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CMR Institute of Technology, Bangalore
DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING
III - INTERNAL ASSESSMENT

Semester: 4-CBCS 2018 Date: 30 Jul 2021
Subject: ENGINEERING STATISTICS AND LINEAR ALGEBRA (18EC44)
Faculty: Dr Meenakshi Krishnan
Time: 01:00 PM - 02:30 PM Max Marks: 50

Instructions to Students:

First question is compulsory. Answer any six from Q2 to Q9.

PART_HYPHEN A
ANSWER_ALL_QUESTIONS

Marks CO PO BT/CL

1. Question 1

1. Define the four fundamental subspaces of a matrix A.

Find the dimension and basis for four fundamental subspaces for

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

[8.0] 1 [1] [3]

PART_HYPHEN B
ANSWER_ANY 6 Question(s)

Marks CO PO BT/CL

2. Question 2

A random process is described by $X(t) = A\cos(w_c t + \theta)$ where A and w_c are constants and θ is a random variable uniformly distributed between $\pm\pi$. Is X(t) wide sense stationary? If not, then why not? If so, the what are mean and autocorrelation function for the random process?

[7.0] 1 [1] [3]

3. Question 3

Define auto covariance and cross covariance of random process. If X(t)is periodic with period T, then show that the autocorrelation function is also periodic with period T.

[7.0] 1 [1] [3]

4. Question 4

Define Ergodic process. GivenX(t) and Y(t) are zero mean, jointly wide sense stationary processes. The random process Z(t) is defined as 3X(t)+2Y(t). Find the correlation functions $R_{XZ}(\tau)$, $R_{ZX}(\tau)$, $R_{YX}(\tau)$, $R_{XY}(\tau)$, $R_{XY}(\tau)$ and $R_{ZX}(\tau)$.

[7.0] 1 [1] [3]

5. Question 5

Define SSS and WSS random processes. Prove that autocorrelation function is an even function.

[7.0] 1 [1] [3]

6. Question 6

Define basis of a vector space. Determine whether the vectors (1, 3, 2), (2, 1, 3) and (3, 2, 1) form a basis for R3

[7.0] 1 [1] [3]

7. Question 7

Define Linear transformation with an example. Let $I: V \to \mathbb{R}$ be the integral mapping $(v)=\int v(t)\,{}_{10}dt$. Show that I is a linear transformation.

[7.0] 1 [1] [3]

8. Question 8

Find the eigen values and eigen vectors of the matrix

$$A=\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

[7.0] 1 [1] [3]

9. Question 9

Reduce the matrix A to upper triangular matrix U and hence find it's determinant

$$A=\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{pmatrix}$$. Check whether 0,is an eigen value of A.

Rectangular Snip

[7.0] 1 [1] [3]

1. $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

A has n column vectors

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Each of these vectors has m components so are in R^m

A has m row vectors

$$\begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix}^T \begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{pmatrix}^T \cdots \begin{pmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{pmatrix}^T$$

Each of these vectors has n components are in R^n

<u>Column Space of A</u> The linear Span of n column vectors of A is known as column space of A and is denoted by $C(A)$ which is clearly a subspace of R^m. Dimension of $C(A)$ is given by r

Null Space The set of all solutions of $AX=0$ where $X=(x_1\, x_2 \ldots x_n)^T$ is called as the null space of A and is denoted by $N(A)$, a subspace of R^n. Dimension of $N(A)$ is $n-r$

Left Null space of A The set of all solutions of $A^TY=0$ where $Y=(y_1, y_2 \ldots y_m)^T$ is known as left null space of A denoted by $N(A^T)$ a subspace of R^m. Dimension of $N(A^T)$ is $m-r$

Row space of A The linear span of n column vectors of A^T is known as row space of A and is denoted by $C(A^T)$ a subspace of R^n. Dimension is r.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

$R_3 \leftrightarrow R_3 - R_1$

$$A \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\dim C(A^T) = 2$.
(row space)

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

$C_4 \leftrightarrow C_4 - C_1$ $\quad$ $C_2 \leftrightarrow C_2 - 2C_4$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\dim C(A) = 2$

