CBCS SCHEME

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18MAT41

Fourth Semester B.E. Degree Examination, Jan./Feb. 2021
Complex Analysis, Probability and Statistical Methods

Time: 3 hrs. Max. Marks: 100

Note: *Answer any FIVE full questions, choosing ONE full question from each module.*

Module-1

1 a. Shat that $W = \log Z, Z \neq 0$ is analytic and hence find $\frac{dw}{dz}$. **(06 Marks)**

b. Derive Cauchy – Riemann equation in Cartesian coordinates. **(07 Marks)**

c. Find the analytic function $f(z) = u + iv$ given $v = e^{-x}[x\cos y + y\sin y]$. **(07 Marks)**

OR

2 a. Show that an analytic function with constant modulus is constant. **(06 Marks)**

b. If $f(z) = u + iv$ is analytic prove that

$$\left[\frac{\partial|f(z)|}{\partial x}\right]^2 + \left[\frac{\partial|f(z)|}{\partial y}\right]^2 = |f'(z)|^2.$$ **(07 Marks)**

c. If $u - v = (x-y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is analytic function, if $z = x + iy$, find f(z) in terms of z. **(07 Marks)**

Module-2

3 a. State and prove Cauchy's Integral formula. **(06 Marks)**

b. Discuss the transformation $W = e^z$. **(07 Marks)**

c. Find the Bilinear transformation which sends points $Z = 0, 1, \infty$ into the points $W = -5, -1, 3$ respectively. What are the invariant points in this transformation? **(07 Marks)**

OR

4 a. Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the line i) $y = x$ ii) $y = x^2$. **(06 Marks)**

b. Evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}dz$, where C is the circle $|z| = 3$. **(07 Marks)**

c. Find the Bilinear transformation that maps the points $Z = -1, i, 1$ onto the points $W = 1, i, -1$. Also find the invariant points. **(07 Marks)**

Module-3

5 a. A random variable X has the following probability function :

x	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

Find i) the value of k ii) $P(x < 6)$ iii) $P(x \geq 6)$ iv) $P(3 < x \leq 6)$. **(06 Marks)**

b. The probability that a pen manufactured by a company be defective is $\frac{1}{10}$. If 12 such pens are manufactured, what is the probability that i) Exactly 2 are defective ii) at least 2 are defective iii) none of them are defective. **(07 Marks)**

c. A sample of 100 battery cells tested to find the length of life produced the following results $\bar{x} = 12$ hours, $\sigma = 3$ hours. Assuming the data to be normally distributed what percentage of battery cells are expected to have life. i) more than 15 hours ii) less than 6 hours iii) between 10 and 14 hours. $[A(1) = 0.3413, A(2) = 0.4772, A(0.67) = 0.2487]$. **(07 Marks)**

OR

6 a. The probability density function of a random variable (y = x) is

$$f(x) = \begin{cases} k\,x^2 & -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$. Evaluate K and find

i) $P(1 \le x \le 2)$ ii) $P(x \le 2)$ iii) $P(x > 1)$. **(06 Marks)**

b. A certain screw making machine produces on average two defective out of 100 and packs them in boxes of 500. Find the probability that the box contains i) three defective ii) At least one defective iii) between 2 and 4 defective. **(07 Marks)**

c. The length of a telephone conversation is an exponential variate with mean 3 minute. Find the probability that a call i) ends in less than 3 minutes ii) between 3 and 5 minutes. **(07 Marks)**

Module-4

7 a. Ten students get the following percentage of marks in two subjects A and B.

Marks in A	78	36	98	25	75	82	90	62	65	39
Marks in B	84	51	91	60	68	62	86	58	53	47

Calculate the rank correlation coefficient. **(06 Marks)**

b. Fit a best fitting parabola $Y = a + bx + cx^2$ for the following data :

x	-2	-1	0	1	2
y	-3.150	-1.390	0.620	2.880	5.378

(07 Marks)

c. The regression lines $4x - 5y + 33 = 0$ and $20x - 9y = 107$. Find i) the mean values of x and y ii) the correlation between x and y iii) the variance of Y given that the variance of X is 9. **(07 Marks)**

OR

8 a. If θ is the acute angle between the lines of regression then show that

$$\tan\theta = \frac{\sigma_x \ \sigma_y}{\sigma_x^2 + \sigma_y^2}\left(\frac{r^2 - 1}{r}\right)$$. Explain the significance when r = 0 and r = ± 1. **(06 Marks)**

b. Find the coefficient of correlation and the equation of regression lines for the following data

x	1	2	3	4	5	6	7	8	9
y	12	11	13	15	14	17	16	19	18

(07 Marks)

c. Fit a best fitting curve in the form $y = ax^b$ for the following data :

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

(07 Marks)

Module-5

9 a. The joint probability distribution for two random variables X and Y is as given below :

X \ Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Find the marginal distribution of X and Y. Also find the Covariance of X and Y. **(06 Marks)**

b. In 324 throws of a six faced die an odd number turned up 181 times. It is reasonable to think that the die is an unbiased at 0.01 level of significance. **(07 Marks)**

c. The nine item of a sample have the following values 45 , 47 , 50 , 52, 48, 47, 49, 53, 51. Does the mean of these differ significantly from the assumed mean of 47.5? **(07 Marks)**

OR

a. Define the following terms :
i) Null hypothesis   ii) Type – I and Type II error   iii) Level of significance. **(06 Marks)**

b. Ten individuals are choosen at random from a population and their heights in inches are found to be 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. Test the hypothesis that the mean height of the universe is 66 inches. (Given $t_{0.05} = 2.262$ for gdf). **(07 Marks)**

c. The theory predicts the proportion of beans in the four groups G1, G2, G3, G4 should be in the ratio 9:3:3:1. In an experiment with 1600 beans the numbers in the four group were 882 , 313 , 287 and 118. Does the experimental result support the theory (at 5% level of significance for 3)? **(07 Marks)**

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CMR

18MAT41, VTU QP, JAN/FEB, 2021

COMPLEX ANALYSIS, PROBABILITY & STATISTICAL METHODS

SOLUTIONS

MODULE-I

1 (a) Show that $w = \log z,\ z \neq 0$ is analytic and hence find $\frac{dw}{dz}$.

