CMR Institute of Technology
Department of ECE
18EC52- Digital Signal Processing
IAT-1 Solution

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INTERNAL ASSESSMENT TEST – I

Sub:	DIGITAL SIGNAL PROCESSING							Code:	18EC52
Date:	11 / 11 / 2021	Duration:	90 mins	Max Marks:	50	Sem:	V	Branch:	ECE

Answer any 5 full questions

		Marks	CO	RBT
1	Why is it necessary to perform frequency domain sampling? Explain the process of frequency domain sampling and obtain an expression for DFT and IDFT.	[10]	CO2	L3
2	Compute the 8-point DFT of the sequence $x[n] = [1,3,5,7]$. Plot the magnitude spectrum and the phase spectrum.	[10]	CO2	L3
3(a)	Compute the 4-point DFT of $x[n] = [1,2,3,4]$ using matrix method. Plot the magnitude spectrum and the phase spectrum.	[06]	CO2	L2
3(b)	Compute the IDFT of $X[k] = [9, -3 + j1.7321, -3 - j1.7321]$ using matrix method.	[04]	CO2	L2

		Marks	CO	RBT
4	The first 5 samples of 8-point DFT of a real 8-point sequence are as follows. $X[k] = [36,\ -4 + 9.6569j,\ -4 + 4j,\ -4 + 1.6569j,\ -4]$. Determine the remaining samples of $X[k]$. Evaluate the following without explicitly determining $x[n]$. i) $x[0]$ ii) $x[4]$ iii) $\sum_{n=0}^{7} x[n]$ iv) $\sum_{n=0}^{7} \lvert x[n]\rvert^2$	[10]	CO2	L3
5	With proof, explain the nature of DFT for the following cases. i) $x(n)$ is real and circularly even ii) $x(n)$ is real and circularly odd iii) x(n) is imaginary and circularly even iv) x(n) is imaginary and circularly odd	[10]	CO2	L2
6(a)	Compute the N-point DFT of the sequence $x(n) = e^{j\frac{2\pi}{N}ln}, 0 \le n \le N-1$	[04]	CO2	L2
6(b)	Compute the N-point DFT of the sequence $x(n) = a^n, 0 \le n \le N-1$, hence evaluate the DFT of $x(n) = 0.5^n, 0 \le n \le 3$.	[06]	CO2	L2

1.

DISCRETE FOURIER TRANSFORMS (DFT)

I. Frequency Domain sampling and Reconstruction of Discrete time signals

To perform frequency domain analysis of a discrete time signal $x(n)$, we compute Discrete Time Fourier Transform (DTFT) $X(\omega)$ of the signal $x(n)$.

But $X(\omega)$ is a continuous function of frequency ω, and therefore it cannot be processed with digital signal processors.

$\therefore$ We consider sampling of $X(\omega)$ which leads to Discrete Fourier Transform (DFT).

Consider an aperiodic discrete-time signal $x(n)$ with DTFT, as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \longrightarrow (1)$$

$X(\omega)$ is periodic with period 2π.

Let us take N equidistant samples in the interval $0 \le \omega < 2\pi$ with spacing $\delta\omega = \frac{2\pi}{N}$.

(i.e) we have to sample $X(\omega)$ at

$$\omega = 0, \frac{2\pi}{N}1, \frac{2\pi}{N}2, \frac{2\pi}{N}3, \dots \frac{2\pi}{N}(N-1)$$

(i.e) at $\omega = \frac{2\pi K}{N}$ where $K = 0, 1, \dots (N-1)$

$\therefore$ If we evaluate eqn (1) at $\omega = \frac{2\pi K}{N}$, we obtain

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi kn}{N}} \; ; \quad k = 0,1,\ldots(N-1) \rightarrow (2)$$

* The above summation can be subdivided into an infinite number of summations, where each sum contains N terms. Thus,

$$X\left(\frac{2\pi k}{N}\right) = \cdots + \sum_{n=-N}^{-1} x(n)\, e^{-j\frac{2\pi kn}{N}} + \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} +$$

$$\sum_{n=N}^{2N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} + \sum_{n=2N}^{3N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} + \cdots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} \rightarrow (3)$$