Null space of A is the set of solutions of $Ax = 0$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$x + 2y + w = 0 \quad y + z = 0 \Rightarrow y = -z \quad x = -2y - w = 2z - w$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2z - w \\ -z \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2z \\ -z \\ z \\ 0 \end{pmatrix} + \begin{pmatrix} -w \\ 0 \\ 0 \\ w \end{pmatrix}$$

$$= z\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\dim N(A) = 2$

The left null space of A is the set of all solutions of $A^T y = 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \leftrightarrow R_2 - 2R_1 \quad R_4 \leftrightarrow R_4 - R_1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x + z = 0 \quad x = -z \quad y = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \dim N(A^T) = 1$$

2 $X = A\cos(\omega t + \theta) \qquad -\pi < \theta < \pi$

pdf $f(\theta) = \begin{cases} \frac{1}{2\pi} \\ 0 \text{ . otherwise} \end{cases}$

$$E(x(t)) = \int_{-\infty}^{\infty} x f_\theta(\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} \frac{A}{2\pi} \cos(\omega t + \theta)\, d\theta$$

$$= \frac{A}{2\pi} \left\{ \sin(\omega t + \theta) \right\}_{-\pi}^{\pi} = 0$$

$E(x(t))$ is constant

$$E(x(t)\, x(t+\tau)) = \int_{-\pi}^{\pi} \frac{A^2}{2\pi} \cos(\omega t + \theta) \cos(\omega t + \theta + \omega\tau)\, d\theta$$

$$= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \left\{ \cos(2\omega t + 2\theta + \tau) + \cos \omega\tau \right\} d\theta$$

$$= \frac{A^2}{2\pi} \cos \omega\tau = R_x(\tau)$$

ACF

$x(t)$ is WSS

Q3 ACF of $x(t)$

$$C_{xx}(t, t+\tau) = E\{[x(t) - \mu_x(t)][x(t+\tau) - \mu_x(t+\tau)]\}$$

$$= R_{xx}(t, t+\tau) - \mu_x(t)\mu_x(t+\tau)$$

CCF of $x(t)$ and $y(t)$

$$C_{xy}(t, t+\tau) = R_{xy}(t, t+\tau) - \mu_x(t)\mu_y(t+\tau)$$

Let the RP be $x(t) = x(t \pm T)$ where T is the time period

$$x(t+\tau) = x(t \pm T + \tau)$$
$$= x(t + \tau \pm T)$$
$$R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$$
$$= E[x(t \pm T)\, x(t + \tau \pm T)]$$
$$R_{xx}(\tau) = R_{xx}(\tau \pm T)$$

ACF is periodic with same period T

Q4 A RP $x(t)$ is said to be SSS if its statistical characteristics do not change with shift in time.

A RP $x(t)$ is said to be WSS if

1) mean $E(x(t)) = \text{constant}$
2) ACF depends only on time difference

We k.T. $R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$

$R_{xx}(-\tau) = E[x(t)\, x(t-\tau)]$

Let $t - \tau = \alpha$ $\quad t = \alpha + \tau$

$R_{xx}(-\tau) = E[x(\alpha+\tau)\, x(\alpha)]$

$= E[x(\alpha)\, x(\alpha+\tau)] = R_{xx}(\tau)$

5. Ergodic processes are signals for which measurements based on a single sample function are sufficient to determine the ensemble statistics.