Soln :- Given $w = \log z$, $\Rightarrow w = \log re^{i\theta}$ $\Rightarrow u + iv = \log r + \log e^{i\theta}$

$\Rightarrow u = \log r$ & $iv = i\theta$ $\Rightarrow v = \theta$ ($\because$ $w = u + iv$ & $z = re^{i\theta}$)

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r} \qquad \frac{\partial v}{\partial r} = 0$$

$$\frac{\partial u}{\partial \theta} = 0 \qquad \frac{\partial v}{\partial \theta} = 1.$$

Clearly the CR eqns $r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$ & $r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}$ are satisfied $\Rightarrow$ $W = \log z$ is analytic

Also we know, $f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r} + i \cdot 0\right) = \frac{1}{re^{i\theta}}$

$$\Rightarrow f'(z) = \frac{1}{z}.$$

(b) CR eqns in cartesian form.

Let $f(z)$ be analytic at a point $z = x + iy$

$\Rightarrow f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and is unique

we know $f(z) = u(x,y) + iv(x,y) \Rightarrow f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$

where $\delta z = \delta x + i\delta y$.

$\therefore f'(z) = \underset{\delta z \to 0}{lt} \dfrac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta x + i\delta y}$

$\Rightarrow f'(z) = \underset{\delta z \to 0}{lt} \dfrac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y} + i \underset{\delta z \to 0}{lt} \dfrac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y} \longrightarrow (A)$

$\because \delta z \to 0$ we have $\delta y = 0$ & $\delta x \to 0$

$\delta y \to 0$ & $\delta x = 0$.

Case i :- Let $\delta y = 0$ & $\delta x \to 0$ in (A),

Then (A) $\Rightarrow f'(z) = \underset{\delta x \to 0}{lt} \dfrac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \underset{\delta x \to 0}{lt} \dfrac{v(x+\delta x, y) - v(x,y)}{\delta x}$

$\Rightarrow f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} \longrightarrow (1)$.

Case(ii): Let $\delta x = 0$ & $\delta y \to 0$ in (A)

Then (A) $\Rightarrow f'(z) = \underset{\delta y \to 0}{lt} \dfrac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \underset{\delta y \to 0}{lt} \dfrac{v(x, y+\delta y) - v(x,y)}{i\delta y}$

$\Rightarrow f'(z) = \dfrac{1}{i}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y} \Rightarrow f'(z) = \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y} \longrightarrow (2)$

(1) & (2) $\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

These are called Cauchy-Riemann equations

1 (c). To find the analytic fn $f(z) = u + iv$ given $v = e^{-x}(x\cos y + y\sin y)$

Soln: $v(x,y) = e^{-x}(x\cos y + y\sin y)$

$$\Rightarrow \frac{\partial v}{\partial x} = e^{-x}[\cos y + 0] + (x\cos y + y\sin y)(-e^{-x})$$

$$= e^{-x}[\cos y - x\cos y - y\sin y] = e^{-x}[(1-x)\cos y - y\sin y]$$

$$\& \frac{\partial v}{\partial y} = e^{-x}[x(-\sin y) + y\cos y + \sin y] = e^{-x}[(1-x)\sin y + y\cos y]$$

Now we know $f'(z) = u_x + iv_x = v_y + iv_x$

$$= e^{-x}[(1-x)\sin y + y\cos y]$$
$$+ i[(1-x)\cos y - y\sin y]e^{-x}$$

By Milne-Thompson method take $x = z$ & $y = 0$ to get

$$f'(z) = e^{-z}[(1-z)\sin 0 + 0\cos 0] + ie^{-z}[(1-z)\cos 0 - 0\sin 0]$$

$$= e^{-z}[0] + ie^{-z}[(1-z) - 0] \Rightarrow f'(z) = ie^{-z}(1-z)$$

$$\Rightarrow f(z) = \int ie^{-z}(1-z)\,dz = i\left[\int e^{-z}dz - \int ze^{-z}dz\right]$$

$$= i\left[\frac{e^{-z}}{-1} - \left\{z\cdot\left(\frac{e^{-z}}{-1}\right) - (1)\left(\frac{e^{-z}}{(-1)^2}\right)\right\}\right] + c$$

$$= i\left[-e^{-z} + ze^{-z} - e^{-z}\right] + c = i\left[e^{-z}(z-2)\right] + c.$$

$$\therefore f(z) = i\left[e^{-z}(z-2)\right] + c$$

2 (a) Let $f(z) = u + iv$ be analytic function

and Let $|f(z)| = \sqrt{u^2+v^2} = $ constant.

$$\Rightarrow u^2 + v^2 = \text{Constant}$$

$$\Rightarrow 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad (\text{differentiating wrt } x \text{ partially}$$

$\Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \rightarrow (1)$

Similarly $u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$ (differentiating $u^2+v^2 = \text{const}$, partially w.r.t y) $\rightarrow (2)$

$\because f(z) = u+iv$ is analytic, u & v satisfy CR eqns $\Rightarrow u_x = v_y$ and $u_y = -v_x$

so (2) becomes $u\left(-\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial u}{\partial x}\right) = 0 \rightarrow (3)$

(1) & (3) $\Rightarrow$

$u u_x + v v_x = 0 \quad \times u$

$u_x v - u v_x = 0 \quad \times v$

$\Rightarrow u^2 u_x + u v \cancel{v_x} = 0$

$\Rightarrow v^2 u_x - \cancel{u} v v_x = 0$

$\Rightarrow u_x (u^2+v^2) = 0$

$\Rightarrow u_x = 0 \quad (\because u^2+v^2 = \text{const})$

$\Rightarrow$ ~~u is a cons~~

IIIly we get $v_x = 0$

$\Rightarrow u_y = 0$

$\Rightarrow u_x = 0$ & $u_y = 0$. Like wise we get $v_x = 0$ & $v_y = 0$.

