 = y(0.1) = 1.0662$

2(c) Using Adams-Bashforth method, find y when $x = 1.4$ given $\frac{dy}{dx} = x^2(1+y)$, $y(1) = 1$; $y(1.1) = 1.233$; $y(1.2) = 1.548$; $y(1.3) = 1.979$

Soln:- Given $x_0 = 1$; $x_1 = 1.1$; $x_2 = 1.2$; $x_3 = 1.3$ } $h = 0.1$
$y_0 = 1$; $y_1 = 1.233$; $y_2 = 1.548$; $y_3 = 1.979$ } $f(x,y) = x^2(1+y)$

$\Rightarrow y_0' = 2$; $y_1' = 2.702$; $y_2' = 3.669$; $y_3' = 5.035$.

By Adams Bashforth method, Predictor $y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$

$\Rightarrow y_4^{(P)} = 1.979 + \frac{0.1}{4}\left[55(5.035) - 59(3.669) + 37(2.702) - 9(2)\right] = 2.573$

$y_4' = x_4^2(1+y_4^{(P)}) = (1.4)^2(1+2.573) = 7.004$

Now $y_4^{(C)} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$

$= 1.979 + \frac{0.1}{24}(9(7.004) + 19(5.035) - 5(3.669) + 2.702)$

$= 2.575$.

$\therefore y_4 = y(1.4) = 2.575$.

Module-2

3 ⓐ Use RK Method (4th order) to solve $\frac{d^2y}{dx^2} = x^3(y + \frac{dy}{dx})$ for $x = 0.1$ correct to 4 decimals, given that $y(0) = 1$ & $y'(0) = 0.5$.

Soln: Let $z = \frac{dy}{dx}$ then $\frac{dz}{dx} = \frac{d^2y}{dx^2} = x^3(y+z)$.

So take $f(x,y,z) = z$ & $g(x,y,z) = x^3(y+z)$; $x_0 = 0$; $y_0 = 1$;
$z_0 = 0.5$; $h = 0.1$.

$\therefore y_1 = y(0.1) = y_0 + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$ where

$k_1 = hf(x_0, y_0, z_0) = hz_0 = 0.1(0.5) = 0.05$.

$l_1 = hg(x_0, y_0, z_0) = 0.1(x_0^3(y_0 + z_0)) = 0.1(0^3(1+0.5)) = 0$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = h(z_0 + \frac{l_1}{2}) = 0.1(0.5 + \frac{0}{2})$
$= 0.05$

$l_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = h\left[(x_0 + \frac{h}{2})^3((y_0 + \frac{k_1}{2}) + (z_0 + \frac{l_1}{2}))\right]$

$= 0.1\left[(0.05)^3\{(1 + \frac{0.05}{2}) + (0.5 + \frac{0}{2})\}\right] = 0.1\left[0.000125\{1.025 + 0.5\}\right]$

$= 0.000019$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = 0.1(z_0 + \frac{l_2}{2}) = 0.1(0.5 + 0.0000095)$
$= 0.05001$

$l_3 = h\,g\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right) = 0.1\left[\left(x_0+\frac{h}{2}\right)^3\left\{\left(y_0+\frac{k_2}{2}\right)+\left(z_0+\frac{l_2}{2}\right)\right\}\right]$

$= 0.1\left[\left(0+\frac{0.1}{2}\right)^3\left\{(1+0.025)+\left(0.5+\frac{0.0019}{2}\right)\right\}\right]$

$= 0.1\left[(0.05)^3\left\{1.025+0.500095\right\}\right] = 0.000019$

$k_4 = h\,f(x_0+h, y_0+k_3, z_0+l_3) = 0.1(z_0+l_3) = 0.1(0.5+0.000019)$

$= 0.1(0.5000019) = 0.0500019$

$y_1 = y_0 + \frac{0.05+2(0.05)+2(0.05001)+0.0500019}{6} = 1.05000365$

$\therefore y_1 = y(0.1) = 1.05$

3b The Legendre's differential equation is $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \rightarrow (1)$

Let $y = \sum_{r=0}^{\infty} a_r x^r \rightarrow (2)$ be a series solution of (1).

$\Rightarrow \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r r x^{r-1}$; $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2}$.

Then

(1) $\Rightarrow (1-x^2)\sum a_r r(r-1)x^{r-2} - 2x\sum a_r r x^{r-1} + n(n+1)\sum a_r x^r = 0.$

$\Rightarrow \sum a_r r(r-1)x^{r-2} - \sum a_r r(r-1)x^r - \sum 2a_r r x^r + n(n+1)\sum a_r x^r = 0.$

Equating the coeffts of powers of x to zero

Coefft of x^{-2}; $a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0.$

Coefft of x^{-1}; $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0.$

Coefft of $x^r (r \geq 0)$; $a_{r+2}(r+2)(r+1) - a_r r(r-1) - 2a_r r + n(n+1)a_r = 0$

$\Rightarrow a_{r+2}(r+2)(r+1) = a_r(r-1) - 2a_r r + n(n+1)a_r$

$\Rightarrow a_{r+2} = a_r\left(r(r-1)+2r-n(n+1)\right) / \{(r+2)(r+1)\}$

$$\Rightarrow a_{r+2} = \left\{\frac{-n(n+1) - r(r+1)}{(r+2)(r+1)}\right\} a_r$$

Let $r=0 \Rightarrow a_2 = \frac{-n(n+1)}{2} a_0$; $a_3 = \frac{-(n-1)(n+2)}{6} a_1$

$$a_4 = -\frac{(n-2)(n+3)}{12} a_2 = \left\{\frac{-(n-2)(n+3)}{12}\right\}\left\{\frac{-n(n+1)}{2}\right\} a_0$$

$$= \frac{n(n+1)(n-2)(n+3)}{24} a_0$$

$$a_5 = -\frac{(n-3)(n+4)}{20} \cdot a_3 = \left\{\frac{-(n-3)(n+4)}{20}\right\}\left\{-\frac{(n-1)(n+2)}{6} a_1\right\}$$

$$= \frac{(n-1)(n+2)(n-3)(n+4)}{120} a_1$$

Substituting these coefficients in expanded form of (2)

We get $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \cdots) + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots)$$

$$\Rightarrow y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 - \cdots\right]$$

$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}x^5 - \cdots\right].$$

$\Rightarrow y = a_0 u(x) + a_1 v(x)$ — series soln of Legendre's eqn

also $y = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots + F(x)$

Where $F(x) = \begin{cases} a_0 & \text{if } n \text{ is even} \\ a_1 x & \text{if } n \text{ is odd} \end{cases}$.

