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Internal Assesment Test - II

Sub:	Transform Calculus, Fourier Series and Numerical Techniques	Code:	18MAT31
Date: 24/01/2022	Duration: 90 mins Max Marks: 50 Sem: 3	Branch:	All (Regular)

First question is compulsory , answer any 6 from Q2 to Q8

		Marks	OBE CO	RBT
1.	Solve the difference equation $u_{n+2} - 3u_{n+1} + 2u_n = 3^n$, with $u_0 = 0 = u_1$ using Z-transform	[8]	CO	L
2.	Find the Z-transform of $2n + \sin\frac{n\pi}{4} + 1$	[7]	CO3	L3
3.	If $U(z) = \frac{2z^2+5z+14}{(z-1)^4}$, then evaluate the values of u_2, u_3	[7]	CO3	L3
4	Employ Taylors' series method to find y at $x = 0.1$ and $x = 0.2$ correct to four places of decimal, given $\frac{dy}{dx} = 3e^x + 2y, y(0) = 0$	[7]	CO4	L3
5.	Using modified Eulers' formula compute $y(1.1)$ correct to five decimal places given that $\frac{dy}{dx} = \frac{1-xy}{x^2}$ and $y = 1$ at $x = 1$ (taking h=0.1)	[7]	CO4	L3
6.	Use 4th order Runge-Kutta method to solve $\frac{dy}{dx} = \frac{1}{(x+y)}, y(0.4) = 1$ at $x = 0.5$ correct to four decimal places	[7]	CO4	L3
7.	Apply Milnes predictor-corrector formula to compute $y(0.4)$, given $\frac{dy}{dx} = 2e^x y$ with	[7]	CO4	L3

x	0.0	0.1	0.2	0.3
y	2.4	2.473	3.129	4.059

8. Solve the differential equation $y' + y + xy^2 = 0$ with the initial values [7] CO4 L3

x	0.0	0.1	0.2	0.3
y	1.0000	0.9008	0.8066	0.7220

by computing the value of $y(0.4)$ by applying Adams-Bashforth predictor corrector formula

Q1 Solve the difference equation $u_{n+2} - 3u_{n+1} + 2u_n = 3^n$, with $u_0 = 0 = u_1$ using Z-transform. (8 marks)

Sol: Let $Z(u_n) = U(z) \Leftrightarrow u_n = Z^{-1}(U(z))$

Given $u_{n+2} - 3u_{n+1} + 2u_n = 3^n$

2 marks

$$\Rightarrow Z(u_{n+2} - 3u_{n+1} + 2u_n) = Z(3^n)$$

$$\Rightarrow Z(u_{n+2}) - 3Z(u_{n+1}) + 2Z(u_n) = \frac{z}{z-3}$$

By linearity of Z and $Z(a^n) = \frac{z}{z-a}$

$Z(u_{n+1}) = z(U(z) - u_0)$

$= zU(z)$ ($\because u_0 = 0$)

$Z(u_{n+2})$

$= z^2\left(U(z) - u_0 - \frac{u_1}{z}\right)$

$= z^2 U(z)$

($\because u_0 = 0 = u_1$)

By left shift rule for Z-transform.

$$\Rightarrow (z^2 - 3z + 2)\, U(z) = \frac{z}{z-3}$$

$$\Rightarrow U(z) = \frac{z}{(z-1)(z-2)(z-3)}$$

3 marks

Let $$\frac{U(z)}{z} = \frac{1}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$\Rightarrow 1 = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

$$\Rightarrow \begin{cases} z = 2 \Rightarrow B = -1 \\ z = 3 \Rightarrow C = \frac{1}{2} \\ z = 1 \Rightarrow A = \frac{1}{2} \end{cases}$$

1-Mark.

$$\therefore \frac{U(z)}{z} = \frac{1}{2}\cdot\frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{2}\cdot\frac{1}{z-3}$$

$$\Rightarrow U(z) = \frac{1}{2}\cdot\frac{z}{z-1} - \frac{z}{z-2} + \frac{1}{2}\cdot\frac{z}{z-3}$$

2-Marks

$$\Rightarrow u_n = Z^{-1}(U(z))$$

$$= \frac{1}{2}Z^{-1}\left(\frac{z}{z-1}\right) - Z^{-1}\left(\frac{z}{z-2}\right) + \frac{1}{2}Z^{-1}\left(\frac{z}{z-3}\right)$$

$$= \frac{1}{2}(1)^n - (2)^n + \frac{1}{2}(3)^n$$

$$\therefore u_n = \frac{1}{2} - 2^n - \frac{1}{2}\cdot 3^n$$

Q2) Find the Z-transform of $2n + \sin\frac{n\pi}{4} + 1$ (7-marks)

