CBCS SCHEME

USN [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]

17MAT11

First Semester B.E. Degree Examination, Jan./Feb. 2021
Engineering Mathematics – I

Time: 3 hrs. Max. Marks: 100

Note: Answer any FIVE full questions, choosing ONE full question from each module.

Module-1

1 a. Find the n^{th} derivative of $$\frac{x}{(x-1)(2x+3)}$$ **(06 Marks)**

b. Find the angle of intersection of the curves $r = \frac{a}{1+\cos\theta}$ and $r = \frac{b}{1-\cos\theta}$. **(07 Marks)**

c. Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point (1, 1). **(07 Marks)**

OR

2 a. If $y = e^{a\sin^{-1}x}$ prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$. **(06 Marks)**

b. Find the pedal equation of $r^2 = a^2 \sec 2\theta$. **(07 Marks)**

c. Find the radius of curvature of the curve $r^n = a^n \sin n\theta$. **(07 Marks)**

Module-2

3 a. Obtain the Taylor's expansion of $\tan x$ about $x = \frac{\pi}{4}$ upto third degree terms. **(06 Marks)**

b. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \left(\frac{2^x+3^x+4^x}{3}\right)^{1/x}$ **(07 Marks)**

c. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ **(07 Marks)**

OR

4 a. If $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$. **(06 Marks)**

b. Obtain the Maclaurin's expansion of $\log(\sec x)$ upto fourth degree terms. **(07 Marks)**

c. If $x + y + z = u$, $y + z = v$, $z = uvw$ find the Jacobian $J\left(\frac{x, y, z}{u, v, w}\right)$ **(07 Marks)**

Module-3

5 a. If $\phi = x^2 + y^2 + z^2$, $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ then find grad ϕ, div $\vec{F}$ and curl $\vec{F}$. **(06 Marks)**

b. A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is the time. Find the components of velocity and acceleration at $t = 1$ in the direction of the vector $i - 3j + 2k$. **(07 Marks)**

c. Prove that $\mathrm{curl}(\phi\vec{A}) = \phi\,\mathrm{curl}\vec{A} + (\mathrm{grad}\phi \times \vec{A})$ **(07 Marks)**

OR

6 a. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at (2, −1, 2) **(06 Marks)**

b. Show that the vector field $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ is irrotational and find ϕ such that $\vec{F} = \nabla\phi$. **(07 Marks)**

c. Prove that $\operatorname{div}(\phi\vec{A}) = \phi \operatorname{div}\vec{A} + (\operatorname{grad}\phi \cdot \vec{A})$ **(07 Marks)**

Module-4

7 a. Obtain the reduction formula for

$$\int_0^{\pi/2} \cos^n x \, dx$$ **(06 Marks)**

b. Solve $\dfrac{dy}{dx} = xy^3 - xy$ **(07 Marks)**

c. A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be the temperature of the body after 40 minutes from the original? **(07 Marks)**

OR

8 a. Evaluate $\displaystyle\int_0^{2a} \frac{x^2}{\sqrt{2ax - x^2}} dx$ **(06 Marks)**

b. Solve $(x^2 + y^2 + x)dx + xy\,dy = 0$ **(07 Marks)**

c. Obtain the orthogonal trajectories of the family of curves $r^n = a \sin n\theta$. **(07 Marks)**

Module-5

9 a. Find the rank of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$ **(06 Marks)**

b. Diagonalize the matrix $\begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$ **(07 Marks)**

c. Using Rayleigh's power method find the largest eigen value and the corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$ taking $(1\ 1\ 1)^T$ as the initial eigen vector. **(07 Marks)**

OR

10 a. Using Gauss-Siedel method, solve

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

using (0, 0, 0) as the initial approximation to the solution. **(06 Marks)**

b. Show that the linear transformation $y_1 = 2x_1 + x_2 + x_3$, $y_2 = x_1 + x_2 + 2x_3$, $y_3 = x_1 - 2x_3$ is regular and find the inverse transformation. **(07 Marks)**

c. Reduce the quadratic form $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$ into the canonical form. **(07 Marks)**

Jan/Feb 2021 Solution 17MAT11

Engineering Mathematics - I

CMR

① (a) Find the n^{th} derivative of $\dfrac{x}{(x-1)(2x+3)}$

⇒ Given

$$y = \frac{x}{(x-1)(2x+3)}$$ is a proper fraction.

