CBCS SCHEME

USN | | | | | | | | | | |

17MAT21

Second Semester B.E. Degree Examination, Jan./Feb. 2021

Engineering Mathematics – II

Time: 3 hrs. Max. Marks: 100

Note: Answer any FIVE full questions, choosing ONE full question from each module.

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

Module-1

1 a. Solve $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^{3x}$ **(06 Marks)**

b. Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 3\text{Sin}x$ **(07 Marks)**

c. Solve by the method of undetermined coefficients
$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$ **(07 Marks)**

OR

2 a. Solve $(D^4 + 4D^3 - 5D^2 - 36D - 36)y = 0$ **(06 Marks)**

b. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ **(07 Marks)**

c. Solve by variation of parameters method $\frac{d^2y}{dx^2} + a^2y = \tan ax$ **(07 Marks)**

Module-2

3 a. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$ **(06 Marks)**

b. Solve $xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\frac{dy}{dx} + xy = 0$ **(07 Marks)**

c. Find the general and singular solution of $y = px - \sin^{-1}p$. **(07 Marks)**

OR

4 a. Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin(2\log(1+x))$ **(06 Marks)**

b. Solve $p^2 + 2py\cot x = y^2$, where $p = \frac{dy}{dx}$ **(07 Marks)**

c. Solve $(px - y)(py + x) = a^2p$ by taking $x^2 = X$ and $y^2 = Y$. **(07 Marks)**

Module-3

5 a. Form the partial differential equation from $xyz = \phi(x + y + z)$ **(06 Marks)**

b. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$ by direct integration. **(07 Marks)**

c. Find all possible solutions of the one-dimensional heat equation $U_t = c^2 U_{xx}$ by the method of separation of variables. **(07 Marks)**

OR

6 a. Form the partial differential equation from $z = f(x + at) + g(x - at)$, where a is a constant. **(06 Marks)**

b. Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$, given that at $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$ and $z = 0$. **(07 Marks)**

c. With suitable assumptions, derive the one dimensional wave equation as $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ **(07 Marks)**

Module-4

7 a. Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y\,dx\,dy$ by changing the order of integration. **(06 Marks)**

b. Evaluate $\int_{-1}^{1} \int_0^{z} \int_{x-z}^{x+z} (x + y + z)\,dx\,dy\,dz$ **(07 Marks)**

c. Show that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ **(07 Marks)**

OR

8 a. Evaluate $\int_0^a \int_y^a \frac{x}{x^2 + y^2}\,dx\,dy$ by changing to polar coordinates. **(06 Marks)**

b. Using double integration find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. **(07 Marks)**

c. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **(07 Marks)**

Module-5

9 a. Find Laplace transform of t(Sinat + Cosat) **(06 Marks)**

b. Find the Laplace transform of the periodic function of period 2a given by

$$f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases}$$ **(07 Marks)**

c. Using convolution theorem find $L^{-1}\left[\frac{s}{(s^2 + a^2)^2}\right]$ **(07 Marks)**

OR

10 a. Express $f(t) = \begin{cases} \text{Cos}t, & 0 < t < \pi \\ \text{Cos}2t, & \pi < t < 2\pi \\ \text{Cos}3t, & t > 2\pi \end{cases}$ in terms of unit-step function and hence find L(f(t)). **(06 Marks)**

b. Find the inverse Laplace transform of

i) $\frac{s^2 - 3s + 4}{s^3}$ and ii) $\frac{s + 2}{s^2 - 4s + 13}$ **(07 Marks)**

c. Solve by Laplace transform method $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$ with $x = 2$, $\frac{dx}{dt} = -1$ at $t = 0$. **(07 Marks)**

* * * * *

Solution of Engineering Mathematics - II
17MAT21
Jan / Feb 2021

$\boxed{\text{Module - 1}}$

Q. 1. (a) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^{3x}$

$(D^3 - 6D^2 + 11D - 6)\, y = e^{3x}$

$y = y_c + y_p$

for y_c, $f(m) = 0$

$m^3 - 6m^2 + 11m - 6 = 0$

$m = 1, 2, 3$

$y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

$y_p = \frac{1}{D^3 - 6D^2 + 11D - 6} e^{3x} = x \frac{1}{3D^2 - 12D + 11} e^{3x}$