Further $a_n = \dfrac{-[n(n+1)-(n-2)(n-1)]}{n(n-1)} a_{n-2} = \dfrac{-(4n-2)}{n(n-1)} a_{n-2}$

or $a_{n-2} = \dfrac{-n(n-1)}{2(2n-1)} a_n$ & $a_{n-4} = \dfrac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)} a_n \cdots$

$\Rightarrow y = a_n\left[x^n - \dfrac{n(n-1)}{2(2n-1)} x^{n-2} + \dfrac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)} x^{n-4} - \cdots + G(x)\right]$

where $G(x) = \begin{cases} a_0/a_n, & n \text{ even} \\ a_1x/a_n & n \text{ odd.} \end{cases}$

If constants a_n are chosen such that $y=1$ when $x=1$, the polynomials obtained are called Legendre Polynomials denoted by $P_n(x)$.

3 c To prove $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$.

Proof :- we have $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\dfrac{x}{2}\right)^{n+2r} \cdot \dfrac{1}{\Gamma(n+r+1)\cdot r!}$

take $n = 1/2 \Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\dfrac{x}{2}\right)^{\frac{1}{2}+2r} \cdot \dfrac{1}{\Gamma(\frac{1}{2}+r+1)\cdot r!}$

$= \sqrt{\dfrac{x}{2}} \sum_{r=0}^{\infty} (-1)^r \left(\dfrac{x}{2}\right)^{2r} \dfrac{1}{r!\,\Gamma(r+\frac{3}{2})}$

$= \sqrt{\dfrac{x}{2}} \left[\dfrac{1}{\Gamma(\frac{3}{2})} - \left(\dfrac{x}{2}\right)^2 \dfrac{1}{\Gamma(\frac{5}{2})} + \left(\dfrac{x}{2}\right)^4 \dfrac{1}{\Gamma(\frac{7}{2})\cdot 2!} - \cdots\right]$

we know $\Gamma(\frac{1}{2}) = \sqrt{\pi}$; $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$; $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3\sqrt{\pi}}{4}$

$\Gamma(\frac{7}{2}) = \frac{5}{2}\Gamma(\frac{5}{2}) = \frac{5}{2}\cdot\frac{3\sqrt{\pi}}{4} = \frac{15\sqrt{\pi}}{8}, \cdots$

Substituting these values we get.

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\cdot\frac{8}{15\sqrt{\pi}\cdot 2} - \cdots\right].$$

$$= \sqrt{\frac{x}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{2}{x}\left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \Rightarrow \boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x}$$

4(a) To compute $y(1.4)$ giving $\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ and

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

Let $y' = \frac{dy}{dx} = z$; $\Rightarrow y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$

Now the DE is $z' = 2x + \frac{z}{2}$

$\Rightarrow z_0' = 2x_0 + \frac{z_0}{2} = 3$; $z_1' = 2x_1 + \frac{z_1}{2} = 3.3598$

$z_2' = 2x_2 + \frac{z_2}{2} = 2(1.2) + \frac{2.6725}{2} = 3.73625$

$z_3' = 2x_3 + \frac{z_3}{2} = 2(1.3) + \frac{3.0657}{2} = 4.13285$.

Thus we have

x	$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$
y	$y_0 = 2$	$y_1 = 2.2156$	$y_2 = 2.4649$	$y_3 = 2.7514$
$y' = z$	$z_0 = 2$	$z_1 = 2.3178$	$z_2 = 2.6725$	$z_3 = 3.0657$
$y'' = z'$	$z_0' = 3$	$z_1' = 3.3598$	$z_2' = 3.73625$	$z_3' = 4.13285$

By Milne's Method, the Predictor $y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$

$\Rightarrow y_4^{(P)} = 2 + \frac{4(0.1)}{3}\left(2(2.3178) - 2.6725 + 2(3.0657)\right)$

$= 3.0793$

and $z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$

$= 2 + \frac{4(0.1)}{3}\left(2(3.3589) - 3.73625 + 2(4.13285)\right)$

$= 3.4996$

and the corrector $y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$

$\Rightarrow y_4^{(C)} = 2.4649 + \frac{0.1}{3}\left(2.6725 + 4(3.0657) + 3.4996\right)$

$= 3.0794$

$\therefore y_4 = y(1.4) = 3.0794$.

4(b) Rodrigue's formula $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \rightarrow (1)$.

Proof:- Let $u = (x^2-1)^n$.

We first show u_n, the n^{th} derivative of u, is a soln of Legendre differential eqn. $(1-x^2)y'' - 2xy' + n(n+1)y = 0 \rightarrow (2)$

Differentiating (2) w.r.t x, we have

$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1} 2x \Rightarrow (x^2-1)u_1 = 2nx(x^2-1)^n$

$\Rightarrow (x^2-1)u_1 = 2nxu$

Differentiating again w.r.t x, we have

$(x^2-1)u_2 + 2xu_1 = 2n(xu_1 + u)$

Now, differentiating n times using Leibnitz Thm.