Considering one finite duration of N samples in the inner summation (i.e changing n as n+lN

$$\cancel{= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}}}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n+lN)\, e^{-j\frac{2\pi k}{N}(n+lN)}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n+lN)\, e^{-j\frac{2\pi kn}{N}}\, e^{-j2\pi k\frac{lN}{N}}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n+lN)\, e^{-j\frac{2\pi kn}{N}} \qquad \left[\because e^{-j2\pi k\frac{lN}{N}} = 1\right]$$

* By interchanging the summations eqn (3) can be written as follows:

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n+lN)\, e^{-j\frac{2\pi kn}{N}} \rightarrow (4)$$

where $k = 0,1,\ldots(N-1)$

consider the signal $\sum_{l=-\infty}^{\infty} x(n+lN)$

This signal is obtained by the periodic repetition of $x(n)$ every N samples and it is periodic with fundamental period N.

$\therefore$ Let $\sum_{l=-\infty}^{\infty} x(n+lN) = x_p(n)$.

$\therefore$ eqn (4) can be written as,

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x_p(n) e^{\frac{-j2\pi kn}{N}} \longrightarrow (5)$$

where $k = 0, 1, \cdots (N-1)$

$x_p(n)$ is periodic and it can be represented in Fourier series as

$$x_p(n) = \sum_{k=0}^{N-1} C_k e^{\frac{j2\pi kn}{N}} ; \; n = 0, 1, \cdots (N-1) \longrightarrow (6)$$

with Fourier coefficients

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{\frac{-j2\pi kn}{N}} ; \; k = 0, 1, \cdots (N-1) \longrightarrow (7)$$

using eq (5), eq (7) may be written as

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{\frac{-j2\pi kn}{N}}$$

$$= \frac{1}{N} X\left(\frac{2\pi k}{N}\right), \quad 0 \leq k \leq N-1 \longrightarrow (8)$$

using (8) equation (6) can be written as.

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{\frac{j2\pi kn}{N}} \longrightarrow (9)$$

$0 \leq n \leq N-1$

equation (9) suggests that $x_p(n)$ can be reconstructed from $X\left(\frac{2\pi k}{N}\right)$.

However it does not imply that we can reconstruct $x(n)$ from $X\left(\frac{2\pi k}{N}\right)$, $0 \le k$

* But if $x(n)$ is of finite Length L, and if $N \ge L$ then

$$x(n) = \sum_{l=-\infty}^{\infty} x(n+lN) \quad , \quad 0 \le n \le L-1$$

$$\Rightarrow x(n) = x_p(n) \quad , \quad 0 \le n \le N-1 \rightarrow (10)$$

But if $N < L$, then

$$x(n) \neq x_p(n)$$

due to time domain aliasing.

Assuming that $N \ge L$ we can write

$$x(n) = x_p(n) \quad , \quad 0 \le n \le N-1 \rightarrow (11)$$

using (11) and denoting $X\left(\frac{2\pi K}{N}\right)$ as $X(K)$

we can write eqn (9) as,

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi Kn}{N}} \rightarrow (12)$$

$$0 \le n \le N-1$$

and eqn (8) may be written as,

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi Kn}{N}} \; ; \quad 0 \le K \le N-1$$

$\rightarrow (13)$

$\therefore$ eqn (13) Represents DFT of the signal $x(n)$ $0 \le n \le$

and eq (12) represents inverse discrete fourier transform (IDFT) of $X(k)$, $0 \le k \le (N-1)$.

2. $x[n] = \{1, 3, 5, 7\}$.

$N = 8$

To compute 8-point DFT, $x[n] = \{1, 3, 5, 7, 0, 0, 0, 0\}$

DFT is $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$; $k = 0, 1, \dots (N-1)$

$X(k) = \sum_{n=0}^{7} x(n) e^{-j\frac{2\pi kn}{8}}$; $k = 0, 1, \dots 7$

$X(k) = \sum_{n=0}^{7} x(n) e^{-j\frac{\pi kn}{4}}$

$= x(0) e^{0} + x(1) e^{-j\frac{\pi k}{4}} + x(2) e^{-j\pi\frac{k2}{4}} + x(3) e^{-j\pi k\frac{3}{4}}$

$+ x(4) e^{-j\pi k\frac{4}{4}} + \dots + x(7) e^{-j\pi k\frac{7}{4}}$

$= 1 + 3e^{-j\frac{\pi k}{4}} + 5e^{-j\pi\frac{k}{2}} + 7e^{-j\pi k\frac{3}{4}} + 0 + 0 + 0 + 0$