Sol) Let $u_n = 2n + \sin\frac{n\pi}{4} + 1$

1-Mark

$$Z(u_n) = Z\left(2n + \sin\frac{n\pi}{4} + 1\right) = 2Z(n) + Z\left(\sin n\frac{\pi}{4}\right) + Z(1)$$

(by linearity of Z)

6-Marks

$$= 2\frac{z}{(z-1)^2} + \frac{z\sin\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1} + \frac{z}{z-1}$$

$$Z(n) = \frac{z}{(z-1)^2}$$

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z(1) = \frac{z}{z-1}$$

$$\therefore Z(u_n) = \frac{2z}{(z-1)^2} + \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}} + \frac{z}{z-1}$$

Q3 If $U(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$, then evaluate the values of u_2, u_3.

7 marks

Sol

clearly $U(z) = \dfrac{\frac{1}{z^4}(2z^2+5z+14)}{\frac{1}{z^4}(z-1)^4} = \dfrac{\frac{1}{z^2}\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{\left(1-\frac{1}{z}\right)^4}$

3 Mark

$= \dfrac{1}{z^2} \dfrac{2-5z^{-1}+14z^{-2}}{(1-z^{-1})^4}$

$\therefore$ we have $u_0 = \lim\limits_{z\to\infty} U(z) = 0$

$u_1 = \lim\limits_{z\to\infty} \left[z(U(z)-u_0)\right]$

$= \lim\limits_{z\to\infty} \dfrac{1}{z} \dfrac{(2-5z^{-1}+14z^{-2})}{(1-z^{-1})^4} = 0$

2-marks

$u_2 = \lim\limits_{z\to\infty} \left[z^2(U(z)-u_0-u_1z^{-1})\right]$

$= \lim\limits_{z\to\infty} \dfrac{2-5z^{-1}+14.z^{-2}}{(1-z^{-1})^4} = \dfrac{2}{1} = 2$

2-Mark

$u_3 = \lim\limits_{z\to\infty} \left[z^3(U(z)-u_0-u_1z^{-1}-u_2z^{-2})\right]$

$= \lim\limits_{z\to\infty} z^3\left[\dfrac{2z^2+5z+14}{(z-1)^4} - \dfrac{2}{z^2}\right]$

$$\lim_{z\to\infty} z^3\left(\frac{13z^3+2z^2+8z-2}{z^2(z-1)^4}\right)=\lim_{z\to\infty} z^3\left(\frac{\frac{1}{z^6}(13z^3+2z^2+8z-2)}{\frac{1}{z^6}\cdot z^2(z-1)^4}\right)$$

$$=\lim_{z\to\infty}\frac{13+2z^{-1}+8z^{-2}-2z^{-3}}{(1-z)^4}$$

$$=13$$

$\therefore$, $u_2=2$, $u_3=13$

(Q4) Employ Taylor's series method to find y at $x=0.1$ and at $x=0.2$ correct to four places of decimal, given $\frac{dy}{dx}=3e^x+2y$, $y(0)=0$ (7 Marks)

Sol:

Here $y'=3e^x+2y$, $x_0=0$, $y_0=0$.

3 Marks

$y'=3e^x+2y$ $\qquad y'_0=3e^{x_0}+2y_0=3e^0+2\times0=3$

$y''=3e^x+2y'$ $\qquad y''_0=3e^{x_0}+2y'_0=3e^0+2\times3=9$

$y'''=3e^x+2y''$ $\qquad y'''_0=3e^{x_0}+2y''_0=3e^0+2\times9=21$

we have

2 Marks

$$y(x)=y_0+\frac{y'_0}{1!}(x-x_0)+\frac{y''_0}{2!}(x-x_0)^2+\frac{y'''_0}{3!}(x-x_0)^3+\cdots$$

$$=0+\frac{3}{1}(x-0)+\frac{9}{2}(x-0)^2+\frac{21}{6}(x-0)^3+\cdots$$

$$=3x+4.5x^2+3.5x^3+\cdots$$

$\therefore\ y(x) \approx 3x + 4.5x^2 + 3.5x^3$

1-Mark $\therefore\ y(0.1) \approx 3\times(0.1) + 4.5(0.1)^2 + 3.5(0.1)^3 = 0.3485$

