29/7/2021

IAT-3 Solution

1) A random variable X has the following probability mass function.

$X = x_i$	0	1	2	3	4	5	6	7	8
$P(x)$	a	3a	5a	7a	9a	11a	13a	15a	17a

$$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

since $\sum P(X) = 1$. for probability mass function.

$$\therefore 81a = 1$$

$$a = \frac{1}{81} \quad \text{(option b)}$$

2) Let X be a random variable with PDF given by $f(x) = \begin{cases} cx^2, & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-1}^{1} cx^2\,dx = 1$$

$$\Rightarrow \left[\frac{cx^3}{3}\right]_{-1}^{1} = 1$$

$$\Rightarrow \frac{c}{3}\left[1^3 - (-1)^3\right] = 1$$

$$\Rightarrow c = \frac{3}{2} \quad \text{(option a)}$$

3) An agent sells life insurance policies to five equally aged, healthy people. According to recent data, the probability of a person living in these conditions for 30 years or more is $\frac{2}{3}$. Calculate the probability after 30 years that atleast 3 people are still living.

$$p = \frac{2}{3} \Rightarrow q = 1-p = \frac{1}{3}$$

$$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5)$$

$$= {}^5C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 + {}^5C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^1$$

$$+ {}^5C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^0$$

$$= 0.791 \qquad \text{(option c)}$$

4) In a particular time duration, one telephone line in every five is engaged in a conversation: What is the probability that when 10 telephone numbers are chosen at random, only two are in use.

$$p = \frac{1}{5} \Rightarrow q = 1-p = \frac{4}{5}.$$

$$P(X=2) = {}^{10}C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^8$$

$$= 0.3020 \qquad \text{(option d)}$$

5) 3 students come to attend the class on an average find the probability for exactly 4 students to attend the classes on the following day.

Average = 3

$\Rightarrow$ Mean $\mu = 3 = m$

$$P(X=4) = \frac{m^x e^{-m}}{x!}$$

$$= \frac{3^4 e^{-3}}{4!}$$

$$= 0.1680 \quad \text{(option b)}$$

6) The life of compressor manufactured by a company is known to be 200 months on an average following an exponential distribution. Find the probability that the life of a compressor of that company is between 100 months and 25 years.

$$\text{Mean} = \frac{1}{\alpha}$$

$$\text{Given } \frac{1}{\alpha} = 200 \Rightarrow \alpha = \frac{1}{200}.$$

We have $f(x) = \alpha e^{-\alpha x}$

$$P(100 < x < 300) = \int_{100}^{300} f(x)\,dx$$

$$= \int_{100}^{300} \frac{1}{200} e^{-\frac{1}{200}x}\,dx$$

$$= 0.3834$$

7) Find z_1 if $P(z<z_1)=0.35$

$$P(z<z_1) = 0.35$$

$$\Rightarrow 0.5 + \phi(z_1) = 0.35$$

$$\Rightarrow \phi(z_1) = 0.35 - 0.5$$

$$= -0.15$$

$$\phi(z_1) = -0.15 = \phi(-0.39)$$

$$z_1 = -0.39 \quad \text{(option a)}$$

8) If $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$, then,

since C_1 and C_2 are 2 simple closed curves with C_2 lying entirely within C_1 and $f(z)$ is analytic on C_1, C_2 and in the region bounded by C_1, C_2. (option c)

9) To fit a normal distribution, mean and standard deviation are required parameters

(option d)

10) From Generalized Cauchy's integral formula,

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

(option d)

11) A travel agency has 2 cars which it hires daily. The number of demands for a car on each day is distributed as a Poisson variate with mean 1.5. Find the probability that on a day a demand is refused.

Mean $m = 1.5$

We have $P(x) = \dfrac{m^x e^{-m}}{x!}$

Probability that on a day a demand is refused

$= P(x > 2) = 1 - \left[P(x=0) + P(x=1) + P(x=2)\right]$

$= 1 - e^{-m}\left(1 + \dfrac{m}{1!} + \dfrac{m^2}{2!}\right)$

$= 0.1912.$ (option b)

12) $2P(x=1) = P(x=2)$

$$P(x) = \frac{m^x e^{-m}}{x!}$$

$$\frac{2m^1 e^{-m}}{1!} = \frac{m^2 e^{-m}}{2!}$$

$$4\cancel{e^{-m}} = m\cancel{e^{-m}}$$

$\Longrightarrow m = 4$

$\Longrightarrow$ Variance $= 4$ (option c)

13) Find the binomial distribution function for $x=5$ which has mean 2 and variance $\frac{4}{3}$.

mean $= 2 = np$

Variance $= \frac{4}{3} = npq$.