$\Rightarrow u$ is Constant, K_1 (say) $\quad\Rightarrow v$ is Const, K_2 (say)

$\therefore f(z) = u+iv =$ constant $\quad (\because K_1 + iK_2$ is constant).

2.(b) Let $f(z) = u+iv$ be analytic & let $|f(z)| = \sqrt{u^2+v^2} = \phi \Rightarrow \phi^2 = u^2+v^2$

$\Rightarrow u$ & v satisfy CR eqns

$\Rightarrow 2\phi\phi_x = 2uu_x + 2vv_x \Rightarrow \phi\phi_x = uu_x + vv_x \rightarrow (1)$. IIIly $\phi\phi_y = uu_y + vv_y \rightarrow (2)$

squaring & adding (1) & (2), $\phi^2(\phi_x^2 + \phi_y^2) = u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y$

$\Rightarrow \phi^2(\phi_x^2+\phi_y^2) = u^2(u_x^2 + (-v_x)^2) + v^2(v_x^2 + u_x^2) + 2u\cancel{vu_x}v_x + 2uv(-v_x)u_x$

$\Rightarrow \phi^2(\phi_x^2+\phi_y^2) = (u_x^2+v_x^2)(u^2+v^2) \Rightarrow \phi_x^2 + \phi_y^2 = u_x^2 + v_x^2$

$\Rightarrow \left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = |f'(z)|^2 \quad (\because f'(z) = u_x + iv_x)$

2.c. If $u-v=(x-y)(x^2+4xy+y^2)$ and $f(z)=u+iv$ is analytic function of $z=x+iy$ find $f(z)$ interms of z.

<u>Soln</u>

$$u-v = x^3+3x^2y-3xy^2-y^3$$

$$u_x - v_x = 3x^2+6xy-3y^2 \quad \text{——} \ (1)$$

$$u_y - v_y = 3x^2-6xy-3y^2$$

But $u_y = -v_x$ and $v_y = u_x$ by CR Equations

$$-v_x - u_x = 3x^2-6xy-3y^2 \quad \text{——} \ (2)$$

(1) + (2) $\quad -2v_x = 6(x^2-y^2)$

or $v_x = 3(y^2-x^2)$

(1) − (2) $\quad 2u_x = 12xy \Rightarrow u_x = 6xy$

$$f'(z) = u_x + i v_x$$

$$f'(z) = 6xy + i\,3(y^2-x^2)$$

put $x=z$ and $y=0$ we get $f'(z) = -3iz^2$

$$f(z) = \int -3iz^2\,dz + c \Rightarrow \boxed{f(z) = -iz^3+c}$$

30. State and prove Cauchy's integral formula

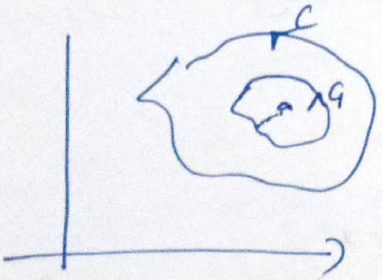
Statement: If $f(z)$ is analytic inside and on a simple closed curve C and if 'a' is any point within C then
$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Proof: Since 'a' is any point within C we shall enclose it by a circle C_1 with $z=a$ as centre and r as radius such that C_1 lies entirely within C.

The function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the annular region b/w C and C_1.

Now as a consequence of Cauchy's thm

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$$



The equation of C_1 circle with centre a and radius r can be written

$|z-a|=r$ $\quad z-a = re^{i\theta}$ or $z = a + re^{i\theta}$ $\quad 0 \le \theta \le 2\pi$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

This is true for any $r>0$ however small, hence $r\to 0$, we get

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a)\,d\theta = i\left[f(a)\right]_0^{2\pi} = 2\pi i f(a)$$

$$\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz}$$

3.b. Discuss the transformation $w = e^z$

Proof: $w = e^z$

$u + iv = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$u = e^x \cos y, \quad v = e^x \sin y$ ——— (1)

We shall find the image in the w-plane corresponding to the straight lines parallel to the Co-ordinate axes in the z-plane. That is x = constant and y = constant.

let us Eliminate x and y Separately from (1)

Squaring and adding we get

$u^2 + v^2 = e^{2x}$ ——— (2)

also by dividing we get $\frac{v}{u} = \tan y$

Case (1): Let $x = c_1$ where c_1 is constant

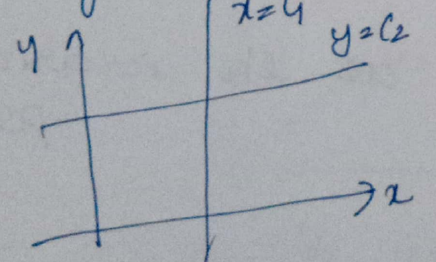
Eq (2) becomes $u^2 + v^2 = e^{2c_1}$ = constant $= r^2$ (say)

This represents a circle with center origin and radius r in the w-plane.