Let

$$y = \frac{x}{(x-1)(2x+3)} = \frac{A}{(x-1)} + \frac{B}{(2x+3)}$$

$$\Rightarrow \quad x = A(2x+3) + B(x-1) \quad \text{--- (1)}$$

setting $x-1=0$, $2x+3=0$ we get $x=1, x=\frac{3}{2}$

put $x=1$ in (1)

$$1 = 5A \Rightarrow \boxed{A = 1/5}$$

put $x = \frac{3}{2}$ in (1)

$$\frac{3}{2} = B\left(\frac{3}{2}-1\right) \Rightarrow \frac{3}{2} = \frac{1}{2}B \Rightarrow \boxed{B=1}$$

$$\therefore \quad y = \frac{1}{5(x-1)} + \frac{1}{(2x+3)}$$

Now $$y_n = \frac{1}{5}\frac{(-1)^n \lfloor n \, 1^n}{(x-1)^{n+1}} + \frac{(-1)^n \lfloor n \, 2^n}{(2x+3)^{n-1}}$$

$$\therefore \quad y_n = (-1)^n \lfloor n \left[\frac{1}{5(x-1)^{n+1}} + \frac{2^n}{(2x+3)^{n+1}}\right] //$$

1. (b) Given

$$r = \frac{a}{1+\cos\theta} \quad ; \quad r = \frac{b}{1-\cos\theta}$$

$\Rightarrow$ $\log r = \log a - \log(1+\cos\theta)$; $\log r = \log b - \log(1-\cos\theta)$

Differentiating these w.r.t θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{(-\sin\theta)}{(1+\cos\theta)} \quad ; \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{(\sin\theta)}{(1-\cos\theta)}$$

$$\cot\phi_1 = \frac{\cancel{2}\sin\theta/_2 \cos\theta/_2}{\cancel{2}\cos^2\theta/_2} \quad ; \quad \cot\phi_2 = -\frac{\cancel{2}\sin\theta/_2\cos\theta/_2}{\cancel{2}\sin^2\theta/_2}$$

$$\cot\phi_1 = \tan\theta/_2 \quad ; \quad \cot\phi_2 = -\cot\theta/_2$$

$$\cot\phi_2 = \cot(-\theta/_2)$$

$$\cot\phi_1 = \cot\left[\frac{\pi}{2} - \frac{\theta}{2}\right]$$

$$\phi_2 = -\theta/_2$$

$$\phi_1 = \frac{\pi}{2} - \frac{\theta}{2} \qquad \phi_2 = -\theta/_2 .$$

$$|\phi_1 - \phi_2| = \left|\frac{\pi}{2} - \frac{\theta}{2} + \frac{\theta}{2}\right| = \frac{\pi}{2}$$

$\Rightarrow$ $\therefore$ the angle of intersection $= \frac{\pi}{2}$ //

CMR

1(c) Given $x^4+y^4=2$

Differentiating with respect to x, we get

$$4x^3+4y^3y_1=0$$

$$y_1=-\frac{x^3}{y^3}$$

$$(y_1)_{(1,1)}=-1$$

differentiating y_1 w.r.t x, we get

$$y_2=-\left\{\frac{y^3(3x^2)-x^3(3y^2y_1)}{y^6}\right\}$$

$$y_2\Big|_{(1,1)}=-\frac{(3+3)}{1}=-6$$

Therefore

$$\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\frac{\{1+(-1)^2\}^{3/2}}{-6}=\frac{2^{3/2}}{-6}=\frac{-\sqrt{2}}{3}$$

$$\boxed{\rho=\frac{\sqrt{2}}{3}}$$

2(a) Let $y = e^{a\sin^{-1}x}$

differentiating w.r.t x, we get

$$y_1 = e^{a\sin^{-1}x} \cdot \frac{a \cdot 1}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

squaring on both sides, we get

$$y_1^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow (1-x^2)\, y_1^2 = a^2y^2$$

Differentiating again w.r.t x, we get

$$(1-x^2)\, 2y_1y_2 + (-2x)\, y_1^2 = a^2 \cdot 2yy_1$$

Dividing on $2y_1$ on both the sides, we get

$$(1-x^2)\, y_2 - xy_1 - a^2y = 0$$

Differentiating n times by using Leibnitz's theorem, we get

$$(1-x^2)\, y_{n+2} + {}^nC_1 (-2x)\, y_{n+1} + {}^nC_2 (-2)\, y_n -$$
$$\{xy_{n+1} + {}^nC_1 (1)\, y_n\} - a^2y_n = 0$$

$$(1-x^2)y_{n+2}-2nxy_{n+1}-2\frac{n(n-1)}{\lfloor 2}y_n-xy_{n+1}-ny_n$$
$$-a^2y_n=0$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-n+n+a^2)y_n=0$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-a^2)=0$$

2(b) Given

$$r^2 = a^2 \sec 2\theta \quad \text{--- (1)}$$

Taking 'log' both sides,

$$2\log r = 2\log a + \log \sec 2\theta$$

diff^n w.r.t 'θ', we get

$$\frac{2}{r}\frac{dr}{d\theta} = 0 + \frac{\sec 2\theta \tan 2\theta \cdot 2}{\sec 2\theta}$$

$$\cot\phi = \tan 2\theta$$

$$\cot\phi = \cot(\pi/2 - 2\theta)$$

$$\Rightarrow \phi = \frac{\pi}{2} - 2\theta$$

Consider $p = r\sin\phi$

$$p = r\sin(\pi/2 - 2\theta)$$

$$p = r\cos 2\theta \quad \text{--- (2)}$$

$$\Rightarrow \frac{r}{p} = \sec 2\theta$$

By (1) & (2)

$$r^2 = a^2 \cdot \frac{r}{p} \quad \Rightarrow \boxed{pr = a^2}$$

Required pedal eq^n.