$[(x^2-1)u_2]_n + 2[xu_1]_n = 2n[xu_1]_n + 2nu_n$.

$\Rightarrow \left[[x^2-1]u_{n+2} + n \cdot 2x u_{n+1} + \frac{n(n-1)}{2} \cdot 2u_n \right]$

$+ 2\left[x u_{n+1} + n \cdot 1 \cdot u_n\right] = 2n\left[x u_{n+1} + n \cdot 1 \cdot u_n\right] + 2nu_n$

$\Rightarrow (x^2-1)u_{n+2} + 2nx u_{n+1} + (n^2-n)u_n + 2xu_{n+1} + 2nu_n$

$= 2nx u_{n+1} + 2n^2 u_n + 2nu_n$

$\Rightarrow (x^2-1)u_{n+2} + 2xu_{n+1} - n^2 u_n - nu_n = 0$

$\Rightarrow (1-x^2)u_{n+2} - 2xu_{n+1} + n(n+1)u_n = 0$.

$\Rightarrow (1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0 \rightarrow (3)$

$\Rightarrow u_n$ is a soln of the Legendre's diff. eqn

$P_n(x)$, a polynomial of degree n, also satisfies Legendre differential eqn. $\Rightarrow$ u_n must be same as $P_n(x)$.

$\Rightarrow P_n(x) = k u_n = k[(x^2-1)^n]_n$

$\Rightarrow P_n(x) = k[(x-1)^n (x+1)^n]_n$

$\Rightarrow P_n(x) = k\Big[(x-1)^n \{(x+1)^n\}_n + n \cdot n(x-1)^{n-1} \{(x+1)^n\}_{n-1}$

$+ \frac{n(n-1)}{2!} n(n-1)(x-1)^{n-2} \{(x+1)^n\}_{n-1} + \cdots \{(x-1)^n\}_n (x+1)^n \Big]$

$\rightarrow (4)$

But $\{(x-1)^n\}_n = n!$

taking $x=1$ in (4)

we get $P_n(1) = k \cdot n! \, 2^n$ & $P_n(1) = 1$ (by defn of $P_n(x)$).

$\Rightarrow \quad k = \frac{1}{n! \, 2^n}$

$\because P_n(x) = k u_n$, we have $P_n(x) = \frac{1}{n! \, 2^n}\{(x^2-1)^n\}_n$

$\Rightarrow \quad P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$, the Rodrigue's formula.

Express $f(x) = 3x^3 - x^2 + 5x - 2$ in terms of Legendree Polynomials.

We have $P_1(x) = x$; $P_0(x) = 1$

$P_2(x) = \frac{3x^2-1}{2}$

$\Rightarrow \quad 3x^2 = 2P_2 + 1$

$\Rightarrow \quad x^2 = \frac{2P_2(x)+1}{3}$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

$5x^3 - 3x = 2P_3(x) \Rightarrow 5x^3 = 2P_3(x) + 3x$

$\Rightarrow \quad x^3 = \frac{2P_3(x) + 3x}{5}$

$\therefore f(x) = 3x^3 - x^2 + 5x - 2 = 3\left(\frac{2P_3(x)+3x}{5}\right) - \left(\frac{2P_2(x)+1}{3}\right) + 5P_1(x) - 2P_0(x)$

$= \frac{6}{5}P_3(x) + \frac{9x}{5} - \frac{2}{3}P_2(x) - \frac{2}{3} - 5P_1(x) - 2P_0(x)$

$= \frac{6}{5}P_3(x) + \frac{9}{5}P_1(x) - \frac{2}{3}P_2(x) - \frac{2}{3}P_0(x) - 5P_1(x) - 2P_0(x)$

$= \frac{6}{5}P_3(x) - \frac{2}{3}P_2(x) + \left(\frac{9}{5} - 5\right)P_1(x) - \left(\frac{2}{3} + 2\right)P_0(x)$

5(a) State and Prove Cauchy-Riemann equations in polar form.

Statement: If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ is analytic at a point z, then there exists four continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfy the equations: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$, called Cauchy-Riemann in polar form.

Proof:- Let $f(z)$ be analytic at a point $z = re^{i\theta}$ and hence by definition, $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and is unique.

In polar form $f(z) = u(r,\theta) + iv(r,\theta)$ and let δz be the increment in z corresponding to increments $\delta r, \delta\theta$ in r, θ

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta z}$$

$$\Rightarrow f'(z) = \lim_{\delta z \to 0} \frac{u(r,\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + i \lim_{\delta z \to 0} \frac{v(r+\delta r, \theta+\delta\theta) - v(r,\theta)}{\delta z} \quad \rightarrow (1)$$

Consider $z = re^{i\theta} \Rightarrow \delta z = \frac{\partial z}{\partial r}\delta r + \frac{\partial z}{\partial \theta}\delta\theta$

$$= \frac{\partial}{\partial r}(re^{i\theta})\delta r + \frac{\partial}{\partial \theta}(re^{i\theta})\delta\theta$$

$$\Rightarrow \delta z = e^{i\theta}\delta r + ire^{i\theta}\delta\theta$$

Since δz tends to zero, we have these two cases.