$\therefore q = \frac{2}{3} \Longrightarrow p = \frac{1}{3}$

$\Longrightarrow n = 6.$

Binomial distribution function is given by

$$P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$$P(x=5) = {}^{6}C_{5} \left(\frac{1}{3}\right)^{5} \left(\frac{2}{3}\right)^{6-5}$$

(option c).

14) If x is exponential variate with mean 6,

$$P(-\infty < x < 10) = \int_0^{10} \frac{1}{6} e^{-\frac{x}{6}} dx$$

$$= -\left(e^{-\frac{x}{6}}\right)\Big|_0^{10}$$

$$= -e^{-\frac{5}{3}} + 1$$

$$= 0.8111 \qquad \text{(option d)}$$

15) The mean weight of 500 students during a medical examination was found to be 50kgs and standard deviation weight 6kgs. Assuming that the weights are normally distributed, find the number of students having weight between 40 & 50kgs.

$\mu = 50$kgs, $\sigma = 6$kgs.

$P(40 < x < 50) = ?$

We know that the standard normal variate

$$z = \frac{x-\mu}{\sigma} = \frac{x-50}{6}.$$

When $x = 40$, $z = -1.6666 \approx -1.67$

$x = 50$, $z = 0$.

$$P(-1.67 < z < 0) = P(0 < z < 1.67) \qquad \text{(option b)}$$

$$= \phi(1.67)$$

$$= 0.4525.$$

The number of students $= 500 \times 0.4525 = 226$

16) Find the value of z such that area to the right of x is 0.24

$\cancel{P(z \geq 0.24)} =$

$$P(x > x_1) = P(z > z_1)$$
$$= 0.5 - \phi(z_1)$$
$$= 0.5 - \phi(2.4)$$
$$= 0.71 \quad \text{(option b)}$$

17) Find the mean.

$$f(x) = \begin{cases} x e^{-x}, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean } \mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x \cdot x e^{-x}\,dx$$

$$= \int_0^1 x^2 e^{-x}\,dx$$

$$= \frac{x^2 e^{-x}}{(-1)} - \int_0^1 \frac{2x e^{-x}}{(-1)}\,dx$$

$$= \left[-x^2 e^{-x} + 2\frac{x e^{-x}}{(-1)} - \frac{(1) e^{-x}}{(-1)(-1)} \right]_0^1$$

$$= 2 - \frac{5}{e} \quad \text{(option b)}$$

18) Find the variance

x	10	20	30	40
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$p(x) > 0$ and $\sum p(x) = 1$

Variance $V = \sum (x-\mu)^2 \cdot p(x)$

$= 225\left(\frac{1}{8}\right) + 25\left(\frac{3}{8}\right) + 25\left(\frac{3}{8}\right) + 225\left(\frac{1}{8}\right)$

$= \frac{600}{8}$

$= 75$ (option b)

19) $\int_C \frac{1}{z^2 - \pi^2} dz$ $C : |z-1| = 3$

$z = -\pi$ lies outside the given circle

$z = +\pi$ lies ~~out~~ inside the given circle.

$\frac{1}{z^2-\pi^2} = \frac{-1}{2\pi(z+\pi)} + \frac{1}{2\pi(z-\pi)}$

$\int_C \frac{1}{z^2-\pi^2} dz = \int_C \left(\frac{-1}{2\pi}\right) \frac{1}{z+\pi} dz + \int_C \frac{1}{2\pi(z-\pi)} dz$

$= 0 + 2\pi i f(\pi) \times \frac{1}{2\pi}$

$= \underline{i}$ (option c)

20) $\int_C \frac{e^{2z}}{(z-2)(z+1)} dz \qquad |z| = 1.5$

Consider

$$\frac{1}{(z-2)(z+1)} = \frac{A}{z-2} + \frac{B}{z+1}$$

$$\frac{1}{(z-2)(z+1)} = \frac{1}{3(z-2)} - \frac{1}{3(z+1)}.$$

$$\int_C \frac{e^{2z}}{(z-2)(z+1)} dz = \frac{1}{3}\int_C \frac{e^{2z}}{z-2} dz - \frac{1}{3}\int_C \frac{e^{2z}}{z+1} dz$$

$$= 0 - \frac{1}{3}\int_C \frac{e^{2z}}{z+1} dz$$

(by Cauchy's theorem)

$$= -\frac{1}{3} 2\pi i \, f(-1)$$

$$= -\frac{2\pi i}{3e^2}$$ (option d)