CMR

2(c) Given

$$r^n = a^n \sin n\theta$$

$\Rightarrow$ $n \log r = n \log a + \log \sin n\theta$

diff-n w.r.t θ we have

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{\cos n\theta}{\sin n\theta} \cdot n$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$$

$$\Rightarrow r_1 = r \cot n\theta$$

Hence $\quad r_2 = r \operatorname{cosec}^2 n\theta . n + r_1 \cot n\theta$

$$r_2 = r \operatorname{cosec}^2 n\theta . n + r \cot^2 n\theta$$

we have

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

$$\rho = \frac{[r^2 + r^2 \cot^2 n\theta]^{3/2}}{r^2 + 2r^2 \cot^2 n\theta - r[r \operatorname{cosec}^2 n\theta . n + r \cot^2 n\theta]}$$

$$\rho = \frac{r^3 \operatorname{cosec}^3 n\theta}{r^2[1+\cot^2 n\theta] - r^2 n \operatorname{cosec}^2\theta}$$

$$= \frac{r^3 \operatorname{cosec}^3 n\theta}{r^2 \operatorname{cosec}^2 n\theta \, [1-n]}$$

$$\boxed{\rho = \frac{r \operatorname{cosec} n\theta}{(1-n)}}$$

3 (a) Given $y = \tan x$

$y = \tan x \Rightarrow y(\frac{\pi}{4}) = 1$

$y_1 = \sec^2 x \Rightarrow y_1(\frac{\pi}{4}) = 2$

$y_1 = 1 + \tan^2 x = 1 + y^2$

$y_2 = 2yy_1 \Rightarrow y_2(\frac{\pi}{4}) = 2.1.2 = 4$

$y_3 = 2yy_2 + 2y_1y_1 \Rightarrow y_3(\frac{\pi}{4}) = 16$

Therefore

$$y = y(\tfrac{\pi}{4}) + \frac{(x-\frac{\pi}{4})}{\lfloor 1} y_1(\tfrac{\pi}{4}) + \frac{(x-\pi/4)^2}{\lfloor 2} y_2(\tfrac{\pi}{4}) + \cdots$$

$$= 1 + (x - \tfrac{\pi}{4})\, 2 + \frac{1}{\lfloor 2}(x-\tfrac{\pi}{4})^2\, 4 + \frac{1}{\lfloor 3}(x-\pi/4)^3\, 16 + \cdots$$

$$\tan x = 1 + 2(x - \tfrac{\pi}{4}) + 2(x-\pi/4)^2 + \frac{8}{3}(x-\pi/4)^3 + \cdots$$

3 (b) Let

$$K = \lim_{x\to 0}\left(\frac{2^x+3^x+4^x}{3}\right)^{1/x} \quad \text{---} \quad 1^\infty$$

Taking log both sides

$$\log K = \lim_{x\to 0}\frac{\log(2^x+3^x+4^x)/3}{x} \quad \text{---} \quad \frac{0}{0}$$

Applying L'Hospital's rule

$$\log K = \lim_{x\to 0}\frac{\frac{3}{2^x+3^x+4^x}\frac{1}{3}\left[2^x\log 2+3^x\log 3+4^x\log 4\right]}{1}$$

$$= \frac{1}{3}\left[\log 2+\log 3+\log 4\right]$$

$$\log K = \frac{1}{3}\log(2.3.4)$$

$$\log K = \log(24)^{1/3}$$

$$\Rightarrow \boxed{K = (24)^{1/3}}$$

CMR

3(c) >> Here we need to convert the given function u into a composite function.

Let $u = f(p, q, r)$ where $p = \frac{x}{y}$, $q = \frac{y}{z}$, $r = \frac{z}{x}$

i.e $\{ u \rightarrow (p, q, r) \rightarrow (x, y, z) \} \Rightarrow u \rightarrow x, y, z$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{1}{y} + \frac{\partial u}{\partial q} \cdot 0 + \frac{\partial u}{\partial r} \left(-\frac{z}{x^2} \right)$$

$$\therefore x \frac{\partial u}{\partial x} = \frac{x}{y} \frac{\partial u}{\partial p} - \frac{z}{x} \frac{\partial u}{\partial r} \quad \text{———} \quad (1)$$

Similarly by symmetry we can write,

$$y \frac{\partial u}{\partial y} = \frac{y}{z} \frac{\partial u}{\partial q} - \frac{x}{y} \frac{\partial u}{\partial p} \quad \text{———} \quad (2)$$

$$z \frac{\partial u}{\partial z} = \frac{z}{x} \frac{\partial u}{\partial r} - \frac{y}{z} \frac{\partial u}{\partial q} \quad \text{———} \quad (3)$$

Thus by adding (1), (2) and (3), we get

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0}$$

4(ii)

$$u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$$

$$\Rightarrow \sin u = \frac{x^2y^2}{x+y}$$

$$\Rightarrow \sin u = \frac{x^4(y^2/x^2)}{x(1+y/x)} = x^3\left[\frac{(y/x)^2}{1+y/x}\right]$$

$$\sin u = x^3 g\left(\frac{y}{x}\right) = f(say)$$

Therefore, f is homogeneous function of degree 3.