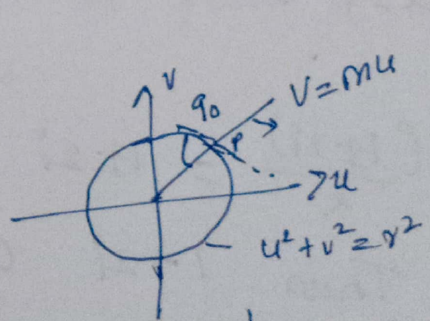
Case 2: let $y = c_2$ where c_2 is a constant

Eq (3) becomes $\frac{v}{u} = \tan c_2 = m$ (say) $\therefore v = mu$

This represents a straight line passing through the origin in the w-plane.



z-plane



w-plane

c. Find the Bilinear transformation which sends points $z = 0, 1, \infty$ into the points $w = -5, -1, 3$ respectively. What are the invariant points in this transformation?

Sol: Here $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ and so that $1/z_3 = 0$

and $w_1 = -5$, $w_2 = -1$, $w_3 = 3$

The reqd transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(z-z_1)(z_2/z_3-1)}{(z/z_3-1)(z_2-z_1)}$$

$$\frac{(w+5)(-4)}{(w-3)(4)} = \frac{(z-0)(-1)}{(-1)(1-0)} \quad \text{or} \quad \frac{-w+5}{w-3} = z$$

$$\text{or} \quad w+5 = -z(w-3)$$

$$w(1+z) = 3z+5 \quad \text{or} \quad w = \frac{3z-5}{z+1}$$

The invariant points are got by setting $w = z$ in the transformation. Thus, for the invariant points, we have

$$z = \frac{3z-5}{z+1} \Rightarrow z^2 + z = 3z - 5 \Rightarrow z^2 - 2z + 5 = 0$$

$$a = 1 \quad b = -2 \quad c = 5 \qquad z = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$

$$z = \frac{2(1 \pm 2i)}{2} = 1 \pm 2i$$

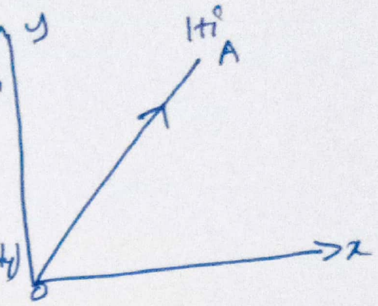
Thus $1+2i$ and $1-2i$ are the invariant points

4.a. Evaluate $\int_0^{1+i}(x^2-iy)\,dz$ along the line

(i) $y=x$ (ii) $y=x^2$

Soln: The parametric Equations of the given straight line $y=x$ are $x=t$, $y=t$ so that $z=x+iy=t+it$

As z-varies from 0 to $1+i$, the parameter t increases from 0 to 1. The given straight line is denoted by OA



$\therefore$ along the given line, the given integral is

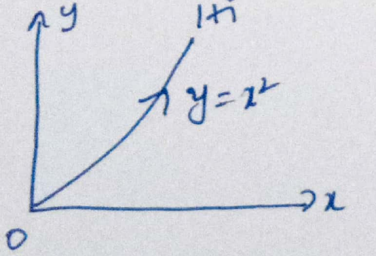
$$I=\int_{z=0}^{1+i}(x^2-iy)\,dz=\int_{z=0}^{z=1+i}(x^2-iy)(dx+idy)$$

$$=\int_{t=0}^{1}(t^2-it)(dt+idt)=(1+i)\int_0^1(t^2-it)\,dt=(1+i)\left[\int_0^1 t^2dt-i\int_0^1 t\,dt\right]$$

$$=(1+i)\left[\frac{1}{3}-\frac{i}{2}\right]=\frac{1}{6}(1+i)(2-3i)=\frac{1}{6}(5-i)$$

(ii) The parametric Equation of the given parabola $y=x^2$ are $x=t$, $y=t^2$ so that $x+iy=t(1+it)$

As z varies from 0 to $1+i$, the parameter t increases from 0 to 1. The path of integration is shown in fig 3.2



$\therefore$ along the given parabola, the given integral

$$I=\int_{z=0}^{1+i}(x^2-iy)\,dz=\int_{z=0}^{1+i}(x^2-iy)(dx+idy)$$

$$\int_{t=0}^{1}(t^2-it^2)(dt+2it\,dt)=(1-i)\int_0^1 t^2(1+2it)\,dt$$

$$(1-i)\left\{\int_1^1 t^2\,dt + 2i\int_0^1 t^3\,dt\right] = (1-i)\left[\frac{1}{3}+\frac{2i}{4}\right]$$

$$= \frac{1}{6}(1-i)(2+3i) = \frac{1}{6}(5+i)$$

4 b) Evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2 (z-2)} dz$, where C is a circle $|z| = 3$ (7M)

Soln:- Consider $\frac{1}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$

$$\Rightarrow 1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2 \quad \text{---} \; (1)$$

Put $z = 1$ in (1)

$$1 = B(-1) \Rightarrow B = -1$$

Put $z = 2$ in (1)

$$1 = C(1) \Rightarrow C = 1$$

Put $z = 0$ in (1),

$$1 = A(2) - (-2) + 1$$

$$1 = 2A + 2 + 1$$

$$\Rightarrow A = -1$$

$$\therefore \frac{1}{(z-2)(z-1)^2} = \frac{-1}{z-1} + \frac{-1}{(z-1)^2} + \frac{+1}{(z-2)}$$

$$f(z) = \sin\pi z^2 + \cos\pi z^2$$

$$\therefore \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2 (z-2)} dz = -\int_C \frac{f(z)}{z-1} dz - \int_C \frac{f(z)}{(z-1)^2} dz + \int_C \frac{f(z)}{z-2} dz.$$

Using Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\,dz = +2\pi i - 2\pi i\, f'(1) + 2\pi i\, f(2)$$

$$= 2\pi i + 4\pi^2 i + 2\pi i$$

$$= 4\pi i + 4\pi^2 i$$

$$= 4\pi i\,(1+\pi).$$

c) Find the bilinear transformation that maps the points $z=-1, i, 1$ onto the points $w=1, i, -1$. Also find the invariant points.