1-Mark $\boxed{y(0.2) \approx 3\times(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 = 0.808}$

or

Let $x_1 = 0.1$, Here $h = x_1 - x_0 = 0.1$

$y_1 = y(x_1) = y(0.1) = 0.3485$

Now $y_1' = 3e^{x_1} + 2y_1 = 3e^{0.1} + 2\times 0.3485 = 4.0125$

$y_1'' = 3e^{x_1} + 2y_1' = 3e^{0.1} + 2\times 4.0125 = 11.3405$

$y_1''' = 3e^{x_1} + 2y_1'' = 3e^{0.1} + 2\times 11.3405 = 25.9965$

$$\therefore\ y(x) = y_1 + \frac{y_1'}{1!}(x-x_1) + \frac{y_1''}{2!}(x-x_1)^2 + \frac{y_1'''}{3!}(x-x_1)^3 + \cdots$$

$$\approx 0.3485 + \frac{4.0125}{1}(x-0.1) + \frac{11.3405}{2}(x-0.1)^2 + \frac{25.9965}{6}(x-0.1)^3$$

$$\therefore\ y(0.2) \approx 0.3485 + \frac{4.0125}{1}(0.2-0.1) + \frac{11.3405}{2}(0.2-0.1)^2 + \frac{25.9965}{6}(0.2-0.1)^3$$

$$= 0.8181$$

Q5) Using modified Euler's formula compute $y(1.1)$ correct to five decimal places given that $\frac{dy}{dx} = \frac{1-xy}{x^2}$ and $y=1$ at $x=1$ (taking $h=0.1$).

Sol

Given that $y' = \frac{1-xy}{x^2}$ and $y(1)=1$

Here $f(x,y) = \frac{1-xy}{x^2}$; $x_0 = 1$, $y_0 = 1$, $h = 0.1$

$x_1 = x_0 + h = 1.1$

To find $y_1 = y(x_1) = y(1.1)$.

3 marks

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 1 + 0.1 f(1,1)$$

$$= 1 + 0.1\left(\frac{1-1\times 1}{1^2}\right) = 1 + 0.1\times 0 = 1$$

$$y_1^{(0)} = 1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left(f(x_0,y_0) + f(x_1, y_1^{(0)})\right)$$

$$= 1 + \frac{0.1}{2}\left(f(1,1) + f(1.1,1)\right)$$

$$= 1 + \frac{0.1}{2}\left(\frac{1-1\times 1}{1^2} + \frac{1-1.1\times 1}{(1.1)^2}\right) = 0.9959$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0,y_0) + f(x_1, y_1^{(1)})\right)$$

$$= 1 + \frac{0.1}{2}\left(f(1,1) + f(1.1, 0.9959)\right)$$

$$= 1 + 0.05\left(\frac{1-1\times 1}{1^2} + \frac{1-1.1\times 0.9959}{(1.1)^2}\right)$$

$$= 0.99605$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left(f(x_0,y_0) + f(x_1, y_1^{(2)})\right)$$

$$= 1 + \frac{0.1}{2}\left(f(1,1) + f(1.1, 0.99605)\right)$$

$$= 1 + 0.05\left(\frac{1-1\times 1}{1^2} + \frac{1-1.1\times 0.99605}{(1.1)^2}\right) = 0.99605$$

4 marks

$\therefore\ y_1 = y(x_1) = y(0.1) \approx y_1^{(3)} = 0.99605$

Q6

Use 4-th order Runge-Kutta method to solve $\frac{dy}{dx} = \frac{1}{x+y}$, $y(0.4) = 1$ at $x = 0.5$ correct to four decimal places (7-Marks)

Sol

2-Mark

Here $f(x,y) = \frac{1}{x+y}$; $x_0 = 0.4$, $y_0 = y(x_0) = 1$;

$h = 0.1$

$x_1 = x_0 + h = 0.5$

To find $y_1 = y(x_1) = y(0.5)$

4-Mark

$K_1 = hf(x_0, y_0) = 0.1 f(0.4, 1) = 0.1 \times \frac{1}{0.4+1}$

$= 0.0714$

$K_1 = 0.0714$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1 f\left(0.4 + \frac{0.1}{2}, 1 + \frac{0.0716}{2}\right)$

$= 0.1 f(0.45, 1.0357) = 0.1 \times \frac{1}{0.45 + 1.0357}$

$= 0.0673$

$K_2 = 0.0673$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1 f(0.45, 1.03365)$

$= 0.1\left(\frac{1}{0.45 + 1.03365}\right) = 0.0674.$

$K_3 = 0.0674$

$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1 f(0.5, 1.0674)$

$= 0.1\left(\frac{1}{0.5 + 1.6674}\right) = 0.0638$

$K_4 = 0.0638$

1-Mark

$\therefore y_1 = y_0 + h = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 1 + \frac{1}{6}[0.714 + 2(0.0673) + 2(0.0674) + 0.0638]$

$= 1.0674.$

$\therefore y_1 = 1.0674 = y(0.5)$

(Q7) Apply Milne's predictor - corrector formula to compute $y(0.4)$, given $\frac{dy}{dx} = 2e^x y$ with

x	0.0	0.1	0.2	0.3
y	2.4	2.473	3.129	4.059

7-Marks

Sol.