Hence by Eulers theorem, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$$

$$x\frac{\partial}{\partial x}\sin u + y\frac{\partial}{\partial y}\sin u = 3\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 3\sin u$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u}$$

4 (b) $y(x) = y(0) + x\,y_1(0) + \frac{x^2}{\lfloor 2} y_2(0) + \frac{x^3}{\lfloor 3} y_3(0) + \frac{x^4}{\lfloor 4} y_4(0) + \cdots \quad (1)$

$y = \log(\sec x) \quad \therefore y(0) = \log 1 = 0$

$y_1 = \frac{\sec x \tan x}{\sec x}$ i.e $y_1 = \tan x \quad \therefore y_1(0) = 0$

$y_2 = \sec^2 x \quad \therefore y_2(0) = 1$

$y_2 = 1 + \tan^2 x$

$y_2 = 1 + y_1^2$

differentiating this w.r.t. x successively we have

$y_3 = 2y_1 y_2 \quad \therefore y_3(0) = 0$

$y_4 = 2(y_1 y_3 + y_2^2) \quad \therefore y_4(0) = 2$

By (1)

$\log(\sec x) = 0 + 0 + \frac{x^2}{2} \cdot 1 + 0 + \frac{x^4}{24} \cdot 2$

Thus $\boxed{\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12}}$

4 (c) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$

It is evident that we should have x, y, z in terms of u, v, w

Consider $x+y+z=u$ — (1) $\quad y+z=v$ — (2) $\quad z=uvw$ — (3)

Using (2) in (1) we have

$x+v=u \quad \therefore \quad x=u-v$

Also by using (3) in (2)

we have, $y+uvw=v \quad \therefore \quad y=v-uvw$

Thus the given data is modified into the form,

$x=u-v, \quad y=v-uvw, \quad z=uvw$

$\therefore \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & (1-uw) & -uv \\ vw & uw & uv \end{vmatrix}$

$\Rightarrow \boxed{J = uv}$

CMR

5 (a) => Given

$$\phi = x^2+y^2+z^2$$

$$\therefore \text{grad}\,\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2+y^2+z^2)\,i + \frac{\partial}{\partial y}(x^2+y^2+z^2)\,j + \frac{\partial}{\partial z}(x^2+y^2+z^2)\,k$$

$$\nabla\phi = 2xi + 2yj + 2zk$$

and

$$\vec{F} = x^2 i + y^2 j + z^2 k$$

so $\text{div}\,\vec{F} = \nabla\vec{F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\left(x^2 i + y^2 j + z^2 k\right)$

$$\nabla\vec{F} = 2x + 2y + 2z = 2(x+y+z)$$

and

$$\text{Curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= i\,[0] - j\,[0] + k\,[0] = \vec{0}$$

$$\boxed{\text{Curl}\,\vec{F} = \vec{0}}$$

5(b) we have

$$\vec{r} = 2t^2\hat{i} + (t^2-4t)\hat{j} + (3t-5)k$$

being the position vector of a particle at time t

so $\vec{v} = \frac{d\vec{r}}{dt} = 4t\hat{i} + (2t-4)\hat{j} + 3K$ is the velocity

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 4\hat{i} + 2\hat{j}$ is a the acceleration.

at $t=1$

$$(\vec{v})_{t=1} = 4\hat{i} - 4\hat{j} + 3K$$

$$(\vec{a})_{t=1} = 4\hat{i} + 2\hat{j}$$

Now the unit vector in the given direction.

$\vec{B} = \hat{i} - 3\hat{j} + 2K$ is $\hat{n} = \frac{\hat{i} - 3\hat{j} + 2K}{\sqrt{1+9+4}}$

$$\hat{n} = \frac{\hat{i} - 3\hat{j} + 2K}{\sqrt{14}}$$

$\therefore$ The required velocity component in the direction of $\vec{B}$.

$$\vec{v}\hat{n} = \frac{(\hat{i} - 3\hat{j} + 2K)}{\sqrt{14}}(4\hat{i} - 4\hat{j} + 3K) = \frac{4+12+6}{\sqrt{14}}$$

$$= \frac{22}{\sqrt{14}}$$

Also the required acceleration component is given direction $\vec{B}$ is.

$$\vec{A}\cdot\hat{n} = (4i+2j)\frac{(i-3j+2k)}{\sqrt{14}} = \frac{4-6}{\sqrt{14}} = \frac{-2}{\sqrt{14}}$$