$X(k) = 1 + 3e^{-j\frac{\pi k}{4}} + 5e^{-j\frac{\pi k}{2}} + 7e^{-j\pi k\frac{3}{4}}$

$k=0$, $X(k) = 1+3+5+7 = 16$

$k=1$, $X(1) = 1 + 3e^{-j\frac{\pi}{4}} + 5e^{-j\pi\frac{1}{2}} + 7e^{-j\pi\frac{3}{4}} = 1 + 3\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$+ 5(-j) + 7\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$= -1.828 - j\,12.07$

$k=2$, $X(2) = 1 + 3e^{-j\pi\frac{2}{4}} + 5e^{-j\pi\frac{2}{2}} + 7e^{-j\pi\frac{6}{4}} = 1 + 3(-j) + 5(-1)$
$+ 7(j)$

$= -4 + j4$

$k=3$, $X(3) = 1 + 3e^{-j\pi\frac{3}{4}} + 5e^{-j\pi\frac{3}{2}} + 7e^{-j\pi\frac{9}{4}} = 1 + 3\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$
$+ 5(j) + 7\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$X(3) = 3.828 + j2.172$

$k = 4, \quad X(4) = X\left(\frac{8}{2}\right) = X\left(\frac{N}{2}\right)$

$= 1 - 3 + 5 - 7 + 0 - 0 + 0 - 0 + 0$

$= -4$

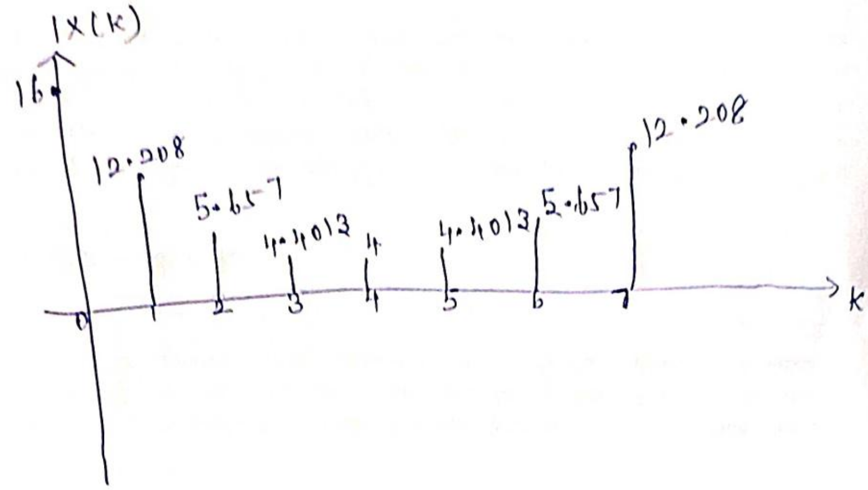
$k = 5, \quad X(5) = X^*(N-k) = X^*(8-5) = X^*(3) = 3.828 - j2.172$

$k = 6, \quad X(6) = X^*(N-6) = X^*(2) = -4 - j4$

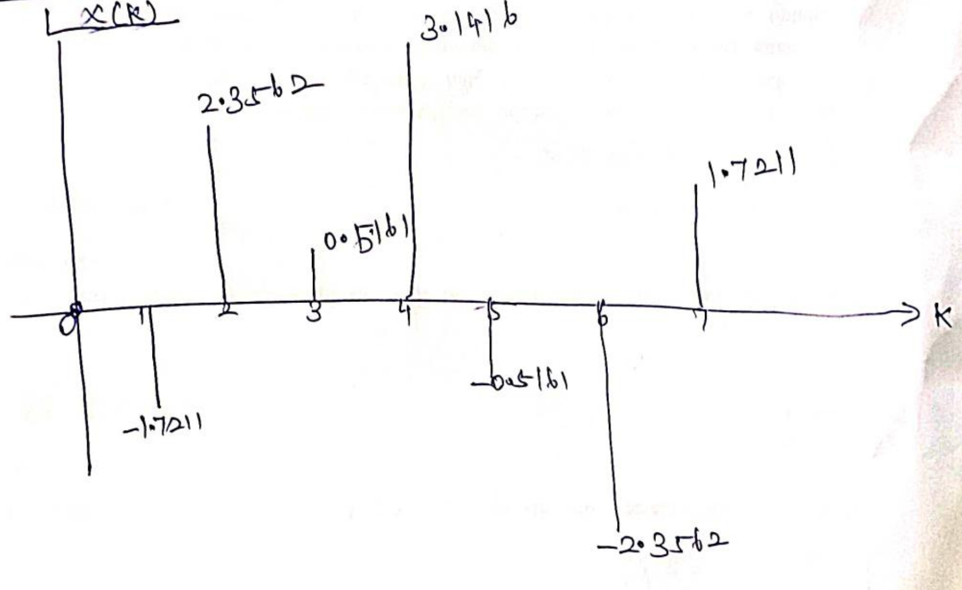
$k = 7, \quad X(7) = X^*(N-7) = X^*(1) = -1.828 + j12.07$