Here $f(x,y) = 2e^x y$

i.e $y' = 2e^x y \Rightarrow y' = f(x,y)$

To find $y_4 = y(x_4) = y(x_3+h) = y(0.4)$

x	y	$y' = f = f(x,y) = 2e^x y$
$x_0 = 0.0$	$y_0 = 2.4$	$y_0' = f_0 = f(x_0,y_0) = f(0.0, 2.4) = 2e^{0.0} \times 2.4 = 4.8$
$x_1 = 0.1$	$y_1 = 2.473$	$y_1' = f_1 = f(x_1,y_1) = f(0.1, 2.473) = 2e^{0.1} \times 2.473 = 5.466$
$x_2 = 0.2$	$y_2 = 3.129$	$y_2' = f_2 = f(x_2,y_2) = f(0.2, 3.129) = 2 \times e^{0.2} \times 3.129 = 7.6435$
$x_3 = 0.3$	$y_3 = 4.059$	$y_3' = f_3 = f(x_3,y_3) = f(0.3, 4.059) = 10.9581$

3-Marks

Here $h = 0.1$ Note that $x_4 = x_3 + h = 0.3 + 0.1 = 0.4$

To find: $y_4 = y(x_4) = y(0.4)$

3-Marks

$y_4^P = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) = 2.4 + \frac{4(0.1)}{3}(2 \times 5.466 - 7.6435 + 2 \times 10.9581)$

$= 5.7606$

Now $f_4^P = f(x_4, y_4^P) = f(0.4, 5.7606) = 2e^{0.4} \times 5.7606$

$= 17.1876$

$y_4^C = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^P) = 3.129 + \frac{0.1}{3}(7.6435 + 4 \times 10.9581 + 17.1876)$

$= 5.3836$

1-Mark

$\therefore y_4 = y(0.4) \simeq y_4^C = 5.3836$

Q.8 Solve the differential equation $y'+y+xy^2=0$ with the initial values

x	0.0	0.1	0.2	0.3
y	1.0000	0.9008	0.8066	0.7220

by computing the value of $y(0.4)$ by applying Adams-Bashforth predictor corrector formulas.

(7-Marks)

Sol (3-Marks)

Given $y'+y+xy^2=0 \Rightarrow y' = -(y+xy^2)$

Here $f(x,y) = -(y+xy^2)$

and $h = 0.1$

x	y	$y' = f = f(x,y) = -(y+xy^2)$
$x_0=0.0$	$y_0=1.0000$	$f_0 = f(x_0,y_0) = f(0.0, 1.0000) = -(1.0000+0.0(1.0000)^2) = -1.0000$
$x_1=0.1$	$y_1=0.9008$	$f_1 = f(x_1,y_1) = f(0.1, 0.9008) = -(0.9008+0.1(0.9008)^2) = -0.9819$
$x_2=0.2$	$y_2=0.8066$	$f_2 = f(x_2,y_2) = f(0.2, 0.8066) = -(0.8066+0.2(0.8066)^2) = -0.9367$
$x_3=0.3$	$y_3=0.7220$	$f_3 = f(x_3,y_3) = f(0.3, 0.7220) = -(0.7220+0.3(0.7220)^2) = -0.8784$

$x_4 = x_3+h = 0.3+0.1 = 0.4$

To find $y_4 = y(x_4) = y(0.4)$

$$y_4^p = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0)$$

$$= 0.7220 + \frac{0.1}{24}(55\times -0.8784 - 59\times -0.9367 + 37\times -0.9819 - 9\times -1)$$

$$= 0.6371$$

$$f_4^p = f(x_4, y_4^p) = f(0.4, 0.6371)$$

$$= -(0.6371 + 0.4(0.6371)^2)$$

$$= -0.7995$$

(4-Md)

$$y_4^c = y_3 + \frac{h}{24}(9f_4^p + 19f_3 - 5f_2 + f_1)$$

$$= 0.7220 + \frac{0.1}{24}(9\times -0.7995 + 19\times -0.8784 - 5\times -0.9367 + (-0.9819))$$

$$= 0.6379$$

$\therefore y_4 \approx y_4^c = 0.6379$

$\therefore y(0.4) \approx 0.6379$