5 (c)

we need to prove $\text{curl}(\phi\vec{A}) = \phi\,\text{curl}\vec{A} + (\text{grad}\,\phi\times\vec{A})$

ii) Let ϕ and $\vec{A} = a_1 i + a_2 j + a_3 k$ be respectively scalar and vector point functions of x, y, z

$$\therefore \quad \phi\vec{A} = (\phi a_1)i + (\phi a_2)j + (\phi a_3)k$$

Now

$$\text{curl}(\phi\vec{A}) = \nabla\times(\phi\vec{A}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi a_1 & \phi a_2 & \phi a_3 \end{vmatrix}$$

$$= \Sigma i\left\{\frac{\partial}{\partial y}(\phi a_3) - \frac{\partial}{\partial z}(\phi a_2)\right\}$$

$$= \Sigma i\left\{\left(\phi\frac{\partial a_3}{\partial y} + \frac{\partial\phi}{\partial y}a_3\right) - \left(\phi\frac{\partial a_2}{\partial z} + \frac{\partial\phi}{\partial z}a_2\right)\right\}$$

$$= \phi \Sigma \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) i + \Sigma \left(\frac{\partial \phi}{\partial y} a_3 - \frac{\partial \phi}{\partial z} a_2\right) i$$

$$= \phi \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \phi (\nabla \times \vec{A}) + \nabla \phi \times \vec{A}$$

Thus $\boxed{\text{Curl}(\phi \vec{A}) = \phi(\text{curl}\vec{A}) + \nabla\phi \times \vec{A}}$

6(a) Let $\phi = x^2+y^2+z^2-9$

$$\nabla \phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\therefore \vec{n}_1 = [\nabla \phi]_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 2k$$

$$(\vec{n}_1) = \sqrt{36} = 6$$

$$\psi = x^2+y^2-z-3$$

$$\nabla \psi = 2xi + 2yj - k$$

$\therefore \quad \vec{n}_2 = [\nabla \phi]_{(2,-1,2)} = 4i - 2j - k$

$|\vec{n}_2| = \sqrt{16+4+1} = \sqrt{21}$

$\therefore$

$$\cos\theta = \frac{(4i-2j+2k)(4i-2j-k)}{6\sqrt{21}}$$

$$\cos\theta = \frac{16+4-2}{6\sqrt{21}} = \frac{18}{6\sqrt{21}} = \frac{3}{\sqrt{21}}$$

$$\theta = \cos^{-1}\left[\frac{3}{\sqrt{21}}\right]$$

6(b) Given

$$\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$$

we have to show that $\text{curl}\,\vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - yz) & (y^2 - zx) & (z^2 - xy) \end{vmatrix}$$

$$= i\left[-x - (-x)\right] + j\left[-y - (-y)\right] + k\left[-z - (-z)\right] = \vec{O}$$

$\therefore$ $\boxed{\nabla \times \vec{F} = \vec{O}}$ $\therefore$ $\vec{F}$ is irrotational.

Now we have to find ϕ such that $\nabla \phi = \vec{F}$

$$\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (x^2 - yz) i + (y^2 - zx) j + (z^2 - xy) k$$

$\Rightarrow$ $\frac{\partial \phi}{\partial x} = x^2 - yz$ $\therefore$ $\phi = \frac{x^3}{3} + xyz + f_1(y, z)$ — (1)

$\Rightarrow$ $\frac{\partial \phi}{\partial y} = y^2 - zx$ $\therefore$ $\phi = \frac{y^3}{3} + xyz + f_2(x, z)$ — (2)

$\Rightarrow$ $\frac{\partial \phi}{\partial z} = z^2 - xy$ $\therefore$ $\phi = \frac{z^3}{3} - xyz + f_3(x, y)$ — (3)

Let us choose

$f_1(y, z) = 0$, $f_2(x, z) = 0$, $f_3(x, y) = c$

from ① & ② & ③

Thus $$\boxed{\phi = \frac{x^3+y^3+z^3}{3} - xyz}$$

6ⓒ Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z and ϕ be a scalar point function of x, y, z.

$\therefore \phi\vec{A} = \phi(a_1 i + a_2 j + a_3 k) = \Sigma(\phi a_1) i$

Now $\text{div}(\phi\vec{A}) = \nabla \cdot (\phi\vec{A})$

$$= \Sigma \frac{\partial}{\partial x} i \cdot \Sigma(\phi a_1) i$$

$$= \Sigma \frac{\partial}{\partial x}(\phi a_1) = \Sigma\left(\phi \frac{\partial a_1}{\partial x} + \frac{\partial \phi}{\partial x} a_1\right)$$

$$\Rightarrow \text{div}(\phi\vec{A}) = \phi \Sigma \frac{\partial a_1}{\partial x} + \Sigma \frac{\partial \phi}{\partial x} i \cdot \Sigma a_1 i$$