Case (i): Let $\delta\theta = 0 \Rightarrow \delta z = e^{i\theta}\delta r$ and $\delta z \to 0 \Rightarrow \delta r \to 0$

Now ① implies

$$f'(z) = \lim_{\delta r \to 0} \frac{u(r+\delta r, \theta) - u(r,\theta)}{e^{i\theta}\delta r} + i \lim_{\delta r \to 0} \frac{v(r+\delta r, \theta) - v(r,\theta)}{e^{i\theta}\delta r}$$

$$= e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]. \to ②$$

Case ii: Let $\delta r = 0$ so that $\delta z = ire^{i\theta}\delta\theta$ and $\delta z \to 0 \Rightarrow \delta\theta \to 0$

Now, ① implies

$$f'(z) = \lim_{\delta\theta \to 0} \frac{u(r, \theta+\delta\theta) - u(r,\theta)}{ire^{i\theta}\delta\theta} + i \lim_{\delta\theta \to 0} \frac{v(r, \theta+\delta\theta) - v(r,\theta)}{ire^{i\theta}\delta\theta}$$

$$= \frac{1}{ire^{i\theta}}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right] = \frac{1}{re^{i\theta}}\left[\frac{1}{i}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right].$$

$$\Rightarrow f'(z) = e^{-i\theta}\left[-\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right]. \to ③$$

Now ② & ③ $\Rightarrow$ $f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = e^{-i\theta}\left[-\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right]$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

These are Cauchy-Riemann eqns in Polar form.

5(b)

$V = e^{-2y}\sin 2x$, to find analytic fn $f(z) = u + iv$

$$\frac{\partial v}{\partial x} = e^{-2y}(2\cos 2x) \;;\quad \frac{\partial v}{\partial y} = \sin 2x\,(-2e^{-2y}).$$

But since $u + iv$ is analytic, u & v satisfy CR eqns

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -2e^{-2y}\sin 2x$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2e^{-2y}\cos 2x .$$

So $f'(z) = u_x + iv_x = -2e^{-2y}\sin 2x + i(2e^{-2y}\cos 2x)$

$\Rightarrow f'(z) = 2e^{-2y}(-\sin 2x + i\cos 2x)$

By Milne's Thompson method, put $x = z$, $y = 0$

$$f'(z) = 2e^{-2(0)}(-\sin 2z + i\cos 2z)$$

$$= 2(1)(-\sin 2z + i\cos 2z) = -2e^{-2z} + c$$

$$\therefore f'(z) = -2e^{-2z} + c.$$

(c) Find the Bilinear transformation that maps $0, i, \infty$ onto $1, -i, -1$; the Bilinear transformation is $w = \frac{az+b}{cz+d} \rightarrow (1)$

$z = 0, w = 1 \Rightarrow 1 = \frac{0+b}{0+d} \Rightarrow b = d$ or $b - d = 0 \rightarrow (2)$

$z = i, w = -i \Rightarrow -i = \frac{ai+b}{ci+d}$

$\Rightarrow ai + b - c + di = 0. \rightarrow (3)$

$z=\infty,\ w=-1 \Rightarrow -1 = \dfrac{a+b/z}{c+d/z}$

$\Rightarrow -1 = \dfrac{a}{c} \Rightarrow a=-c \Rightarrow a+c=0 \rightarrow (4)$

$(3)+(4) \Rightarrow (1+i)a+b+id=0 \rightarrow (5)$

Solving (2) & (5) we get $0\cdot a+1b-1d=0$

$(1+i)a+1b+id=0.$

By cross multiplication,

$$\frac{a}{\begin{vmatrix}1 & -1\\ 1 & i\end{vmatrix}} = \frac{-b}{\begin{vmatrix}0 & -1\\ 1+i & i\end{vmatrix}} = \frac{d}{\begin{vmatrix}0 & 1\\ 1+i & 1\end{vmatrix}}.$$

$$\frac{a}{i+1} = \frac{-b}{1+i} = \frac{d}{-(1+i)};\ \Rightarrow \frac{a}{1} = \frac{b}{-1} = \frac{d}{-1}$$

$\Rightarrow a=1;\ b=-1\ \&\ d=-1 \Rightarrow c=-a=-1$

$\therefore w = \dfrac{z-1}{-z-1} \Rightarrow w = \dfrac{1-z}{1+z}$ is the required BLT;

6a) State and prove Cauchy's Theorem on Complex Integration

Statement:- If $f(z)$ is analytic at every point inside and on a simple closed curve C then

Proof: Let $f(z)=u+iv$

$\Rightarrow \int_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx-vdy)+i\int(vdx+udy)$

By Green's thm "If $M(x,y)$ and $N(x,y)$ are two real valued functions having continuous first order partial derivatives in a region R bounded by curve C, then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

But since $f(z) = u+iv$ is analytic, u & v satisfy CR eqns

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and hence we have

$$\int_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= 0$$

$\Rightarrow \int_C f(z)\,dz = 0$. This proves Cauchy's theorem.

6(b) Evaluate $I = \int_C \frac{z^2+5}{(z-2)(z-3)}\,dz$ where $C: |z| = \frac{5}{2}$

Let us consider $\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3} = \frac{A(z-3)+B(z-2)}{(z-2)(z-3)}$

$\Rightarrow A(z-3) + B(z-2) = 1$

put $z = 2 \Rightarrow -A + B(0) = 1 \Rightarrow A = -1$

$z = 3 \Rightarrow A(0) + B = 1 \Rightarrow B = 1$

$\Rightarrow \frac{1}{(z-2)(z-3)} = \frac{-1}{z-2} + \frac{1}{z-3}$

$$\Rightarrow I = \int_C \frac{z^2+5}{(z-2)(z-3)}\,dz = \int_C \frac{-(z^2+5)}{(z-2)}\,dz + \int_C \frac{(z^2+5)}{z-3}\,dz = I_1 + I_2$$

We know by Cauchy's Integral formula, $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$.