$= x \frac{1}{27 - 36 + 11} e^{3x} = \frac{x}{2} e^{3x}$

$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} + \frac{x}{2} e^{3x}$

(b) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 3\sin x$

$y = y_c + y_p$

for y_c, $f(m) = 0 \Rightarrow m^2 + 4m + 4 = 0$

$(m+2)^2 = 0 \Rightarrow m = -2, -2$

$y_c = (c_1 + c_2 x) e^{-2x}$

$$y_p = \frac{1}{D^2 + 4D + 4} 3\sin x = 3 \frac{1}{-1 + 4D + 4} \sin x$$

$$= 3 \frac{1}{4D+3} \sin x = 3 \frac{4D-3}{16D^2 - 9} \sin x$$

$$= -\frac{3}{25}(4D-3)\sin x = -\frac{12}{25}\sin x + \frac{9}{25}\cos x$$

(c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$

$\Rightarrow (D^2 + 2D + 4)y = 2x^2 + 3e^{-x}$

$m^2 + 2m + 4 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4-16}}{2}$

$m = -1 \pm \sqrt{3}$

$y_c = e^{-x}\{c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\}$

Let $y_p = a + bx + cx^2 + de^{-x}$

a, b, c, d are such that

$y_p'' + 2y_p' + 4y_p = 2x^2 + 3e^{-x}$

$$2c + de^{-x} + 2b + 4cx - 2de^{-x} +$$
$$4a + 4bx + 4cx^2 + 4de^{-x} = 2x^2 + 3e^{-x}$$

$$(4a+2b+2c) + (4b+4c)x + 4cx^2 +$$
$$\cancel{e^{-x}}(\cdot\, 3de^{-x} = 2x^2 + 3e^{-x}$$

$$\Rightarrow \quad 4a+2b+2c = 0 \quad \Rightarrow a = 0$$
$$4b+4c = 0 \quad \Rightarrow b = -1/2$$
$$4c = 2 \quad \Rightarrow c = 1/2$$
$$3d = 3 \quad \Rightarrow d = 1$$

$$\therefore \quad y_p = \frac{x^2}{2} - \frac{x}{2} + e^{x}$$

Q2 (a) $(D^4 + 4D^3 - 5D^2 - 36D - 36)y = 0$

$f(m) = 0 \Rightarrow m^4 + 4m^3 - 5m^2 - 36m - 36 = 0$

$m = \pm 3, -2, -2$

$$y = C_1 e^{3x} + C_2 e^{-3x} + (C_3 + C_4 x)e^{-2x}$$

(b) $$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$$

$$(D^2 + D)y = x^2 + 2x + 4$$

$f(m) = 0 \Rightarrow m^2 + m = 0 \Rightarrow m = 0, -1$

$y_c = C_1 + C_2 e^{-x}$

$$y_p = \frac{1}{D^2 + D} (x^2 + 2x + 4)$$

$$D + D^2 \overline{) x^2 + 2x + 4} \left(\frac{x^3}{3} + 4x \right.$$

$x^2 + 2x$

4

$4 + 0$

0

$$y = C_1 + C_2 e^{-x} + \frac{x^3}{3} + 4x$$

(c) $(D^2 + a^2) y = \tan ax$

$m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$y_c = C_1 \cos ax + C_2 \sin ax$

$y = A \cos ax + B \sin ax$

$$A = -\int \frac{y_2 \phi}{W} dx + k_1$$

$$B = \int \frac{y_1 \phi}{w} dx + k_2 , \quad w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$w = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

$$A = -\int \frac{\cos ax \cdot \tan ax}{a} dx + k_1$$

$$= -\frac{1}{a}\left[\frac{-\cos ax}{a}\right] + k_1 = \frac{\cos ax}{a^2} + k_1$$

$$B = \int \frac{\sin ax \cdot \tan ax}{a} dx + k_2$$

$$= \frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax} dx + k_2$$

$$= \frac{1}{a}\left\{\int \sec ax \, dx - \int \cos ax \, dx\right\} + k_2$$

$$= \frac{1}{a^2}\log(\sec ax + \tan ax) - \frac{\sin ax}{a^2} + k_2$$

$$y = k_1 \cos ax + k_2 \sin ax + \frac{\cos^2 ax}{a^2} - \frac{\sin ax}{a^2} \log(\sec ax + \tan ax) - \frac{\sin^2 ax}{a^2}$$

$\boxed{\text{Module-2}}$

Q3. (a) $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Let $\log x = z \Rightarrow x = e^z$

$x\frac{d}{dx} \equiv D$, $x^2\frac{d^2}{dx^2} \equiv D(D-1)$, $D = \frac{d}{dz}$

$[D(D-1) - D + 1]\, y = z$

$[D^2 - 2D + 1]\, y = z$

$f(m) = m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

$y_c = (C_1 + C_2 z)\, e^z$

$y_p = \frac{1}{D^2 - 2D + 1}(z)$

$$1-2D+D^2 \,)\, z \,(\, z+2$$

$$z-2$$

$$+$$

$$2$$

$$2$$

$$\times$$

$$y=(c_1+c_2 z)e^{z}+z+2$$

$$\Rightarrow y=(c_1+c_2\log x)x+\log x+2$$

b) $$xy\left(\frac{dy}{dx}\right)^2-(x^2+y^2)p+xy=0$$

$$xyp^2-(x^2+y^2)p+xy=0$$

$$p=\frac{(x^2+y^2)\pm\sqrt{(x^2+y^2)^2-4x^2y^2}}{2xy}$$

$$=\frac{(x^2+y^2)\pm(x^2-y^2)}{2xy}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