Thus $$\boxed{\text{div}(\phi\vec{A}) = \phi(\text{div}\,\vec{A}) + \text{grad}\,\phi \cdot \vec{A}}$$

7(a) Evaluation of $\int_0^{\pi/2} \cos^n x \, dx$

Let

$$I_n = \int_0^{\pi/2} \cos^n x \, dx$$

W.K.T

$$I_n = \int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2} \quad \text{--- (1)}$$

So

$$I_n = \int_0^{\pi/2} \cos^n x \, dx = \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$$

But $\cos(\pi/2) = 0 = \sin 0$

Thus

$$I_n = \frac{n-1}{n} I_{n-2} \quad \text{--- (2)}$$

we use this recurrence relation to find I_{n-2} by simply replacing n by $(n-2)$.

i.e, $$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

Hence

$$I_n = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} I_{n-4}$$, by back substitution.

similarly from (2)

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

CMR

Hence $I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} I_{n-6}$, again by back sub^n.

Continuing like this, the reduction process will end up as follows

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} I_1 \text{ if } n \text{ is odd}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} I_0 \text{ if } n \text{ is even.}$$

But

$$I_1 = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = 1 - 0 = 1$$

$$\boxed{I_1 = 1}$$

and

$$I_0 = \int_0^{\pi/2} \cos^0 x \, dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

Thus we have,

$$\int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} & \text{if } n \text{ is odd.} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2}\frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

7(b) Given

$$\frac{dy}{dx} = xy^3 - xy$$

$$\frac{dy}{dx} + xy = xy^3 \quad \text{— Bernoulli's eqn.} \quad (1)$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x$$

Put $\frac{1}{y^2} = t \Rightarrow \frac{-2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dt}{dx}$$

By (1)

$$-\frac{1}{2}\frac{dt}{dx} + xt = x$$

$$\Rightarrow \frac{dt}{dx} - 2xt = -2x \quad \text{— linear in } t$$

$\therefore$ $P = -2x$ $\quad Q = -2x$

$$\Rightarrow I.F = e^{\int -2x\,dx} = e^{-2x^2/2} = e^{-x^2}$$

Solution

$$\frac{1}{y^2} \times IF = \int Q \times IF\, dx + C$$

$$\frac{1}{y^2} \times e^{-x^2} = \int -2x\, e^{-x^2}\, dx + C$$

Let

$-x^2 = u$

$-2x\,dx = du$

$$\Rightarrow \frac{1}{y^2} \times e^{-x^2} = \int e^{u}\, du + C$$

$$\Rightarrow \frac{e^{-x^2}}{y^2} = e^{u} + C$$

$$\Rightarrow \boxed{\frac{e^{-x^2}}{y^2} = e^{-x^2} + C}$$

7 (c) Let T be the temperature at time t, then

$$\frac{dT}{dt} = K(T - T_A) = (T-40)$$

with $T(0) = 80$ and $T(20) = 60$

$$\frac{dT}{T-40} = K\,dt$$

$$\log(T-40) = kt + C$$

$$T-40 = e^{kt+C} = C_1 e^{kt}, \quad C_1 = e^C$$

$$T(t) = 40 + C_1 e^{kt} \quad (1)$$

$$T(0) = 40 + C_1$$

$$80 = 40 + C_1 \Rightarrow C_1 = 40$$

equation (1) becomes

$$T(t) = 40 + 40e^{kt}$$

$$T(20) = 40 + 40\, e^{20k}$$

$$60 = 40 + 40\, e^{20k}$$

$$20 = 40\, e^{20k}$$

$$\Rightarrow \frac{20}{40} = e^{20k}$$

$$\Rightarrow 20k = \log \frac{1}{2}$$

$$\Rightarrow K = \frac{1}{20} \log(0.5) = -0.03465$$

Substituting this in eq^n (2), we get

$$T(t) = 40 + 40e^{-0.03465t}$$

The temperature of the body after 40 mins $\Rightarrow T(4) = 60.00294°C$

CMR

8 (a) Let

$$I_1 = \int_0^{2a} \frac{x^2 dx}{\sqrt{2ax - x^2}}$$

put

$$x = 2a\sin^2\theta \quad \therefore dx = 4a\sin\theta\cos\theta d\theta,$$

θ varies from 0 to $\pi/2$

Also

$$\sqrt{2ax - x^2} = \sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta}$$
$$= 2a\sin\theta\cos\theta$$

So

$$I = \int_0^{\pi/2} \frac{4a^2\sin^4\theta}{2a\sin\theta\cos\theta} (4a\sin\theta\cos\theta d\theta)$$

$$I = 8a^2 \int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$I = 8a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$ By reduction formula.