K	X(K)	|X(K)|	∠X(K)
0	16	16	0
1	$-1.828 - j12.07$	12.208	-1.7211
2	$-4 + j4$	5.657	2.3562
3	$3.828 + j2.172$	4.4013	0.5161
4	-4	4	3.1416
5	$3.828 - j2.172$	4.4013	-0.5161
6	$-4 - j4$	5.657	-2.3562
7	$-1.828 + j12.07$	12.208	1.7211

Magnitude spectrum



Phase spectrum



3.(a) $x[n] = [1, 2, 3, 4]$

$N=4$ Matrix method

Using matrix method, DFT is given by,

$$X_N = W_N \cdot x_N$$

Here $N=4$; $\therefore k, n = 0, 1, 2, 3.$

$$X(k) = \begin{array}{c|c} & k = 0 \quad\quad 1 \quad\quad 2 \quad\quad 3 \\ \begin{array}{c} n=0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \end{array}$$

$$W_N^k = e^{\frac{-j2\pi k}{N}}$$

$$W_4^0 = e^0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi(1)}{4}} = e^{\frac{-j\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_4^2 = e^{\frac{-j2\pi(2)}{4}} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

$$W_4^3 = e^{-j2\pi\left(\frac{3}{4}\right)} = e^{\frac{-j6\pi}{4}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = j$$

$$W_4^4 = e^{-j2\pi\left(\frac{4}{4}\right)} = e^{-j2\pi} = 1$$

$$W_4^5 = W_4^4 \, W_4^1 = (1)(-j) = -j$$

$$W_4^6 = W_4^4 \, W_4^2 = (1)(-1) = -1$$

$$W_4^9 = W_4^4 \, W_4^5 = (1)(-j) = -j$$

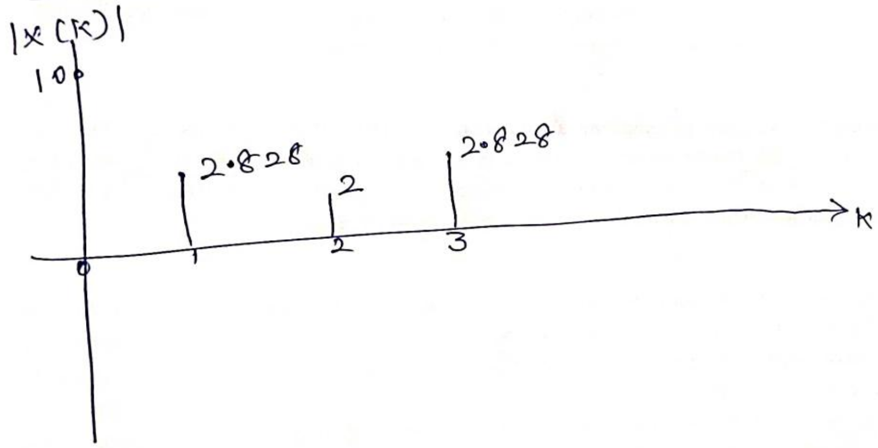
$$\therefore X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+2+3+4 \\ 1-j2-3+j4 \\ 1-2+3-4 \\ 1+j2-3-j4 \end{bmatrix}$$