On integration

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$(y^2 - x^2 - c) = 0$$

$$\log y = \log x + \log c$$

$$y = cx$$

$$y - cx = 0$$

$\therefore \quad (y^2 - x^2 - c)(y - cx) = 0$

(c) $\quad y = px - \sin^{-1}(p)$

Given eq^n is clairaut's eq^n

$\therefore$ The G.S. is $\quad y = cx - \sin^{-1}(c)$

for S.S. diff. partially w.r.t. c

$$0 = x - \frac{1}{\sqrt{1-c^2}} \Rightarrow x^2 = \frac{1}{1-c^2}$$

$$1 - c^2 = \frac{1}{x^2} \Rightarrow c^2 = 1 - \frac{1}{x^2} \Rightarrow c = \frac{\sqrt{x^2-1}}{x}$$

$$y = \sqrt{x^2-1} - \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$$

Q7. (a) $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin\{2\log(1+x)\}$

$\log(1+x) = z \Rightarrow x = e^z - 1$

$(1+x)\frac{d}{dx} \equiv D$ $\qquad D \equiv \frac{d}{dz}$

$(1+x^2)\frac{d^2}{dx^2} \equiv D(D-1)$

$[D(D-1) + D + 1]\, y = \sin 2z$

$(D^2+1)\, y = \sin 2z$

$y = y_c + y_p$

$f(m) = 0 \Rightarrow m^2+1=0 \Rightarrow m = \pm i$

$y_c = C_1 \cos z + C_2 \sin z$

$y_p = \frac{1}{D^2+1}\sin 2z = -\frac{1}{3}\sin 2z$

$y = C_1 \cos z + C_2 \sin z - \frac{1}{3}\sin 2z$

$y = C_1 \cos \log(1+x) + C_2 \sin(\log(1+x)) - \frac{1}{3}\sin[2\log(1+x)]$

(b) $p^2 + 2py\cot x - y^2 = 0$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= -y\cot x \pm y\operatorname{cosec} x$$

$\frac{dy}{dx} = -y(\cot x - \operatorname{cosec} x)$	$\frac{dy}{dx} = -y(\cot x + \operatorname{cosec} x)$
$+\frac{dy}{y} = -(\cot x - \operatorname{cosec} x)\,dx$	$+\frac{dy}{y} = -(\cot x + \operatorname{cosec} x)\,dx$

On integration of L.H.S

$-\log y = -\log\sin x + \log(\operatorname{cosec} x + \cot x) + \log c$

$c(\operatorname{cosec} x + \cot x)\sin x = 1/y$.

$$c(1+\cos x) = \frac{1}{y}$$

On integrating R.H.S.

$-\log y = \log\sin x - \log(\operatorname{cosec} x + \cot x)$

$+ \log c$

(c) $(px-y)(py+x)=a^2p$,

letting $x^2=X$, $y^2=Y$

~~2xdx=~~ $p=\frac{dy}{dx}=\frac{dy}{dY}\frac{dY}{dX}\frac{dX}{dx}$

$p=\frac{x}{y}\frac{dY}{dX}=\frac{\sqrt{X}}{\sqrt{Y}}P$

$\left[\frac{\sqrt{X}}{\sqrt{Y}}P.\sqrt{X}-\sqrt{Y}\right]\left[\frac{\sqrt{X}}{\sqrt{Y}}P.\sqrt{Y}+\sqrt{X}\right]=a^2\frac{\sqrt{X}P}{\sqrt{Y}}$

$\frac{[PX-Y]}{\sqrt{Y}}[P+1]\sqrt{X}=a^2\frac{\sqrt{X}}{\sqrt{Y}}P$

$PX-Y=\frac{a^2P}{P+1}\Rightarrow Y=PX-\frac{a^2P}{P+1}$

which is clairaut's eqn

$\therefore$ The G.S. $\Rightarrow$ $Y=cX-\frac{a^2c}{c+1}$

$\therefore$ $y^2=cx^2-\frac{a^2c}{c+1}$ Ans

Module - 3

Q 5(a) $xyz = \phi(x+y+z)$ — ①

$(xp+z)y = (1+p)\phi'(x+y+z)$ — ②

$(yq+z)x = (1+q)\phi'(x+y+z)$ — ③

from ② & ③ $\dfrac{(xp+z)y}{(yq+z)x} = \dfrac{1+p}{1+q}$

$(xyp+yz)(1+q) = (xyq+xz)(1+p)$

$xyp + yz + \cancel{xypq} + yzq = xyq + xz + \cancel{xypq} + pxz$

$\boxed{xp(y-z) + yq(y-x) = z(x-z)}$ Ans

(b) $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

$\Rightarrow \dfrac{\partial^2 z}{\partial x^2} + 6xy^3 + \cos(2x-y) = f(x)$

$\Rightarrow \dfrac{\partial z}{\partial x} + 3x^2y^3 + \dfrac{\sin(2x-y)}{2} = \int f(x)\,dx + g(y)$

Q5(c)

Find the solution of one dimensional heat equation using variable separable method.