Soln:- Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

For $z=-1$, $w=1$, $\quad 1 = \dfrac{-a+b}{-c+d}$

$$-c+d = -a+b$$

$$-a+b+c-d = 0 \quad \text{——— (1).}$$

For $z=i$, $w=i$, $\quad i = \dfrac{ai+b}{ci+d}$

$$-c+di = ai+b$$

$$ai+b+c-di = 0 \quad \text{——— (2)}$$

For $z=1$, $w=-1$, $-1 = \dfrac{a+b}{c+d}$

$$-c-d = a+b$$

$$a+b+c+d = 0 \quad \text{——— (3)}$$

(1) + (3) $\Rightarrow$ $2b+2c=0$

$$b+c=0 \Rightarrow b=-c \quad \text{——— (4)}$$

(3) $\times i \Rightarrow$ $ai+bi+ci+\cancel{di}=0$

$+$ (2) $\quad ai+b+c-\cancel{di}=0$

$$2ia+b(1+i)+c(1+i)=0. \quad \text{——— (5)}$$

Solving (4) & (5),

$$0.a+b+c=0$$

$$2ia+b(1+i)+c(1+i)=0$$

$$\frac{a}{\begin{vmatrix} 1 & 1 \\ 1+i & 1+i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & 1 \\ 2i & 1+i \end{vmatrix}} = \frac{c}{\begin{vmatrix} 0 & 1 \\ 2i & 1+i \end{vmatrix}} = k$$

$$\frac{a}{0} = \frac{-b}{-2i} = \frac{c}{-2i} = k$$

$$a=0, \quad b=2ik, \quad c=2ik$$

Substituting in (3)

$$0+2ik+2ik+d=0$$

$$\Rightarrow d=-4ik$$

$\therefore \; w = \dfrac{2ik}{2ikz - 4ik}$

$w = \dfrac{i}{i(z-2)} = \dfrac{1}{z-2}$

$w = \dfrac{1}{z-2}$ is the required Bilinear transformation.

To find the invariant points,

$z = \dfrac{1}{z-2}$

$z^2 - 2z - 1 = 0$

$z = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$

$= \dfrac{2 \pm \sqrt{4+4}}{2}$

$z = 1 \pm \sqrt{2}$

$\therefore \; z = 1+\sqrt{2},\; 1-\sqrt{2}$ are the invariant points.

5) a) A random variable X has the following probability function :

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) value of k (ii) $P(x<6)$ (iii) $P(x \geq 6)$ (iv) $P(3<x\leq 6)$

Soln: We must have $P(x) \geq 0$ and $\sum P(x) = 1$

The first condition is satisfied for $k \geq 0$ and we have to find k such that $\sum P(x) = 1$.

i.e., $0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$$10k^2 + 9k - 1 = 0 \implies k = \frac{1}{10} \text{ and } k = -1.$$

For $k = -1$, the first condition fails.

$$\therefore k = \frac{1}{10}.$$

Hence

x	0	1	2	3	4	5	6	7
$p(x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{17}{100}$

Now (i) $P(x < 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$$= 0 + \frac{1}{10} + \frac{1}{5} + \frac{1}{5} + \frac{3}{10} + \frac{1}{100}$$

$$= \frac{81}{100} = 0.81$$

(ii) $P(x \geq 6) = P(6) + P(7)$

$$= \frac{1}{50} + \frac{17}{100}$$

$$= \frac{19}{100} = 0.19$$

(iii) $P(3 < x \leq 6) = P(4) + P(5) + P(6)$

$$= \frac{3}{10} + \frac{1}{100} + \frac{1}{50}$$

$$= \frac{33}{100}$$

$$= 0.33$$

b) The probability that a pen manufactured by a company be defective is $\frac{1}{10}$. If 12 such pens are manufactured what is the probability that (i) exactly 2 are defective (ii) atleast 2 are defective (iii) none of them are defective.

Soln: Probability of a defective pen is $p = \frac{1}{10} = 0.1$

Probability of a nondefective pen $q = 1-p = 1-0.1 = 0.9$

We have $P(x) = {}^nC_x \, p^x q^{n-x}$ where $n = 12$.

(i) Probability that exactly 2 are defective

$$P(x=2) = {}^{12}C_2 \,(0.1)^2 (0.9)^{10}$$

$$= 0.2301$$

(ii) Probability that atleast 2 defectives

$$1 - [P(x=0) + P(x=1)]$$

$$= 1 - \left[{}^{12}C_0 (0.1)^0 (0.9)^{12} + {}^{12}C_1 (0.1)^1 (0.9)^{11}\right]$$

$$= 0.341$$

(iii) Probability that no defectives

$$P(x=0) = {}^{12}C_0 \,(0.1)^0 (0.9)^{12}$$

$$= 0.2824.$$

c) A sample of 100 battery cells tested to find the length of life produced the following results $\bar{x} = 12$ hours, $\sigma = 3$ hours. Assuming the data to be normally distributed what percentage of battery cells are expected to have life.
(i) more than 15 hours (ii) less than 6 hours (iii) between 10 and 14 hours $\left[A(1) = 0.3413,\ A(2) = 0.4772,\ A(0.67) = 0.2487\right]$.