$$X(k) = \begin{bmatrix} 10 \\ -2+j2 \\ -2 \\ -2-j2 \end{bmatrix}$$

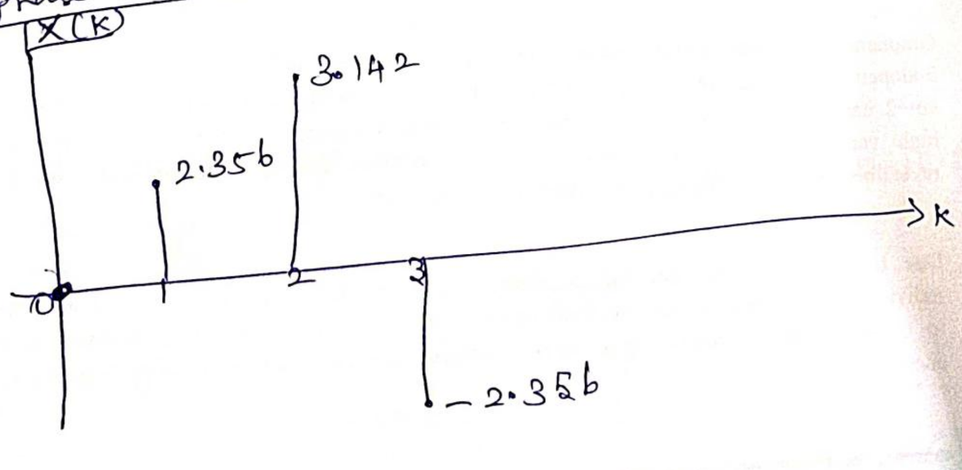
$$\therefore X(k) = \{10, -2+j2, -2, -2-j2\}.$$

k	X(k)	|X(k)|	∠X(k)
0	10	10	0
1	−2+j2	2.828	2.356
2	−2	2	3.142
3	−2−j2	2.828	−2.356

Magnitude spectrum



Phase spectrum



3.(b) $X[k] = [9, -3+j1.7321, -3-j1.7321]$

Matrix method

As per given sequence $N = 3$

As per matrix method, IDFT is,

$$x_N = \frac{1}{N}\left[W_N^* X_N\right]$$

$$x(n) = \frac{1}{3} \begin{array}{c|c} & \begin{array}{ccc} k=0 & 1 & 2 \end{array} \\ \begin{array}{c} n=0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^4 \end{bmatrix}^* \end{array} \begin{bmatrix} 9 \\ -3+j1.7321 \\ -3-j1.7321 \end{bmatrix}$$

$W_3^0 = e^0 = 1$

$W_3^1 = e^{-j2\pi\left(\frac{1}{3}\right)} = \cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3} = -0.5 - j0.866$

$\therefore W_3^{1*} = -0.5 + j0.866$

$W_3^2 = e^{-j2\pi\left(\frac{2}{3}\right)} = \cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3} = -0.5 + j0.866$

$\therefore W_3^{2*} = -0.5 - j0.866$

$W_3^3 = e^{-j2\pi\left(\frac{3}{3}\right)} = W_3^0 = 1 \Rightarrow W_3^{3*} = 1$

$W_3^4 = W_3^3\, W_3^1 = (1)(-0.5 - j0.866) = -0.5 - j0.866$

$\therefore W_3^{4*} = -0.5 + j0.866$

$$x(n) = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -0.5+j0.866 & -0.5-j0.866 \\ 1 & -0.5-j0.866 & -0.5+j0.866 \end{bmatrix}\begin{bmatrix} 9 \\ -3+j1.7321 \\ -3-j1.7321 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 9-3+\cancel{j1.7321}-3-\cancel{j1.7321} \\ 9+1.5-\cancel{j0.86605}-\cancel{j2.598}-1.499+1.5+\cancel{j0.86605}+\cancel{j2.598}-1.499 \\ 9+1.5-\cancel{j0.86605}+\cancel{j2.598}+1.499+1.5+\cancel{j0.86605}-\cancel{j2.598}+1.499 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix}$$

$\therefore\ x(n) = \{1, 3, 5\}$.

4. First 5 samples of 8-point DFT of a real 8-point sequence are

$X[k] = \{36,\ -4+j9.6569,\ -4+j4,\ -4+j1.6569,\ -4\}$.