Solution ⇒ The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{——— (1)}$$

Let $u(x,t) = X(x)\,T(t)$ be the solution of equation (1).

Then $\frac{\partial}{\partial t}(XT) = c^2 \frac{\partial^2}{\partial x^2}(XT) \Rightarrow X\frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$

$$\Rightarrow \frac{1}{c^2 T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k \text{ (let)}$$

Now, we can separate the variables and get two equations

$$\frac{dT}{dt} = c^2 kT \quad ; \quad \frac{d^2X}{dx^2} = kX$$

$$(D - c^2 k)T = 0 \quad ; \quad (D^2 - k)X = 0$$

Case-(i) when $k = 0$

$$\frac{dT}{dt} = 0 \qquad\qquad \frac{d^2X}{dx^2} = 0$$

$T = c_1$

$$\frac{dX}{dx} = c_2$$

$$X = c_2 x + c_3$$

∴ The solution is given by

$$u(x,t) = c_1(c_2 x + c_3) = Ax + B.$$

$A = c_1 c_2 \quad B = c_1 c_3$

Case (ii) when k is positive i.e. $k = p^2$ (say)

$\Rightarrow (D - p^2c^2)T = 0$; $(D^2 - p^2)X = 0$

$\Rightarrow m - p^2c^2 = 0$ $\qquad m^2 - p^2 = 0$

$m = p^2c^2$ $\qquad m = \pm p$

$T = c_1 e^{p^2c^2t}$ $\qquad X = c_2 e^{px} + c_3 e^{-px}$ [$\because$ roots are real & distinct]

$\Rightarrow u(x,t) = c_1 e^{p^2c^2t}\{c_2 e^{px} + c_3 e^{-px}\}$

$= e^{p^2c^2t}\{A_1 e^{px} + B_1 e^{-px}\}$, $A_1 = c_1c_2$, $B_1 = c_1c_3$

Case-(iii) when k is negative i.e. $k = -p^2$ (say)

$(D + p^2c^2)T = 0$ $\qquad (D^2 + p^2)X = 0$

A.E. is given by $\qquad m^2 + p^2 = 0$

$m + p^2c^2 = 0$ $\qquad m = \pm ip$

$m = -p^2c^2$

$X = c_2 \cos px + c_3 \sin px$

$T = c_1 e^{-p^2c^2t}$

$\Rightarrow u(x,t) = c_1 e^{-p^2c^2t}\{c_2 \cos px + c_3 \sin px\}$

$= e^{-p^2c^2t}\{A' \cos px + B' \sin px\}$, $A' = c_1c_2$, $B' = c_1c_3$

Q6.(a) $z = f(x+at) + g(x-at)$

$p = f'(x+at) + g'(x-at)$

$\frac{\partial z}{\partial t} = a f'(x+at) - a g'(x-at)$

$r = \frac{\partial p}{\partial x} = f''(x+at) + g''(x-at)$

$\frac{\partial^2 z}{\partial t^2} = a^2 f''(x+at) + a^2 g''(x-at)$

$\Rightarrow \boxed{\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}}$ Ans

(b) $\frac{\partial^2 z}{\partial x^2} = a^2 z \Rightarrow \frac{d^2 z}{dx^2} - a^2 z = 0$

$(D^2 - a^2) z = 0$

$m = \pm a$

$z = c_1 e^{ax} + c_2 e^{-ax}$

$z = f(y) e^{ax} + g(y) e^{-ax}$

Given that $x = 0, z = 0$

$0 = f(y) + g(y) \Rightarrow f(y) = -g(y)$

$x = 0, \frac{\partial z}{\partial x} = a \sin y$

$\therefore \frac{\partial z}{\partial x} = a f(y) e^{ax} - a g(y) e^{-ax}$

$a \sin y = a f(y) - a g(y)$

$\Rightarrow a \sin y = -2a g(y) \Rightarrow g(y) = -\frac{\sin y}{2}$

$f(y) = \frac{\sin y}{2}$

$\Rightarrow z = \sin y \left(\frac{e^{ax} + e^{-ax}}{2}\right) = \sin y \sinh ax$