Soln:- $\mu = 12$ hours, $\sigma = 3$ hours,

$$z = \frac{x - \mu}{\sigma}$$

(i) For $x = 15$, $z = \frac{15-12}{3} = 1$

(ii) For $x = 6$, $z = \frac{6-12}{3} = -2$

(iii) For $x = 10$, $z = -0.67$

$x = 14$, $z = 0.67$

$P(z > 1)$ = area under the standard normal curve to the right of $z = 1$

$= 0.5 - 0.3413 = 0.1587$

$P(z < -2)$ = area under the standard normal curve to the left of $z = -2$

$= 0.5 - 0.4772 = 0.0228$

$P(-0.67 < z < 0.67) = 2 \times 0.2486$

$= 0.4972$.

15.87% batteries have life more 15 hours &
2.28% batteries have life less than 6 hours

6(a) $P(x)$ is pdf if $P(x) \geq 0$ & $\int_{-\infty}^{\infty} P(x)\,dx = 1$

we note that $P(x) \geq 0$ if $k > 0$

$$\int_{-3}^{3} kx^2\,dx = 1 \Rightarrow \left[\frac{kx^3}{3}\right]_{x=-3}^{3} = 1$$

$$\frac{k}{3}\left[3^3 + 3^3\right] = 1$$

$$k\,[9+9] = 1 \Rightarrow \boxed{k = \frac{1}{18}}$$

(i) $P(1 \leq x \leq 2) = \int_1^2 \frac{x^2}{18}\,dx = \left[\frac{x^3}{54}\right]_1^2 = \frac{1}{54}(8-1) = \frac{7}{54}$

(ii) $P(x \leq 2) = \int_{-3}^{2} \frac{1}{18}x^2\,dx$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{2} = \frac{1}{54}\left[8+27\right] = \frac{35}{54}$$

(iii) $P(x > 1) = \int_1^3 \frac{1}{18}x^2\,dx$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_1^3 = \frac{1}{54}(27-1) = \frac{13}{27}$$

(b) ~~Given~~ Let x be the defective screw.

$p = \frac{2}{100} = 0.02$ (small) $\quad n = 500$ (large)

$m = np = 10$

we consider poisson distribution.

$$P(x) = \frac{m^x e^{-m}}{x!}$$

$$p(x) = \frac{10^x e^{-10}}{x!} \rightarrow \text{distribution function.}$$

(i) three defectives $p(3) = \frac{10^3 e^{-10}}{3!} = 0.0076$

(ii) At least one defective $P(x \geq 1)$

$P(x \geq 1) = 1 - P(0)$

$= 1 - \frac{10^0 e^{-10}}{0!}$

$= 1 - e^{-10}$

$= 0.999$

(iii) between 2 and 4 defective

$P(2 \leq x \leq 4) = P(2) + P(3) + P(4)$

$= e^{-10}\left[\frac{10^2}{2!} + \frac{10^3}{3!} + \frac{10^4}{4!}\right]$

$= e^{-10}\left[\frac{100}{2} + \frac{1000}{6} + \frac{10000}{24}\right]$

$= 0.0287$

6 (c) Given Mean $\mu = \frac{1}{\alpha} = 3 \Rightarrow \frac{1}{\alpha} = 3 \Rightarrow \boxed{\alpha = \frac{1}{3}}$

Exponential distribution

- $f(x) = \alpha . e^{-\alpha x}, \quad 0 < x < \infty$

$\boxed{f(x) = \frac{1}{3} e^{-x/3}}$

(i) ends in less than 3 minutes

$P(x < 3) = \int_0^3 f(x)\, dx = \int_0^3 \frac{1}{3} e^{-x/3}\, dx$

$= \left[\frac{1}{\cancel{3}} \cdot (-\cancel{3}) e^{-x/3}\right]_0^3$

$= -\left[e^{-x/3}\right]_0^3$

$= -\left[e^{-1} - e^0\right] = 1 - e^{-1} = 1 - \frac{1}{e}$

$\boxed{P(x < 3) = 0.6321}$

(ii) $P(3 < x < 5) = \int_3^5 f(x)\, dx$

$$= \int_3^5 \frac{1}{3} e^{-x/5} dx$$

$$= \left[\frac{1}{3}(-3) e^{-x/5}\right]_3^5 = -\left[e^{-x/5}\right]_3^5$$

$$= -\left[e^{-1} - e^{-3/5}\right] = e^{-3/5} - e^{-1}$$

$$= \frac{1}{e^{3/5}} - \frac{1}{e} = 0.5488 - 0.3678$$

$$= 0.181$$

7(a) We prepare the table consisting of the given data along with the ranks assigned according to their order of magnitude. In the subject x, 98 will be awarded rank 1, 90 as rank 2 & so on.