Thus $$\boxed{I = \frac{3\pi a^2}{2}}$$

8(b) Given

$$(x^2+y^2+x)\,dx + xy\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x^2+y^2+x)}{xy}$$

$$\frac{dy}{dx} = -\frac{(x^2+x)}{xy} - \frac{y^2}{xy}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = -\frac{(x+1)}{y}$$

Multiply by y on both sides,

$$\Rightarrow y\frac{dy}{dx} + \frac{y^2}{x} = -(x+1)$$

put $y^2 = t \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \frac{1}{2}\frac{dt}{dx} + \frac{t}{x} = -(x+1)$$

$$\Rightarrow \frac{dt}{dx} + \frac{2t}{x} = -2(x+1)$$ ..... linear in t

$P = \frac{2}{x}$ $\quad Q = -2(x+1)$

$I.F = e^{\int P\,dx} = e^{\int \frac{2}{x} dx} = e^{2\log x} = x^2$

Solution

$t \times x^2 = \int -2(x+1) \times x^2 \, dx + C$

$\Rightarrow y^2 x^2 = -2\int (x^3 + x^2)\, dx + C$

$\Rightarrow x^2 y^2 = -2\left[\frac{x^4}{4} + \frac{x^3}{3}\right] + C$

$\Rightarrow \boxed{x^2 y^2 = -\frac{1}{6}\left[3x^4 + 4x^3\right] + C}$

8 (c) >> $r^n = a \sin n\theta$

$\Rightarrow n \log r = \log a + \log \sin n\theta$

differentiating w.r.t θ we have,

$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{n \cos n\theta}{\sin n\theta}$ or $\frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we have

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot n\theta \quad \text{or} \quad -r\frac{d\theta}{dr} = \cot n\theta$$

$\therefore \tan n\theta \, d\theta = \frac{dr}{-r}$ by separating the variable

$$\int \frac{dr}{r} + \int \tan n\theta \, d\theta = c$$

i.e. $\log r + \frac{\log(\sec n\theta)}{n} = c$

$\Rightarrow \quad n\log r + \log(\sec n\theta) = nc$

$\Rightarrow \quad \log(r^n \sec n\theta) = \log b$

$\Rightarrow \quad \boxed{r^n \sec n\theta = b}$

9 (a)

$$A = \begin{array}{c} \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 4R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - 3R_2$,

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have two non-zero rows

so.

Rank $= \rho(A) = 2$

9(b) Let $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} (-19-\lambda) & 7 \\ -42 & (16-\lambda) \end{vmatrix} = 0$

$\Rightarrow \quad \lambda^2 + 3\lambda - 10 = 0$

$(\lambda - 2)(\lambda + 5) = 0$

i.e $\lambda = 2, -5$ are the eigen values of A.

Now consider $[A - \lambda I][X] = [0]$

$$\begin{bmatrix} (-19-\lambda) & 7 \\ -42 & (16-\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e $(-19-\lambda)x + 7y = 0$

$-42x + (16-\lambda)y = 0$

Case I: Let $\lambda = 2$

we get $\quad -14x + 7y = 0 \quad$ and $\quad -42x + 14y = 0$

$$y = 3x \quad \text{or} \quad \frac{y}{3} = \frac{x}{1}$$

$\therefore \; X_1 = (1, 3)'$ is the eigen vector correspond to $\lambda = 2$.

Case II: Let $\lambda = -5$

we get $\quad -14x + 7y = 0$ and $-42x + 21y = 0$

i.e $\quad y = 2x \quad$ or $\quad \frac{y}{2} = \frac{x}{1}$

$\therefore \; X_2 = (1, 2)'$ is the eigen vector correspond to $\lambda = -5$

Modal Matrix $P = [x_1 \; x_2] = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

we have $\quad |P| = 2 - 3 = -1$ and $P^{-1} = \dfrac{\text{Adj } P}{|P|}$

$$P^{-1} = -\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

CMR

Now $P^{-1}AP = D = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$

Thus $P^{-1}AP = D = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$ is the diagonal matrix

or $P^{-1}AP = \text{Diag}(2, -5)$

9(c) By data $X^{(0)} = [1, 1, 1]'$ is the initial eigenvector.

$AX^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix}$

$= \lambda^{(1)} X^{(1)}$

$AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 7.35 \\ -2.67 \\ 4.01 \end{bmatrix} = 7.33 \begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix}$

$= \lambda^{(2)} X^{(2)}$

$$AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 7.82 \\ -3.63 \\ 4.01 \end{bmatrix} = 7.82 \begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix}$$

$$= \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 7.94 \\ -3.89 \\ 3.99 \end{bmatrix} = 7.94 \begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix}$$

$$= \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 7.98 \\ -3.97 \\ 3.99 \end{bmatrix} = 7.98 \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix}$$

$$= \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix}$$

$$= \lambda^{(6)} X^{(6)}$$

Thus the dominant eigen value 8
and the corresponding eigen vector is $[1, -0.5, 0.5]$

//

CMR

10(a) The equations are diagonally dominant and hence we first write them in the following form.