Derivation of One dimensional wave Equation

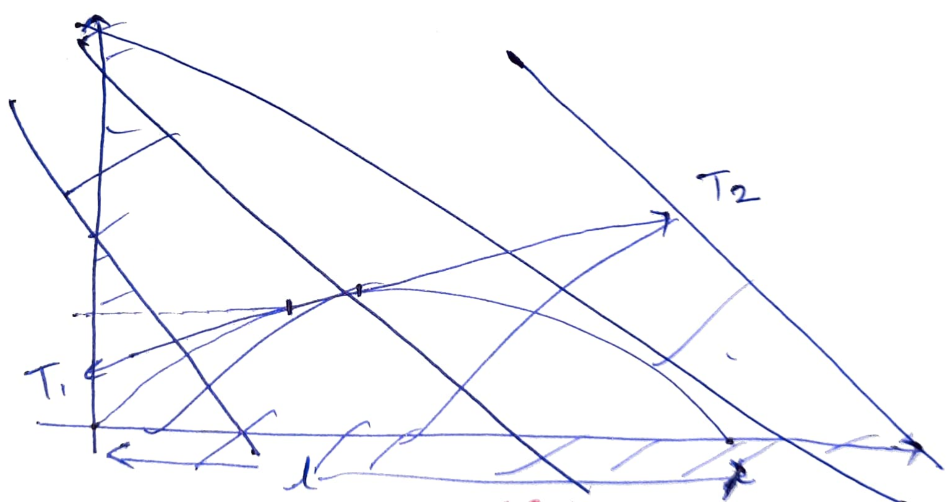
Consider a flexible string tightly streched between two fixed points at a distance 'l' apart. Let ρ be the mass per unit length of the string.

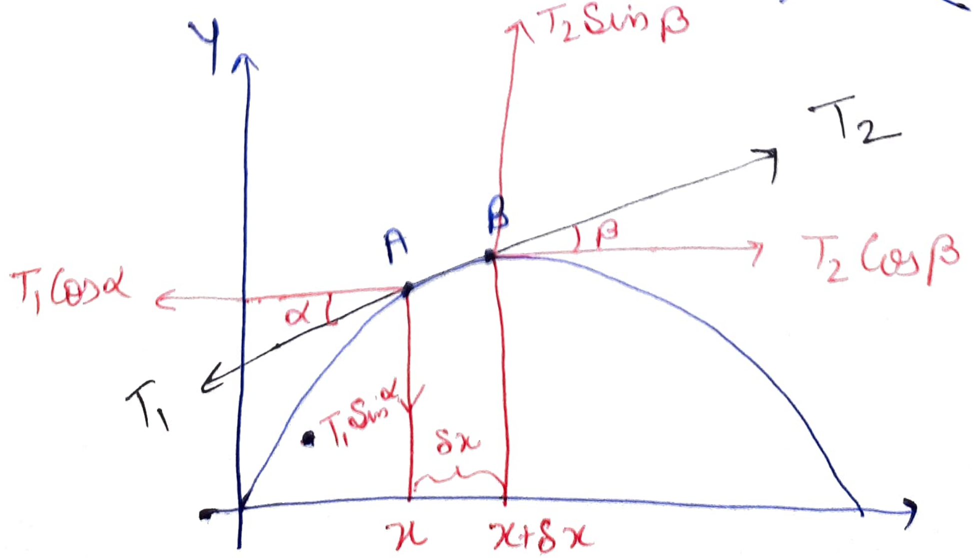
We assume the following conditions

i) The tension T of the string is same throughout.

ii) The effect of gravity is ignored due to large tension.

iii) The motion of string is in small transverse vibration (i.e. in perpendicular direction, no vibration horizontally





Now, we consider a small element AB of length δx.

Let T_1 and T_2 be the tensions at points A and B and α and β are the angles made by T_1 and T_2 with horizontal

∵ There is no motion in the horizontal direction, the horizontal components will cancel each other.

∴ By figure, $T = T_1 \cos\alpha = T_2 \cos\beta$ —— (1)

Hence, the resultant force acting vertically upwards is $T_2 \sin\beta - T_1 \sin\alpha$.. —— (2)

By Newton's ~~law~~ second law of motion.

force = mass × acceleration

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho \delta x) \frac{\partial^2 u}{\partial t^2}$$ [where u is the displacement along 'x']

Dividing throughout by 'T'.

$$\frac{T_2}{T} \sin\beta - \frac{T_1}{T} \sin\alpha = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2}$$

By (1)

$$\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha} = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{P}{T}\delta x \frac{\partial^2 u}{\partial t^2}$$

$\because$ $\tan\beta$ and $\tan\alpha$ represents slopes at $B(x+\delta x)$ and $A(x)$ respectively