$I_1 = \int \frac{f(z)}{z-a}\,dz$; where $f(z) = -(z^2+5) \Rightarrow f(a) = -(2^2+5) = -9$
$a = 2$

$I_2 = \int \frac{f(z)}{z-a}\,dz$; where $f(z) = -(z^2+5)$; $f(a) = -(3^2+5) = -14$.
$a = 3$

$\therefore I = 2\pi i\left((-9)+(-14)\right) = -26\pi i$

Discuss the Transformation $W = z + \frac{1}{z} \rightarrow (1)$

Soln :- Consider $w = z + \frac{1}{z}$; Take $z = re^{i\theta}$

$$\Rightarrow w = u + iv = z + \frac{1}{z} = re^{i\theta} + \frac{1}{re^{i\theta}} = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\Rightarrow u = \left(r + \frac{1}{r}\right)\cos\theta \quad \& \quad v = \left(r - \frac{1}{r}\right)\sin\theta \longrightarrow (2).$$

Eliminating θ we get

$$\frac{u}{r+\frac{1}{r}} = \cos\theta \quad \& \quad \frac{v}{r-\frac{1}{r}} = \sin\theta$$

$$\Rightarrow \cos^2\theta + \sin^2\theta = 1 \Rightarrow \left(\frac{u}{r+\frac{1}{r}}\right)^2 + \left(\frac{v}{r-\frac{1}{r}}\right)^2 = 1$$

$$\Rightarrow \frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = 1. \longrightarrow (3).$$

Eliminating r we get, $\frac{u}{\cos\theta} = r + \frac{1}{r}$; $\frac{v}{\sin\theta} = r - \frac{1}{r}$

$$\Rightarrow \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2 = 4$$

$$\frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1 \longrightarrow (4)$$

We know $|z| = r = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = r^2$

a circle with centre at origin & radius r.

and $\text{amp } z = \theta \Rightarrow \tan^{-1}\frac{y}{x} = \theta$ or $\frac{y}{x} = \tan\theta$.

Case i :- Let r = const

Then (3) $\Rightarrow \frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$ where $a = r+\frac{1}{r}$ & $b = r - \frac{1}{r}$

an ellipse in w-plane with foci $(\pm\sqrt{a^2-b^2}, 0) = (\pm 2, 0)$

$\therefore$ $|z| = r$ = Constant, a circle with centre at o & radius r in z-plane is mapped onto an ellipse in the w-plane by $w = z + \frac{1}{z}$ with foci $(\pm 2, 0)$

Case 2 :- Let θ = Const

Eq.n (4) is of the form

$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$ where $A = 2a\cos\theta$ $B = 2a\sin\theta$.

a hyperbola in w-plane with foci $(\pm\sqrt{A^2+B^2}, 0) = (\pm 2, 0)$.

$\therefore$ Straight line passing thro' origin in the z-plane maps onto a hyperbola in the w-plane with foci $(\pm 2, 0)$ under the mapping $w = z + \frac{1}{z}$,

5) No of transistors in the box = 100

Defective transistors in the box = 20

Good transistors = 100 - 20 = 80.

Let x be the r.v denoting the no of defective transistors

No of transistors drawn at random = 10.

$\therefore$ x can take value from 0 to 10.

i) Probability that all are defective = $P(x=10) = \dfrac{20C_{10} \cdot 80C_0}{100C_{10}}$.

ii) Probability that atleast one is defective = 1 - Prob. that no defects

$= 1 - P(x=0) = 1 - \dfrac{20C_0 \, 80C_{10}}{100C_{10}}$.

iii) Probability that all are good = $P(x=0) = \dfrac{20C_0 \, 80C_{10}}{100C_{10}}$

iv) Probability that atmost 3 defectives = $P(x \leq 3)$

$= P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= \dfrac{20C_0 80C_{10}}{100C_{10}} + \dfrac{20C_1 80C_9}{100C_{10}} + \dfrac{20C_2 \, 80C_8}{100C_{10}} + \dfrac{20C_3 \, 80C_7}{100C_{10}}$.

) To show Mean = Std Deviation for an exponential Distribution

Exponential Distribution is $f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$

where $\alpha > 0$

Mean $= \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} x \cdot \alpha e^{-\alpha x} \, dx$

$= \alpha \int_0^{\infty} x e^{-\alpha x} \, dx = \alpha \left[x \cdot \dfrac{e^{-\alpha x}}{-\alpha} - (1) \dfrac{e^{-\alpha x}}{(-\alpha)^2} \right]_0^{\infty}$

CMR

$\Rightarrow \mu = \alpha\left[0 - \frac{1}{\alpha^2}(0-1)\right] = \frac{1}{\alpha}$ $\quad \because \lim_{x\to\infty} \frac{x}{e^{\alpha x}} \to 0$ by L'Hospital's Rule

$\therefore \boxed{\mu = \frac{1}{\alpha}}$

$\sigma^2 = \text{Variance} = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_0^{\infty} (x-\mu)^2 \alpha e^{-\alpha x}\,dx$

$= \alpha\left[(x-\mu)^2\left(\frac{e^{-\alpha x}}{-\alpha}\right) - 2(x-\mu)\left(\frac{e^{-\alpha x}}{(-\alpha)^2}\right) + 2\left(\frac{e^{-\alpha x}}{(-\alpha)^3}\right)\right]_0^{\infty}$

$= \alpha\left[\frac{-1}{\alpha}(0-\mu^2) - \frac{2}{\alpha^2}(0-(-\mu)) - \frac{2}{\alpha^3}(0-1)\right]$

$\because \lim_{x\to\infty} \frac{x}{e^{\alpha x}} \to 0$ and

$\lim_{x\to\infty} \frac{x^2}{e^{\alpha x}} \to 0$ by L'Hospital's Rule

$= \alpha\left[\frac{\mu^2}{\alpha} - \frac{2\mu}{\alpha^2} + \frac{2}{\alpha^3}\right]$ But we have $\mu = \frac{1}{\alpha}$

So $\sigma^2 = \alpha\left[\frac{1}{\alpha^3} - \frac{2}{\alpha^3} + \frac{2}{\alpha^3}\right] = \frac{1}{\alpha^2} \Rightarrow \boxed{\sigma = \frac{1}{\alpha}}$

$\therefore \boxed{\mu = \sigma = \frac{1}{\alpha}}$ for an exponential distribution.