$$x = \frac{1}{20}\left[17 - y - 2z\right]$$

$$y = \frac{1}{20}\left[-18 - 3x + z\right]$$

$$z = \frac{1}{20}\left[25 - 2x + 3y\right]$$

We start with the trial solution $x=0, y=0, z=0$

I st Iteration

$$x^{(1)} = \frac{17}{20} = 0.85$$

$$y^{(1)} = \frac{1}{20}\left[-18 - 3(0.85)\right] = -1.0275$$

$$z^{(1)} = \frac{1}{20}\left[25 - 2(0.85) + 3(-1.0275)\right]$$

$$= 1.0109$$

II nd Iteration

$$x^{(2)} = \frac{1}{20}\left[17 - (-1.0275) + 2(1.0109)\right]$$

$$= 1.0025$$

$$y^{(2)} = \frac{1}{20}\left[-18 - 3(1.0025) + 1.0109\right] =$$

$$-0.9998$$

$$z^{(2)} = \frac{1}{20}\left[25 - 2(1.0025) + 3(-0.9998)\right]$$

$$= 0.9998$$

IIIrd iteration

$$x^{(3)} = \frac{1}{20}[17 - (-0.9998) + 2(0.9998)]$$
$$= 0.99997 \approx 1$$

$$y^{(3)} = \frac{1}{20}[-18 - 3(0.99997) + 0.9998] =$$
$$-1.0000055 \approx -1$$

$$z^{(3)} = \frac{1}{20}[25 - 2(0.99997) + 3(-1.0000055)]$$
$$= 1.0000022 \approx 1$$

Thus $x=1,\ y=-1,\ z=1$ is required solution.

10(b) i) $y_1 = 2x_1 + x_2 + x_3$

$y_2 = x_1 + x_2 + 2x_3$

$y_3 = x_1 - 2x_3$

The given transformation in the matrix form is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

i.e $Y = AX$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

Now $|A| = 2(-2) - 1(-2) + 1(-1)$
$= -4 + 2 - 1 = -\cancel{3} \neq 0$

$\Rightarrow$ The transformation is regular.

we compute

$$A^{-1} = \frac{1}{|A|} \text{Adj-}A$$

$$\text{Adj-}A = \begin{bmatrix} +(-2) & -(-2) & +1 \\ -(-4) & +(-5) & -3 \\ +(-1) & -(-1) & +1 \end{bmatrix}$$

Hence

$$A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Inverse transformation is given by $X = A^{-1}Y$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ +1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Thus

$x_1 = 2y_1 - 2y_2 - y_3$

$x_2 = -4y_1 + 5y_2 + 3y_3$

$x_3 = y_1 - y_2 - y_3$

10 (c) The Symmetric matrix of the QF is

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

» The characteristic eqn of A is $|A - \lambda I| = 0$

i.e $$\begin{vmatrix} (8-\lambda) & -6 & 2 \\ -6 & (7-\lambda) & -4 \\ 2 & -4 & (3-\lambda) \end{vmatrix} = 0$$

on expanding we have,

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$\lambda = 0, 3, 15$ are the eigent values of A.

we now form the system of equation.

$$(8-\lambda)x - 6y + 2z = 0$$

$$-6x + (7-\lambda)y - 4z = 0$$

$$2x - 4y + (3-\lambda)z = 0 \quad \text{—— (1)}$$

CMR

Case I: Let $\lambda = 0$, (1)

$$8x + 6y + 2z = 0 \quad \text{--- (i)}$$
$$-6x + 7y - 4z = 0 \quad \text{--- (ii)}$$
$$2x - 4y + 3z = 0 \quad \text{--- (iii)}$$

Applying the rule of cross multiplication for (i) & (ii)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

So $x = 1,\ y = 2,\ z = 2$

Thus the eigen vector X_1 corresponding to the eigen value $\lambda = 0$ is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case-II: Let $\lambda = 3$, (1)

$$5x - 6y + 2z = 0 \quad \text{--- (iv)}$$
$$-6x + 4y - 4z = 0 \quad \text{--- (v)}$$
$$2x - 4y + 0z = 0 \quad \text{--- (vi)}$$

From (iv) & (v) $\dfrac{x}{24-8} = \dfrac{-y}{-20+12} = \dfrac{z}{20-36}$

$\therefore (x, y, z) = (2, 1, -2)$

Case III : Let $\lambda = 15$, (1)

$-7x - 6y + 2z = 0$ —— (VII)

$-6x - 8y - 4z = 0$ —— (VIII)

$2x - 4y + 12z = 0$ —— (IX)

we have (VII) & (VIII)

$$\frac{x}{24+16} = \frac{-y}{28+12} = \frac{z}{56-36}$$

$\therefore$ $(x, y, z) = (2, -2, 1)$

Thus $X_3 = (2, -2, 1)$ is the eigen vector corr^{n} to $\lambda = 15$

The model matrix P consisting normalized eigent vectors is

$$P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$P^{-1} = P'$ since P is an orthogonal matrix

we have $D = \mathrm{Diag}(0, 3, 15) = P'AP$

The Canonical form is $3y_2^2 + 15y_3^2$