$\therefore \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ $\quad \tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{P}{T}\delta x \frac{\partial^2 u}{\partial t^2}$$

$$\lim_{\delta x \to 0} \left\{ \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} \right\} = \frac{P}{T}\frac{\partial^2 u}{\partial t^2}$$

L.H.S. is the derivative of $\frac{\partial u}{\partial x}$ i.e. $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

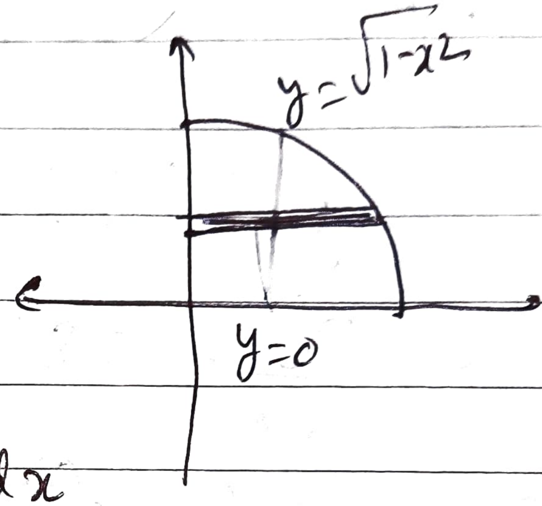
$$\frac{\partial^2 u}{\partial x^2} = \frac{P}{T}\frac{\partial^2 u}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial t^2} = \frac{T}{P}\frac{\partial^2 u}{\partial x^2}$$

$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$ is the one dimensional wave equation where $c^2 = \frac{T}{P}$.

Module - 4

Q7. (a) Given $I = \int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy$

after changing order



$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} x^3 y \, dy \, dx$$

$$= \int_0^1 x^3 \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \frac{1}{8}\int_0^1 x^2(1-x^2)\,dx$$

$$= \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{15}$$

(b) $I = \int_{-1}^{1} \int_0^{z} \int_{x-z}^{x+z} (x+y+z) \, dy \, dx \, dz$

$$= \int_{-1}^{1}\int_{0}^{z} \left\{ xy + \frac{y^2}{2} + yz \right\}_{y=x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \left\{ x(x+z) + \frac{(x+z)^2}{2} + z(x+z) - x(x-z) - \frac{(x-z)^2}{2} - z(x-z) \right\} dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \left\{ 2xz + 2xz + 2z^2 \right\} dx\,dz$$

$$= \int_{-1}^{1} \left\{ 2x^2 z + 2xz^2 \right\}_{0}^{z} dz = \int_{-1}^{1} 4z^3\,dz$$

$$= \left[z^4 \right]_{-1}^{1} = 1 - 1 = 0$$

7 (c) By the defination of Beta and Gamma functions, we have

$$\beta(m,n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{--- (1)}$$

$$\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx \quad \text{--- (2)}$$

$$\Gamma m = 2\int_0^\infty e^{-y^2} y^{2m-1} dy \quad \text{--- (3)}$$

$$\therefore \cancel{\Gamma m + \Gamma n} \quad \Gamma(m+n) = 2\int_0^\infty e^{-r^2} r^{2(m+n)-1} dr \quad \text{--- (4)}$$

$$\Gamma m . \Gamma n = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} dx dy \quad \text{--- (5)}$$

put $x = r\cos\theta$, $y = r\sin\theta$ $dx dy = r dr d\theta$

r is varies from 0 to ∞ and

θ is varies from 0 to $\pi/2$

From eq (5)

$$\Gamma m . \Gamma n = 4\int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1} r dr d\theta$$

$$= \left[2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} dr\right]\left[2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta\right]$$

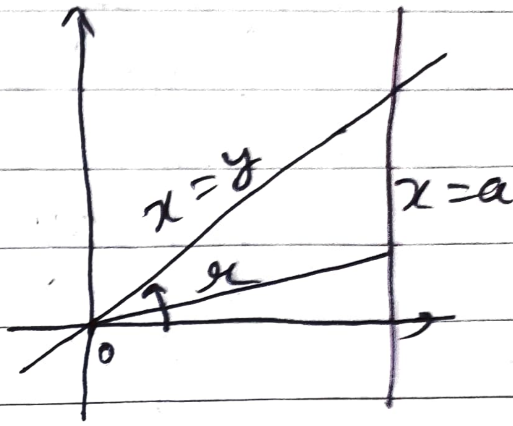
$= \Gamma m+n \cdot \beta(m,n)$ from eq (1) & (4)

$$\boxed{\therefore \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}}$$

Q8(a)