7c The Joint Probability density is

X \ Y	0	1	2	3
0	0	1/8	1/4	1/8
1	1/8	1/4	1/8	0

CMR

To write Marginal Distributions of X and Y, E(X), E(Y) & E(XY)

Marginal Distribution of X

$X = x$	0	1
$f(x)$	$0+\frac{1}{8}+\frac{1}{4}+\frac{1}{8}$ $=\frac{1}{2}$	$\frac{1}{8}+\frac{1}{4}+\frac{1}{8}+0$ $=\frac{1}{2}$

Marginal Distribution of Y

$Y = y$	0	1	2	3
$g(y)$	$0+\frac{1}{8}$ $=\frac{1}{8}$	$\frac{1}{8}+\frac{1}{4}$ $=\frac{3}{8}$	$\frac{1}{4}+\frac{1}{8}$ $=\frac{3}{8}$	$\frac{1}{8}+0$ $=\frac{1}{8}$

$$E(X) = \sum x f(x) = \left(0\times\frac{1}{2}\right)+\left(1\times\frac{1}{2}\right) = \frac{1}{2}$$

$$E(Y) = \sum y\, g(y) = \left(0\times\frac{1}{8}\right)+\left(1\times\frac{3}{8}\right)+\left(2\times\frac{3}{8}\right)+\left(3\times\frac{1}{8}\right)$$

$$= 0 + \frac{3}{8}+\frac{6}{8}+\frac{3}{8} = \frac{12}{8} = \frac{3}{2}$$

$$E(XY) = \sum_{i,j} x_i y_j J_{ij} = (0\times0\times0)+\left(0\times1\times\frac{1}{8}\right)+\left(0\times2\times\frac{1}{4}\right)+\left(0\times3\times\frac{1}{8}\right)$$

$$+\left(1\times0\times\frac{1}{8}\right)+\left(1\times1\times\frac{1}{4}\right)+\left(1\times2\times\frac{1}{8}\right)+(1\times3\times0)$$

$$= 0+0+0+0+0+\frac{1}{4}+\frac{1}{4} = \frac{1}{2}$$

8 To find Mean & Variance of Binomial Distribution.

We know the probability function $P(x)$ of Binomial Distribution is $P(x) = {}^nC_x\, p^x q^{n-x}$

& Mean $= \sum x P(x) = \mu$

$V = \sigma^2 = \text{Variance} = \sum x^2 P(x) - \mu^2$

$$\text{Mean} = \sum x P(x) = \sum_{x=0}^{n} x\, {}^nC_x\, p^x q^{n-x} = \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{x \cdot (n-1)!\,(n)}{(x-1)!\,(x) \cdot [(n-1)-(x-1)]!}\, p \cdot p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \cdot \sum_{x=1}^{n} {}^{(n-1)}C_{x-1}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \cdot (q+p)^{n-1} = np\,(1)^{n-1} = np.$$

$\therefore$ Mean $= \mu = np$.

$$\text{Variance} = V = \sum_{x=0}^{n} x^2 P(x) - \mu^2 = \sum_{x=0}^{n} [x(x-1)+x]\, P(x) - \mu^2$$

$$= \sum_{x=0}^{n} (x(x-1) P(x) + \sum_{x=0}^{n} x P(x) - \mu^2 = \sum_{x=0}^{n} x(x-1)\, nC_x p^x q^{n-x} + \mu - \mu^2$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x} + \mu - \mu^2$$

$$= \sum_{x=0}^{n} \frac{x(x-1)\,(n-2)!\,(n-1)(n)}{(x-2)!\,(x-1)(x)((n-2)-(x-2))!}\, p^2 p^{x-2} q^{(n-2)-(x-2)} + np - n^2p^2$$

$(\because \mu = np)$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!\; p^{x-2} q^{(n-2)-(x-2)}}{(x-2)!\,((n-2)-(x-2))!} + np - n^2p^2$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{(n-2)}C_{x-2} \cdot p^{x-2} q^{(n-2)-(x-2)} + np - n^2p^2$$

$$= (n^2-n)p^2\,(q+p)^{n-2} + np - n^2p^2$$

$= (n^2p^2 - np^2)(1)^{n-2} + np - n^2p^2$

$= n^2p^2 - np^2 + np - n^2p^2$

$= np(1-p) = npq \qquad \because p+q=1$

$\therefore$ Variance $= V = \sigma^2 = npq$.

8 b) No of students who took the exam = 500.

Mean $= \mu = 40\%$.

Std Deviation $\sigma = 10\%$.

No of students passed if 50% is pass mark

$= 500 \times P(x \geq 50) = 500 \times P\left(\frac{x-\mu}{\sigma} \geq \frac{50-40}{10}\right)$

$= 500 \times P(z \geq 1)$

$= 500 \times \{0.5 - P(0 < z < 1)\}$.

$= 500 \times \{0.5 - 0.3413\}$

$= 500 \times 0.1587 = 79.35 \approx 79$.

For 350 students to pass, let the minimum mark be $x_1\%$.

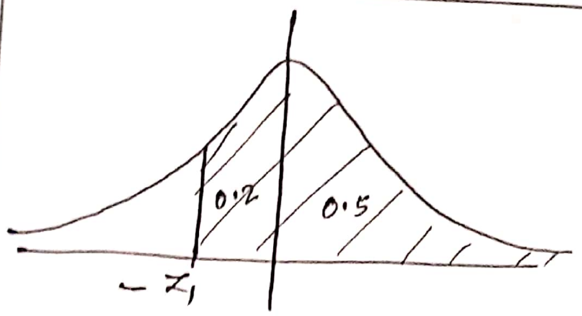
then $350 = 500 \times P(x \geq x_1) \Rightarrow \frac{350}{500} = P\left(\frac{x-\mu}{\sigma} \geq \frac{x_1 - 40}{10}\right)$