Since $x(n)$ is a real sequence,

$$X(k) = X^*(N-k)$$

So, $X(5) = X^*(8-5) = X^*(3) = -4-j1.6569$

$k=6,\ X(6) = X^*(8-6) = X^*(2) = -4-j4$

$k=7,\ X(7) = X^*(8-7) = X^*(1) = -4-j9.6569$

$\therefore X(k) = \{36,\ -4+j9.6569,\ -4+j4,\ -4+j1.6569,\ -4,\ -4-j1.6569,\ -4-j4,\ -4-j9.6569\}.$

(i) $x[0] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)$

$= \frac{1}{8}\sum_{k=0}^{7} X(k)$

$= \frac{1}{8}\left[X(0)+X(1)+\cdots+X(7)\right]$

$= \frac{1}{8}\left[36-4-4-4-4-4-4-4\right]$

$= \frac{1}{8}\left[8\right] = \cancel{\frac{12}{8}} = \cancel{\frac{3}{2}} = \frac{8}{8} = 1$

(ii) $x[4] = \frac{1}{8}\left[X(0)-X(1)+X(2)-X(3)+X(4)-X(5)+X(6)-X(7)\right]$

$= \frac{1}{8}\left[36+4-4+4-4+4-4+4\right]$

$= \frac{1}{8}\left[40\right] = \frac{40}{8} = 5.$

(iii) $\sum_{n=0}^{7} x[n]$

Since $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}};\quad 0 \le k \le N-1$

If $k=0$, $X(0) = \sum_{n=0}^{7} x(n)$

$$\therefore \sum_{n=0}^{7} x(n) = X(0) = 36$$

(iv) $\sum_{n=0}^{7} |x(n)|^2$

As per parseval's theorem,

$$\sum_{n=0}^{7} |x(n)|^2 = \frac{1}{8} \sum_{k=0}^{7} |X(k)|^2$$

$$= \frac{1}{8} \sum_{k=0}^{7} X(k) X^*(k)$$

$$= \frac{1}{8} \left[X(0) X^*(0) + X(1) X^*(1) + \cdots + X(7) X^*(7) \right]$$

$$= \frac{1}{8} \left[1296 + 109.255 + 32 + 18.745 + 16 + 18.745 + 32 + 109.255 \right]$$

$$= \frac{1}{8} \left[1632 \right] = 204$$

5.

(IV) SYMMETRY PROPERTIES OF THE DFT

Let us assume that both the sequence $x(n)$ and its DFT $X(k)$ are complex valued.

$$\therefore x(n) = x_r(n) + j x_i(n) \rightarrow (1)$$

$$X(k) = X_R(k) + j X_I(k) \rightarrow (2)$$

DFT equation is,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \rightarrow (3)$$

use eqn (1) in eqn (3)

$$\therefore X(k) = \sum_{n=0}^{N-1} [x_r(n) + j x_i(n)] e^{-j2\pi kn/N}$$

where $0 \leq k \leq N-1$

$$= \sum_{n=0}^{N-1} [x_r(n) + j x_i(n)] \left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$= \sum_{n=0}^{N-1} \left[x_r(n) \cos\left(\frac{2\pi kn}{N}\right) + x_i(n) \sin\left(\frac{2\pi kn}{N}\right) - j x_r(n) \sin\left(\frac{2\pi kn}{N}\right) + j x_i(n) \cos\left(\frac{2\pi kn}{N}\right)\right]$$

$$X_R(k) + jX_I(k) = \sum_{n=0}^{N-1} \left[x_r(n) \cos\left(\frac{2\pi kn}{N}\right) + x_i(n) \sin\left(\frac{2\pi kn}{N}\right)\right] + j\left[x_i(n) \cos\left(\frac{2\pi kn}{N}\right) - x_r(n) \sin\left(\frac{2\pi kn}{N}\right)\right]$$

Separating real part and Imaginary part, we get,

$$\boxed{X_R(k) = \sum_{n=0}^{N-1} x_r(n) \cos\left(\frac{2\pi kn}{N}\right) + x_i(n) \sin\left(\frac{2\pi kn}{N}\right) \rightarrow (4) \quad 0 \leq k \leq N-1}$$

$$\boxed{X_I(k) = \sum_{n=0}^{N-1} x_i(n) \cos\left(\frac{2\pi kn}{N}\right) - x_r(n) \sin\left(\frac{2\pi kn}{N}\right) \rightarrow (5) \quad 0 \leq k \leq N-1}$$

(i) Real-valued sequences:

If $x(n)$ is real, then

$$x_i(n) = 0 \longrightarrow (6)$$

$$\therefore X_R(k) = \sum_{n=0}^{N-1} x_r(n) \cos\left(\frac{2\pi kn}{N}\right) \longrightarrow (7)$$

$0 \le k \le N-1$

$$\therefore X_I(k) = -\sum_{n=0}^{N-1} x_r(n) \sin\left(\frac{2\pi kn}{N}\right) \longrightarrow (8)$$

(ii) Real and Even sequences

If $x(n)$ is real and circularly even, then $x_r(n) \sin\left(\frac{2\pi kn}{N}\right)$ will be circularly odd and $\sum_{n=0}^{N-1} x_r(n) \sin\left(\frac{2\pi kn}{N}\right) = 0$

$$\therefore X_I(k) = 0 \quad [\text{from eqn } (8)]$$

$$\therefore X(k) = X_R(k)$$

$$= \sum_{n=0}^{N-1} x_r(n) \cos\left(\frac{2\pi kn}{N}\right)$$

where $0 \le k \le N-1$

$$\therefore X(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi kn}{N}\right) \longrightarrow (9)$$

Hence if $x(n)$ is real and circularly even, then $X(k)$ is real and circularly even.

(iii) Real and odd sequence.

If $x(n)$ is real and circularly odd, then $x_r(n)\sin\left(\frac{2\pi kn}{N}\right)$ will be circularly even and $x_r(n)\cos\left(\frac{2\pi kn}{N}\right)$ will be circularly odd.

$$\therefore \sum_{n=0}^{N-1} x_r(n)\cos\left(\frac{2\pi kn}{N}\right) = 0.$$

$$\therefore X_R(k) = 0 \qquad \text{[from eqn (7)]}$$

$$\therefore X(k) = X_R(k) + jX_I(k)$$

using eqn (8) here

$$= -j\sum_{n=0}^{N-1} x_r(n)\sin\left(\frac{2\pi kn}{N}\right)$$

$$= -j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right) \longrightarrow (10)$$

$\therefore$ If $x(n)$ is real and circularly odd, then $X(k)$ is imaginary and circularly odd.

(iv) Imaginary sequence

If $x(n)$ is purely imaginary, then

$$x_r(n) = 0 \longrightarrow (11).$$

$\therefore$ Eq (4) and (5) may be modified as follows:

$$X_R(k) = \sum_{n=0}^{N-1} x_i(n)\sin\left(\frac{2\pi kn}{N}\right) \longrightarrow (12)$$

$0 \le k \le N-1$

$$X_I(k) = \sum_{n=0}^{N-1} x_i(n)\cos\left(\frac{2\pi kn}{N}\right) \longrightarrow (13).$$

$0 \le k \le N-1$

(V) Purely Imaginary and even sequences.

If $x(n)$ is purely imaginary and circular even, then $x_i(n)\sin\left(\frac{2\pi kn}{N}\right)$ will be circularly odd.

$$x_i(n)\sin\left(\frac{2\pi kn}{N}\right)=0 \longrightarrow (14)$$

$$\therefore X_R(K)=0 \quad (\text{from eqn } 12) \longrightarrow (15)$$

$$\therefore X(K)=X_R(K)+jX_I(K)$$
$$=0+j\,X_I(K)$$
$$=jX_I(K)$$
$$=j\sum_{n=0}^{N-1} x_i(n)\cos\left(\frac{2\pi kn}{N}\right) \longrightarrow (16)$$

Hence if $x(n)$ is purely imaginary and circular even, then $X(K)$ is purely imaginary and circularly even.

(VI) Purely Imaginary and odd sequences.

If $x(n)$ is purely imaginary, and circularly odd, then $x_i(n)\cos\left(\frac{2\pi kn}{N}\right)$ will be circularly ~~odd~~ even

$$x_i(n)\cos\left(\frac{2\pi kn}{N}\right)=0.$$

$$\therefore X_I(K)=0 \quad (\text{from ean } 13)$$

$$\therefore X(K)=X_R(K)+jX_I(K)$$
$$=X_R(K)$$
$$=\sum_{n=0}^{N-1} x_i(n)\sin\left(\frac{2\pi kn}{N}\right) \longrightarrow (17)$$

Hence if $x(n)$ is purely imaginary and circularly odd, then $X(k)$ will be real and circularly odd.