$$I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$$

changing to polar



$x = r\cos\theta,\ y = r\sin\theta$

$dx\,dy \rightarrow r\,d\theta\,dr$

$r = 0$ to $r = a\sec\theta$

$\theta = 0$ to $\theta = \pi/4$

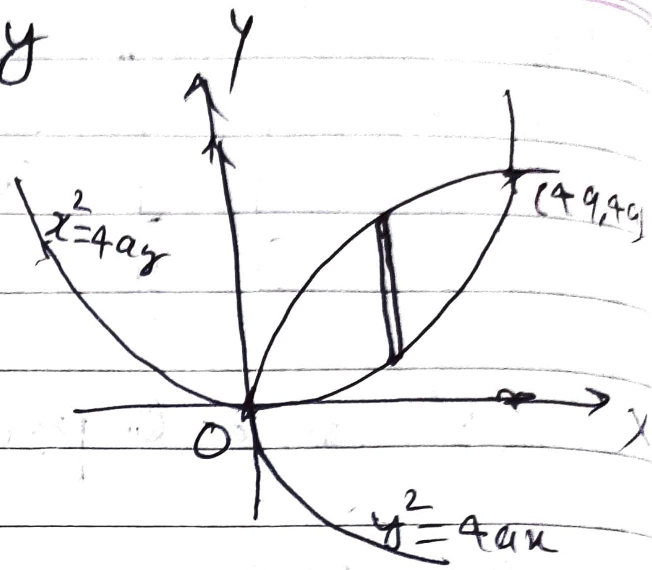
$$I = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r\cos\theta}{r^2} \cdot r\,d\theta\,dr$$

$$= \int_0^{\pi/4} \cos\theta \left\{ \int_0^{a\sec\theta} dr \right\} d\theta$$

$$= \int_0^{\pi/4} \cos\theta\,(a\sec\theta)\,d\theta = a\int_0^{\pi/4} d\theta =$$

$$\boxed{I = \frac{\pi a}{4}}$$ ans

Q8(b) $y^2 = 4ax$ & $x^2 = 4ay$



$$\text{Area} = \iint dx\,dy$$

$$= \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$

$$= \int_0^{4a} [y]_{x^2/4a}^{2\sqrt{ax}} dx = \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a}\right] dx$$

$$= \left[2\sqrt{a}\cdot\frac{x^{3/2}}{(3/2)} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \left[\frac{4}{3}\cdot\sqrt{a}\cdot(4a)(2\sqrt{a}) - \frac{(4a)^3}{12a}\right]$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2 \text{ sq unit}$$

(8) (c) By the defination of Gamma function,

$$\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1}\,dx$$

$$\therefore \Gamma\tfrac{1}{2} = 2\int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-y^2}\,dy$$

$$\therefore \left(\Gamma\tfrac{1}{2}\right)^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = 4I \text{ (say)} \quad \text{—— (1)}$$

put $x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2+y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$

Since x, y varies from 0 to ∞, 'r' also varies from 0 to ∞. In the 1st quadrant 'θ' varies from 0 to $\pi/2$. Hence from eq (1)

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

put $r^2 = t$

$\Rightarrow r\,dr = dt/2$

$t \to 0$ to ∞

$$\therefore I = \int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[e^{-t}\right]_0^\infty d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}(0-1)\,d\theta = \frac{1}{2}\left[\theta\right]_0^{\pi/2}$$

$$= \pi/4$$

$\boxed{\therefore I = \frac{\pi}{4}}$ $\to$ (1) gives $\boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$

Module- 5

Q9(a) $L\{t[\sin(at)+\cos(at)]\}$

$f(t)=\sin at+\cos at$

$$L[f(t)] = \frac{a}{s^2+a^2} + \frac{s}{s^2+a^2}$$

$$L\{t\,f(t)\} = -\frac{d}{ds}\left[\frac{a}{s^2+a^2} + \frac{s}{s^2+a^2}\right]$$

$$= -\left[\frac{-2s}{(s^2+a^2)^2} + \frac{(s^2+a^2)-s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{2s}{(s^2+a^2)^2} - \frac{(a^2-s^2)}{(s^2+a^2)^2} = \frac{(s^2+2s-a^2)}{(s^2+a^2)^2}$$

Q9(b) We have $T=2a$ and

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a t\, e^{-st}\, dt + \int_a^{2a} (2a-t)\, e^{-st}\, dt \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[\frac{-t e^{-st}}{s} \right]_0^a - \left(\frac{e^{-st}}{s^2} \right)_0^a \right\}$$

$$\frac{1}{1-e^{-2as}} \left[\frac{(2a-t)\, e^{-st}}{-s} + \frac{(e^{-st})}{s^2} \right]_a^{2a}$$

$$= \frac{1}{1-e^{-2as}} \left[\cancel{-\frac{a}{s} e^{-as}} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \cancel{\frac{a}{s} e^{-as}} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{s^2(1-e^{-2as})} \left[1 - 2e^{-as} + e^{-2as} \right]$$

$$= \frac{(1-e^{-as})^2}{s^2(1+e^{-as})(1-e^{-as})} = \frac{(1-e^{-as})}{s^2(1+e^{-as})} \quad \text{Ans}$$