Mark in x	Rank(x)	Marks in y	Rank(y)	$d = x-y$	$d^2 = (x-y)^2$
78	4	84	3	1	1
36	9	51	9	0	0
98	1	91	1	0	0
25	10	60	6	4	16
75	5	68	4	1	1
82	3	62	5	-2	4
90	2	86	2	0	0
62	7	58	7	0	0
65	6	53	8	-2	4
39	8	47	10	-2	4

$\Sigma d^2 = 30$

Rank correlation $\rho = 1 - \frac{6 \Sigma d^2}{n(n^2-1)}$, Here $n = 10$

$$\rho = 1 - \frac{6(30)}{10(10^2-1)} = 0.81818 \approx 0.82$$

$\therefore \boxed{\rho = 0.82}$

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(b) $y = a + bx + cx^2$ Second degree parabola

Associated normal equations are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

$(n=5)$

The relevant table is as follows

x	y	xy	x^2y	x^2	x^3	x^4
-2	-3.15	6.30	-12.60	4	-8	16
-1	-1.39	1.39	-1.39	1	-1	1
0	0.62	0	0	0	0	0
1	2.88	2.88	2.88	1	1	1
2	5.378	10.756	21.512	4	8	16
$\Sigma=0$	$\Sigma=4.338$	$\Sigma=21.326$	$\Sigma x^2y=10.402$	$\Sigma x^2=10$	$\Sigma x^3=0$	$\Sigma x^4=34$

Above normal equations becomes

$5a + 10c = 4.338$ ← (1)

$10b = 21.326 \Rightarrow \boxed{b = 2.1326}$

$10a + 34c = 10.402$ ← (2)

$2\times(1) - (2) \Rightarrow -14c = -1.726$

$\boxed{c = 0.1233}$

From (1) $5a = 4.338 - 10c$

$$a = \frac{4.338 - 10(0.1233)}{5}$$

$\boxed{a = 0.6210}$

∴ Equation of best fit is $y = a + bx + cx^2$

$$\boxed{y = 0.6210 + (2.1326)x + 0.1233x^2}$$

7) c) $4x - 5y + 33 = 0, \quad 20x - 9y = 107.$

$4\bar{x} - 5\bar{y} + 33 = 0$
$20\bar{x} - 9\bar{y} = 107$

Solving
$\bar{x} = 13, \ \bar{y} = 17$

$y = \frac{4}{5}x + \frac{33}{5}$

$\Rightarrow r\frac{\sigma_y}{\sigma_y} = \frac{4}{5}$

$b_{yx} = \frac{4}{5}$

& $x = \frac{9}{20}y + \frac{107}{9}$

$\Rightarrow r\frac{\sigma_x}{\sigma_y} = \frac{9}{20}$

Multiplying we get

$r^2 = \frac{4}{5} \times \frac{9}{20} = 0.36$

$\Rightarrow r = 0.6$

Variance of y when variance of x is 9
$\therefore$ S.D $= 3$

$\sigma_y = \frac{\sigma_x b_{yx}}{r} \Rightarrow \sigma_y = \frac{\cancel{9}\,3 \cdot 4/5}{0.6} = \cancel{108}\ 4$

S.D of $Y = 4$

$\therefore$ Variance $= 16$

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8) a $\tan\theta = \frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}\left(\frac{r^2-1}{r}\right)$

θ is acute, the angle btw lines $y = m_1x+c_1$ & $y = m_2x+c_2$ is given by

$$\tan\theta = \frac{m_2-m_1}{1+m_1m_2}$$

$y-\bar{y} = \frac{r\sigma_y}{\sigma_x}(x-\bar{x})$ & $(x-\bar{x}) = \frac{r\sigma_x}{\sigma_y}(y-\bar{y})$

Taking slopes $m_1 = \frac{r\sigma_y}{\sigma_x}$ & $m_2 = \frac{\sigma_y}{r\sigma_x}$

$$\tan\theta = \frac{\frac{\sigma_y}{r\sigma_x} - \frac{r\sigma_y}{\sigma_y}}{1+\frac{r\sigma_y}{\sigma_x}\cdot\frac{\sigma_y}{r\sigma_x}} = \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\left(\frac{1-r^2}{r}\right)$$

If $r = \pm 1$, $\tan\theta = 0 \Rightarrow \theta = 0$

$\Rightarrow$ Regression lines coincide

Also if $r=0$, $\tan\theta = \infty \Rightarrow \theta = \pi/2$

$\therefore$ lines are perpendicular.

$\Rightarrow$ No correlation

b) Find r & regression line.

1	2	3	4	5	6	7	8	9	$\therefore \bar{x} = \frac{\Sigma x}{n} = 5$
12	11	13	15	14	17	16	19	18	$\therefore \bar{y} = \frac{\Sigma y}{n} = 15$

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	X^2	Y^2	XY
1	12	-4	-3	16	9	12
2	11	-3	-4	9	16	12
3	13	-2	-2	4	4	4
4	15	-1	0	1	0	0
5	14	0	-1	0	1	0
6	17	1	2	1	4	2
7	16	2	1	4	1	2
8	19	3	4	9	16	12
9	18	4	3	16	9	12
				$\Sigma X^2 = 60$	$\Sigma Y^2 = 60$	$\Sigma XY = 56$

$$Y = \frac{\Sigma XY}{\Sigma X^2} \cdot X \qquad\qquad X = \frac{\Sigma XY}{\Sigma Y^2} Y$$

$$y - 15 = \frac{56}{50}(x - 5) \qquad\qquad x - 5 = \frac{56}{60}(y - 15$$

$$\therefore y = 0.93x + 10.35 \qquad\qquad x = 0.93y - 8.95$$

$$\& \ r = \pm\sqrt{(\text{coeff of } x)(\text{coeff of } y)} = 0.93$$

8(c) $y = ax^b$

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

$$\log y = \log a + b \log_e x$$

$$Y = A + bX, \quad A = \log_e a, \quad X = \log_e x$$

$$\Sigma Y = nA + b\Sigma X$$

$$\Sigma XY = A\Sigma X + b\Sigma X^2, \quad n = 5$$

x	y	$X = \log_e x$	$Y = \log_e y$	XY	X^2
1	0.5	0	-0.6931	0	0
2	2	0.6931	0.6931	0.4804	0.4804
3	4.5	1.0986	1.5041	1.6524	1.2069
4	8	1.3863	2.0794	2.8827	1.9218
5	12.5	1.6094	2.5257	4.0649	2.5902
		4.7874	6.1092	9.0804	