$\Rightarrow \frac{350}{500} = P(z \geq z_1) = 0.5 + P(0 < z < z_1)$ where $z_1 = \frac{x_1 - 40}{10}$

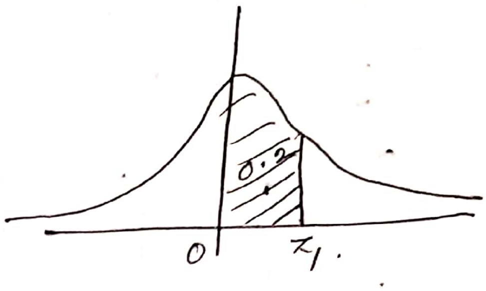
$\Rightarrow P(0 < z < z_1) = -(0.5 - 0.7) = +0.2$

from Normal tables, $P(0 < z < 0.53) = 0.2 \Rightarrow \frac{x_1 - 40}{10} = .53$

$\Rightarrow x_1 = 45.3$ or $x_1 = 45\%$.



$z_1 > 0$.



iii) No of students who scored $\geq 60\%$.

$= 500 \times P(X \geq 60) = 500 \times P\left(\frac{X-\mu}{\sigma} \geq \frac{60-40}{10}\right) = 500 \times P(Z \geq 2)$

$= 500 \times (0.5 - P(0 < Z < 2)) = 500 \times (0.5 - 0.4772)$

$= 500 \times 0.0228 = 11.4 \approx \underline{11}$

3(a) Given

x_i	0	1
$f(x_i)$	0.7	0.3

y_i	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Find the Joint Probability distribution of X & Y. If X & Y are independent random Variables. Also find $E(X)$, $E(Y)$ & $Cov(X,Y)$

Soln:- Since X & Y are independent random Variables

$J_{ij} = f(x_i)\, g(y_j)$

$J_{11} = f(x_1)\, g(y_1) = 0.7 \times 0.3 = 0.21$

$J_{12} = f(x_1)\, g(y_2) = 0.7 \times 0.5 = 0.35$

$J_{13} = f(x_1)\, g(y_3) = 0.7 \times 0.2 = 0.14$

$J_{21} = f(x_2)\, g(y_1) = 0.3 \times 0.3 = 0.09$

$J_{22} = f(x_2)\, g(y_2) = 0.3 \times 0.5 = 0.15$

$J_{23} = f(x_2)\, g(y_3) = 0.3 \times 0.2 = 0.06$

So the Joint Probability Distribution is

X \ Y	−2	5	8	f(x)
0	0.21	0.35	0.14	0.7
1	0.09	0.15	0.06	0.3
g(y)	0.3	0.5	0.2	1

$E(X) = \sum x f(x) = (0 \times 0.7) + 1(0.3) = 0 + 0.3 = 0.3$

$E(Y) = \sum y\, g(y) = (-2 \times 0.3) + (5 \times 0.5) + (8 \times 0.2)$

$= -0.6 + 2.5 + 1.6 = 3.5$

$E(XY) = \sum_{i,j} x_i y_j J_{ij} = (0 \times -2 \times 0.21) + (0 \times 5 \times 0.35) + (0 \times 8 \times 0.14)$
$+ (1 \times -2 \times 0.09) + (1 \times 5 \times 0.15) + (1 \times 8 \times 0.06)$

$= 0 + 0 + 0 + (-0.18) + 0.75 + 0.48$

$= 1.05$

$Cov(X, Y) = E(XY) - E(X)E(Y) = 1.05 - (3.5)(0.3) = 0$

9(a). Sample 1

$\bar{x}_1 = $ Rs 210

$s_1 = $ Rs 10

$n_1 = 100$

Sample 2

$\bar{x}_2 = $ Rs 220.

$s_2 = $ Rs 12

$n_2 = 150$.

Step 1:- Null hypothesis H_0: There is no significant difference between the average incomes.

Alternative hypothesis H_1:- There is significant difference between the average incomes

Step2:- Level of significance $\alpha = 0.05$

Step3:- Criterion to Reject Ho:

Reject Ho if $|Z| > Z_{\frac{\alpha}{2}}$ ie $|Z| > Z_{0.025}$ ie $|Z| > 1.96$.

Step4:- calculation:

$$|Z| = \frac{|\bar{x}_1 - \bar{x}_2|}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \left|\frac{210-220}{\sqrt{\frac{100}{100} + \frac{144}{150}}}\right| = \left|\frac{10}{\sqrt{1+0.96}}\right| = \left|\frac{10}{\sqrt{1.96}}\right|$$

$$= \frac{10}{1.4} = 7.142$$

Step5:- Clearly $|Z| > 1.96$

So we reject Ho.

ie There is significant difference between the incomes

9b Given No of patients = 12 ($n < 30$), Small Sample

Changes in blood pressure: 5, 2, 8, -1, 3, 0, 6, -2, 1, 5, 0, 4

Step1:- Null hypothesis Ho: The stimulus does not increase the blood pressure

Alternative hypothesis:- The Stimulus will increase the blood pressure

Step 2: Level of significance = α = 0.05 (given)

Step3:- Criterion to reject Ho:- Reject Ho if $t > t_\alpha$ for 11 degrees of freedom.

ie Reject Ho if $t > 2.201$

Step 4: Calculation: $t = \frac{\bar{x}-\mu}{s/\sqrt{n}}$.