6. (a)

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi ln}{N}} e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-l)n} \longrightarrow (1).$$

Let $\alpha = e^{-j\frac{2\pi}{N}(k-l)} \longrightarrow (2)$

we know that

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \frac{1-\alpha^N}{1-\alpha} & \text{for } \alpha \neq 1 \\ N & \text{for } \alpha = 1 \end{cases}$$

If $k=l$, then $\alpha = e^{-j\frac{2\pi}{N}(k-l)}$

$$= e^{-j\frac{2\pi}{N}(l-l)} = e^0 = 1$$

If $k \neq l$, $\alpha \neq 1$

If $k=l$, $X(k) = N$, since $\alpha = 1$

If $k \neq l$, $X(k) = \frac{1 - e^{-j\frac{2\pi}{N}(k-l)N}}{1 - e^{-j\frac{2\pi}{N}(k-l)}}$

$$= \frac{1 - e^{-j2\pi(k-l)}}{1 - e^{-j\frac{2\pi}{N}(k-l)}}$$

$$= 0.$$

$$\therefore X(k) = \begin{cases} N, & \text{for } k=l \\ 0, & \text{for } k \neq l \end{cases}$$

$$\boxed{X(k) = N\,\delta(k-l)}$$

If $k=l$, then

$$X(k) = N\,\delta(l-l)$$
$$= N\,\delta(0)$$
$$= N.$$

6. (b)

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

$0 \le k \le N-1$

$$= \sum_{n=0}^{N-1} a^n e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} \left(a e^{\frac{-j2\pi k}{N}}\right)^n \longrightarrow (1).$$

we know that,

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \frac{1-\alpha^N}{1-\alpha} \; ; & \text{if } \alpha \neq 1 \\ N \; ; & \text{if } \alpha = 1 \end{cases} \longrightarrow (2).$$

∴ eqn (1) can be simplified as follows

$$\underset{0 \le k \le N-1}{X(k)} = \frac{1-\left(a e^{\frac{-j2\pi k}{N}}\right)^N}{1-a e^{\frac{-j2\pi k}{N}}}$$

$$= \frac{1-a^N e^{-j2\pi \frac{k}{N} \cdot N}}{1-a e^{\frac{-j2\pi k}{N}}}$$

$$= \frac{1-a^N e^{-j2\pi k}}{1-a e^{\frac{-j2\pi k}{N}}}$$

$$= \frac{1-a^N}{1-a e^{\frac{-j2\pi k}{N}}} \; ; \quad (\because e^{-j2\pi k} = 1 \text{ always})$$

$\longrightarrow (3)$

Given $x(n) = 0.5^n$, $0 \le n \le 3$.

$\therefore a = 0.5, \ N = 4.$

$\therefore$ eqn (3) becomes

$$X(k) = \frac{1 - (0.5)^4}{1 - 0.5e^{-j\frac{2\pi k}{4}}}$$

$0 \le k \le 3$

$$= \frac{1 - 0.0625}{1 - 0.5e^{-j\frac{\pi k}{2}}} = \frac{0.9375}{1 - 0.5e^{-j\frac{\pi k}{2}}}$$

$$k=0, \quad X(0) = \frac{0.9375}{1 - 0.5} = 1.875$$

$$k=1, \quad X(1) = \frac{0.9375}{1 - 0.5e^{-j\pi\frac{1}{2}}} = \frac{0.9375}{1 - j(0.5)} =$$

$$= \frac{0.9375 \ (1+j0.5)}{(1-j0.5)(1+j0.5)} = 0.75 - j0.375$$

$$k=2, \quad X(2) = \frac{0.9375}{1 - 0.5e^{-j\pi\frac{2}{2}}} = \frac{0.9375}{1 + 0.5} = \frac{0.9375}{1.5}$$

$$= 0.625$$

$$k=3, \quad X(3) = \frac{0.9375}{1 - 0.5e^{-j\pi\frac{3}{2}}} = \frac{0.9375}{1 + j0.5} = \frac{0.9375 \ (1-j0.5)}{(1+j0.5)(1-j0.5)}$$

$$= 0.75 + j0.375$$

$$\therefore X(k) = \{1.875, \ 0.75 - j0.375, \ 0.625, \ 0.75 + j0.375\}.$$