$Z(t) = 3X(t) + 2Y(t)$

$\mu_x = 0 \quad \mu_y = 0 \quad R_z(\tau) = E[Z(t)\, Z(t+\tau)]$

$= E[\{3x(t) + 2Y(t)\}\{3X(t+\tau) + 2Y(t+\tau)\}]$

$= E[9x(t)x(t+\tau) + 6Y(t)x(t+\tau) + 6x(t)Y(t+\tau) + 4Y(t)x(t+\tau)]$

$= 9R_x(\tau) + 6R_{yx}(\tau) + 6R_{xy}(\tau) + 4R_y(\tau)$

$R_{zx}(\tau) = E[\{3x(t) + 2Y(t)\}\, x(t+\tau)]$

$= E[3x(t)\, x(t+\tau) + 2Y(t)\, x(t+\tau)]$

$= 3R_x(\tau) + 2R_{yx}(\tau)$

$R_{XZ}(\tau) = $ ~~$E[\{3X(t) + 2Y(t)\} X(t)]$~~

$= E[X(t)\{3X(t+\tau) + 2Y(t+\tau)\}]$

$= E[3X(t)X(t+\tau) + 2X(t)Y(t+\tau)]$

$= 3R_X(\tau) + 2R_{XY}(\tau)$

$R_{ZY}(\tau) = E[\{3X(t) + 2Y(t)\} Y(t+\tau)]$

$= 3E[X(t)Y(t+\tau) + 2Y(t)Y(t+\tau)]$

$= 3R_{XY}(\tau) + 2R_Y(\tau)$

$R_{YZ}(\tau) = E[Y(t)\{3X(t+\tau) + 2Y(t+\tau)\}]$

$= 3E[Y(t)X(t+\tau)] + 2E[Y(t)Y(t+\tau)]$

$= 3R_{YX}(\tau) + 2R_Y(\tau)$

Q6. A ~~set~~ v.s V is said to be of finite dimension n or n dimnsl if $\exists$ n l.i vectors $v_1, v_2 \ldots v_n$ which span V

$S = \{v_1, v_2 \ldots v_n\}$ is called a basis of V

eg $S = \{(1,1)\ (-1,1)\}$ is the basis for V

Let $v_1 = (1, 3, 2)$ $v_2 = (2, 1, 3)$ $v_3 = (3, 2, 1)$ $\in R^3$

Consider the l.c $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\alpha_1, \alpha_2, \alpha_3 \in R$

$\alpha_1(1\ 3\ 2) + \alpha_2(2\ 1\ 3) + \alpha_3(3\ 2\ 1) = (0\ 0\ 0)$ — (1)

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$ — (2)

$3\alpha_1 + \alpha_2 + 2\alpha_3 = 0$ — (3)

$2\alpha_1 + 3\alpha_2 + \alpha_3 = 0$ — (4)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & -4 & -6 \end{pmatrix} \quad \begin{matrix} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad R_3 \leftrightarrow R_3 - 2R_2$$

rank = 3

$\{v_1\ v_2\ v_3\}$ are l.i and forms the basis

$\dim R^3 = 3$

7. Let U and V be 2 v.s. $T: U \to V$ be a L.T iff $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ $\forall\, u, v \in U$ $\alpha, \beta \in F$

$L(v) = \int v(t)\, dt$

Let $f(t)$ $g(t)$ be 2 integrable functions. a, b are scalars

$$L(af+bg) = \int (af+bg)\,dt$$
$$= a\int f\,dt + b\int g\,dt$$
$$= aL(f) + bL(g)$$

L is a L.T

8. $$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Characteristic eqn $|A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\{(2-\lambda)(1-\lambda) - 0\} = 0$$
$$(2-\lambda)^2(1-\lambda) = 0$$

$\lambda = 1, 2, 2$ are the eigen values

To find the eigen vectors consider the matrix eqn $(A-\lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{--- (1)}$$

Case 1 $\lambda = 1$

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x - y + z = 0 \Rightarrow y = z$

$-x = 0$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ k \\ k \end{pmatrix}$$

Case 2 $\lambda = 2$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x + z = 0 \quad x = -z$

$n = 3 \quad m = 1$ choose 2 unknowns

$z = k_1 \quad x = -k_1 \quad y = k_2$

$$X = \begin{pmatrix} -k_1 \\ k_2 \\ k_1 \end{pmatrix} \quad \begin{pmatrix} -k_1 \\ 0 \\ k_1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} (k_1 = 1)$$

$$X = \begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (k = 1)$$

Case 3 $\lambda = 2$

9. $$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{pmatrix}$$

$R_2 \leftrightarrow R_2 - 2R_1$

$R_3 \leftrightarrow R_3 + R_1$

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 0 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

$R_3 \leftrightarrow R_3 - R_2 \quad R_4 \leftrightarrow R_4 - R_2$

$\det A = (1)(2)(3)(6) = 36$

The eigen values are 1, 2, 3 and 6

0 is not the eigenvalue