Q9 (c) Let $\bar{f}(s) = \frac{1}{s^2+a^2}$; $\bar{g}(s) = \frac{s}{s^2+a^2}$

$\therefore$ $f(t) = \frac{\sin at}{a}$, $g(t) = \cos at$

By convolution

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_{u=0}^{t} \frac{\sin au}{a} \cdot \cos(at-au)\, du$$

$$= \frac{1}{2a} \int_0^t [\sin at + \sin(2au - at)]\, du$$

$$= \frac{1}{2a} \sin at\, (u)_0^t - \left(\frac{\cos(2au-at)}{2a}\right)_0^t$$

$$= \frac{t \sin at - \cos at + \cos at}{2a} = \frac{t \sin at}{2a}$$

Q10(a) $f(t) = \cos t + (\cos 2t - \cos t)\, u(t-\pi) + (\cos 3t - \cos 2t)\, u(t-2\pi)$

$$L f(t) = L\{\cos t\} + L\{(\cos 2t - \cos t)\, u(t-\pi)\}$$
$$+ L\{(\cos 3t - \cos 2t)\, u(t-2\pi)\}$$

Let $F(t-\pi) = \cos 2t - \cos t$

$G(t-2\pi) = \cos 3t - \cos 2t$

$F(t) = \cos 2t + \cos t \; ; \; G(t) = \cos 3t - \cos 2t$

$\bar{F}(s) = \dfrac{s}{s^2+4} + \dfrac{s}{s^2+1} \; ; \; \bar{G}(s) = \dfrac{s}{s^2+9} - \dfrac{s}{s^2+4}$

$L\left[F(t-\pi)u(t-\pi)\right] = e^{-\pi s}\,\bar{F}(s)$ and

$L\left[G(t-2\pi)\,u(t-2\pi)\right] = e^{-2\pi s}\,\bar{G}(s)$

$L\left[F(t-\pi)u(t-\pi)\right] = e^{-\pi s}\,\bar{F}(s)$ and

$L\left[G(t-2\pi)u(t-2\pi)\right] = e^{-2\pi s}\,\bar{G}(s)$

$L\left[(\cos 2t + \cos t)\,u(t-\pi)\right] = e^{-\pi s}\left[\dfrac{s}{s^2+4} + \dfrac{s}{s^2+1}\right]$

$L\left[(\cos 3t - \cos 2t)\,u(t-2\pi)\right] = e^{-2\pi s}\left[\dfrac{s}{s^2+9} - \dfrac{s}{s^2+4}\right]$

$\therefore \; L\left[f(t)\right] = \dfrac{s}{s^2+1} + e^{-\pi s}\left[\dfrac{s}{s^2+4} + \dfrac{s}{s^2+1}\right]$

$+ \; e^{-2\pi s}\left[\dfrac{s}{s^2+9} - \dfrac{s}{s^2+4}\right]$

Q10(b) i) $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left\{-\frac{3}{s^2}\right\} + L^{-1}\left(\frac{4}{s^3}\right)$

$$= 1 - 3t + \frac{4t^2}{2!} = 1-3t+2t^2$$

ii) $L^{-1}\left[\frac{s+2}{(s^2-4s+13}\right] = L^{-1}\left\{\frac{s+2}{(s-2)^2+9}\right\}$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+9} + \frac{4}{(s-2)^2+9}\right\}$$

$$= e^{2s}\, L^{-1}\left\{\frac{s}{s^2+9}\right\} + 4e^{2s}\, L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= e^{2s}\left[\cos 3t\right] + \frac{4}{3} e^{2s} \sin 3t$$

Q10(c) $L[x''-2x'+x] = L[e^{t}]$

$$s^2\bar{x}(s) - s\,x(0) - x'(0) - 2s\,\bar{x}(s) + 2x(0) + \bar{x}(s) = \frac{1}{s-1}$$

$$\Rightarrow [s^2-2s+1]\,\bar{x}(s) - 1 + 4 = \frac{1}{s-1}$$

$$(s-1)^2\,\bar{x}(s) = \frac{1}{(s-1)^2} - 3$$

$$\bar{x}(s) = \frac{1}{(s-1)^4} - \frac{3}{(s-1)^2}$$

$$L^{-1}\{\bar{x}(s)\} = L^{-1}\left\{\frac{1}{(s-1)^4}\right\} - 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$x(t) = e^t L^{-1}\left\{\frac{1}{s^4}\right\} - 3e^t L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$x(t) = e^t\left\{\frac{t^3}{3!} - 3t\right\}$$