Given $\mu = 0$ (no increase in blood pressure)

$$\bar{x} = \frac{5+2+8+(-1)+3+0+6+(-2)+1+5+0+4}{12} = \frac{31}{12} = 2.5833$$

$$s = \sqrt{\frac{\sum_{i=1}^{n}(\bar{x}-x_i)^2}{n-1}} = \sqrt{\frac{(2.5833-5)^2+(2.5833-2)^2+\cdots+(2.5833-4)^2}{12-1}}$$

$$= \sqrt{9.538} = 3.088$$

$\therefore s = 3.088$; $n = 12$

$$\therefore t = \frac{2.5833-0}{\left(3.088/\sqrt{12}\right)} = 2.8979$$

Step 5:- Conclusion/Decision:-

Clearly $2.8979 > 2.201$ $(t > t_\alpha)$

So reject H_0 ie Accept H_1

$\therefore$ So the stimulus increases the blood pressure

9c Stochastic Matrix: A square matrix P is said to be a regular stochastic matrix if all the entries of some power P^n are positive

Ex:- If $A = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$. none of the entries in A^2 is zero

$\therefore$ A is a regular stochastic matrix of order 2

To find the unique fixed probability vector of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$.

ie to find $v = (x, y, z) \ni x+y+z=1$ & $vA = v$.

$\Rightarrow z = 1-x-y$

$$\Rightarrow [x \; y \; z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x \; y \; z]$$

$$\Rightarrow \left[\frac{y}{6}, \; x+\frac{y}{2}+\frac{2z}{3}, \; \frac{y}{3}+\frac{z}{3}\right] = [x \; y \; z]$$

$$\Rightarrow \frac{y}{6} = x \; ; \quad x+\frac{y}{2}+\frac{2z}{3} = y ; \quad \frac{y}{3}+\frac{z}{3} = z.$$

$$\Rightarrow y = 6x ; \quad 6x+3y+4z = 6y ; \quad y-2z=0.$$

Using $y=6x$ & $y=2z$ in $6x-3y+4z=0$, we get

$$6x - 3(6x) + 4(1-x-6x) = 0 \Rightarrow \boxed{x = 1/10}.$$

$$y = 6x = 6/10 ; \quad z = 1-x-y = 1-\frac{1}{10}-\frac{6}{10} = 3/10$$

∴ The unique fixed probability vector v is

$$v = (1/10, \; 6/10, \; 3/10).$$

CMR

10(a). **Type I error:-** It is an error that occurs during the hypothesis testing process a null hypothesis Ho is rejected though it is accurate and should not have been rejected.

Type II error:- It is an error that occurs during the hypothesis testing process a null hypothesis Ho is accepted though it is false and should have been rejected.

Level of significance:- It is the fixed probability of wrong elimination of null hypothesis Ho when it (Ho) is true. It is the probability of committing an error of type I.

Confidence limits:-

Confidence limits for the mean are an internal estimate for the mean.

For large sample, these limits are $\bar{x} \pm Z_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$ $Z_{\frac{\alpha}{2}} = 2.58$ for $\alpha = 1\%$, $= 1.96$ for $\alpha = 5\%$.

For small sample these limits are $\bar{x} \pm t_{\alpha} \frac{s}{\sqrt{n}}$.

Null hypothesis: It is a conjecture used in Statistics which is to be tested for rejection assuming that it is true. It is denoted by Ho.

10 b. Marks of 11 students in a maths test before and after coaching.

Marks (Before coaching) Test I	23	20	21	18	18	20	18	17	23	16	19
Marks (After coaching) Test II	24	19	18	20	20	22	20	20	23	20	17

To test whether students are benefited or not?

Step 1:- Null hypothesis H_0: Students are not benefited from Extra coaching ($\bar{x}_1 = \bar{x}_2$)

Alternative hypothesis H_1:- Students got benefited from Extra coaching

Step 2:- Level of significance $\alpha = 0.05$

Step 3:- Criterion to Reject H_0:- Reject H_0 if $|t| > t_\alpha$

ie Reject H_0 if $t > 2.228$ (given)

Step 4:- Calculation:

$$\bar{x}_1 = \frac{23+20+21+18+18+20+18+17+23+16+19}{11} = \frac{213}{11} = 19.36$$

$$\bar{x}_2 = \frac{24+19+18+20+20+22+20+20+23+20+17}{11} = \frac{223}{11} = 20.27$$

$s_1^2 = \frac{1}{n}\sum(x_1 - \bar{x}_1)^2$ =

$$s^2 = \frac{\Sigma(x_i - \bar{x}_1)^2 + \Sigma(x_i - \bar{x}_2)^2}{n_1 + n_2 - 2} = \frac{52.5456 + 42.1819}{20} = \frac{94.7275}{20}$$

$= 4.7364$

$\Rightarrow s = 2.1763$.

$$\therefore |t| = \left|\frac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right| = \left|\frac{19.36 - 20.27}{2.1763\sqrt{\frac{1}{11} + \frac{1}{11}}}\right| \quad \frac{0.91}{2.1763(\sqrt{0.18182})}$$

$$= \frac{0.91}{0.928} = 0.00098.$$

Step 5: Conclusion/Decision:

$\because t \ngtr t_\alpha$ we can't reject H_0

So there is a significant in the maths marks after extra coaching

10 (c). A always throws the ball to B.
B always throws the ball to C
C can throw the ball to A or B.
Now, C is the first person to throw the ball
To find probabilities after 3 throws .. i) A has the ball
ii) B has the ball
iii) C has the ball.

Since C is throwing the ball initially, the initial probability vector is $p^{(0)} = (0, 0, 1)$.

and the transition probability matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow P^2 = P \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Likewise $P^3 = P \cdot P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$; $P_4 = P \cdot P^3 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

$$P^5 = P \cdot P^4 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} \end{bmatrix}$$

$\because$ all entries of P^5 are non-zero

P is a regular stochastic matrix

But we want to find probabilities after 3 throws

so $p^{(3)} = p^{(0)} P^3 = (0 \; 0 \; 1) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{4} \; \frac{1}{4} \; \frac{1}{2} \right)$

After three throws,

$\therefore$ The Probability that A has the ball $= \frac{1}{4}$

The Probability that B has the ball $= \frac{1}{4}$

The probability that C has the ball $= \frac{1}{2}$