$$\Sigma X^2 = 6.1993$$

$$5A + 4.7874b = 6.1092$$

$$4.7874A + 6.1993b = 9.0804$$

$$A = -0.69315, \quad b = 2$$

$$\log_e a = A \Rightarrow a = e^A = e^{-0.69315} = 0.5$$

$$\boxed{y = 0.5x^2}$$

9) a

X \ Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

$P_1 = 0.1 + 0.2 + 0 + 0.3 = 0.6$; $P_2 = 0.2 + 0.1 + 0.1 + 0 = 0.4$

$Q_1 = 0.1 + 0.2 = 0.3$; $Q_2 = 0.2 + 0.1 = 0.3$

$Q_3 = 0 + 0.1 = 0.1$; $Q_4 = 0.3 + 0 = 0.3$

X	1	2
P(X)	0.6	0.4

Y	-2	-1	4	5
P(Y)	0.3	0.3	0.1	0.3

Mean of X & Y are $\mu_x = \sum x_i p(x_i) = 1(0.6) + 2(0.4) = 1.4$

$\mu_y = \sum y_i p(y_i) = (-2)(0.3) + (-1)(0.3)$

$+ 4(0.1) + 5(0.3) = 1.0$

$E(XY) = \sum_{i,j} x_i y_j J_{ij}$

$= (1)(-2)(0.1) + 1(-1)(0.2) + 1 \times 4 \times 0 + 1 \times 5(0.3)$

$+ 2(-2)(0.2) + 2(-1)(0.1) + 2 \times 4 \times 0.1 + 2 \times 5 \times 0$

$E(XY) = 0.9$; $(E(X) = 1.4)$ & $E(Y) = 1$

$\text{cov}(X, Y) = E(XY) - E(X)E(Y)$

$= 0.9 - (1.4)(1) = -0.5$

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9(b) Prob. of the turn up of an odd no. is

$$p = \frac{3}{6} = \frac{1}{2}$$

$$\therefore \quad q = 1 - p = \frac{1}{2}$$

Expected no. of successes $= \frac{1}{2} \times 324 = 162$

Observed " " " $= 181$

$\therefore$ Difference $= 181 - 162 = 19$

Consider $z = \dfrac{x - np}{\sqrt{npq}} = \dfrac{19}{\sqrt{324 \times \frac{1}{2} \times \frac{1}{2}}} = \dfrac{19}{9} = 2.11$

$z = 2.11 < 2.58$ (At 1% level of significance)

$\therefore$ The die is unbiased.

9(c) Nine items have following values

45, 47, 50, 52, 48, 47, 49, 53, 51 , we've $\mu = 47.5$ & $n = 9$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{442}{9} = 49.11$$

$$s^2 = \frac{1}{n-1} \Sigma (x - \bar{x})^2$$

$$= \frac{1}{8} \left[(45-49.11)^2 + (47-49.11)^2 + (50-49.11)^2 + (52-49.11)^2 + (48-49.11)^2 + (47-49.11)^2 + (49-49.11)^2 + (53-49.11)^2 + (51-49.11)^2 \right]$$

$= \frac{1}{8}(54.8889) = 6.8611 > t_{0.05} = 2.306$ at 8 d.o.f.

$\therefore$ Hyp. is rejected.

$\boxed{10(a)}$ (i) Null Hypothesis - The hypothesis formulated for the purpose of its rejection under the assumption that it's true is called Null Hyp. and denoted by H_0

(ii) Type-I and Type-II error - A type-I error occurs if an investigator rejects a null hypothesis that is actually true in the population; a type-II error occurs if the investigator fails to reject a null hyp. that is actually false in the population

(iii) Level of Significance - The prob. level, below which leads to the rejection of the hypo. is known as the significance level.

$\boxed{10(b)}$ We've $\mu = 66, \quad n = 10$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{678}{10} = 67.8$$

$$s^2 = \frac{1}{n-1} \Sigma (x - \bar{x})^2$$

$$s^2 = \frac{1}{9}\left[(63-67.8)^2 + \dots + (71-67.8)^2\right] = 9.067 \quad \therefore s = 3.011$$

We've $\quad t = \dfrac{\bar{x}-\mu}{s}\sqrt{n} = \dfrac{(67.8-66)}{3.011}\sqrt{10} = 1.89 < 2.262$

$\therefore$ The hyp. is accepted at 5% level of significance

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$\boxed{10c}$ Let us take the hyp. that these fig. support to the general result in the ratio 9 : 3 : 3 : 1

The expected frequencies in the respective groups are

$\frac{9}{16} \times 1600 = 900$; $\frac{3}{16} \times 1600 = 300$, $\frac{3}{16} \times 1600 = 300$, $\frac{1}{16} \times 1600 = 100$

we've

O_i	882	313	287	118
E_i	900	300	300	100

$$\chi^2 = \Sigma \frac{(O_i - E_i)^2}{E_i}$$

$$= \frac{(18)^2}{900} + \frac{(13)^2}{300} + \frac{(13)^2}{300} + \frac{(18)^2}{100}$$

$$= \frac{(18)^2}{900} + \frac{2(13)^2}{300} + \frac{(18)^2}{100}$$

$$= \frac{324}{900} + \frac{338}{300} + \frac{324}{100} = 4.73$$

$$\chi^2 = 4.73 < \chi^2_{0.05} = 7.81 \text{ for } 3 \text{ d.o.f.}$$

$\therefore$ Hypothesis is accepted.