CMR
INSTITUTE OF
TECHNOLOGY

USN ☐☐☐☐☐☐☐☐☐☐

Internal Assesment Test-II									
Sub:	Electromagnetic Waves							Code:	18EC55
Date:	20/12 /2021	Duration:	90 mins	Max Marks:	50	Sem:	5th	Branch:	ECE(A,B,C,D)
Solutions									

			OBE	
		Marks	CO	RBT
1.	Derive Maxwell's equation of electrostatics $\nabla . D = \rho_V$.	[10]	CO2	L2

Divergence of a vector & Divergence theorem: from Gauss's law:

Net outward flux $\psi = \oint_s \vec{D} \cdot \vec{ds}$

$$\boxed{\psi = \oint \vec{D} \cdot \vec{ds} = Q_{enc}}$$

Definition of divergence

Divergence of D is defined as the net outward flux per unit volume as the volume shrinks to zero

$$\text{div } \vec{D} = \vec{\nabla} \cdot \vec{D} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{D} \cdot \vec{ds}}{\Delta v}$$

It gives the measure of how much the field diverges or emanates from that point.

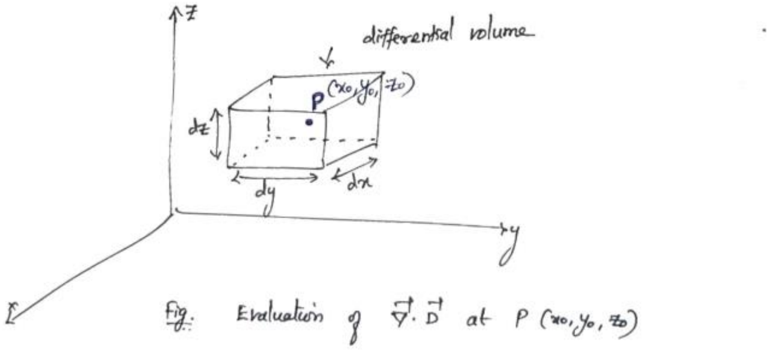


Fig. Evaluation of $\vec{\nabla} \cdot \vec{D}$ at $P(x_0, y_0, z_0)$

$$\oint_s \vec{D} \cdot \vec{Ds} = \left(\iint_{\text{front}} + \iint_{\text{back}} + \iint_{\text{left}} + \iint_{\text{Right}} + \iint_{\text{top}} + \iint_{\text{bottom}} \right) \vec{D} \cdot \vec{ds} .$$

Three dimensional Taylor series expansion of D_x at P,

$$\boxed{\begin{aligned} D_x(x,y,z) &= D_x(x_0,y_0,z_0) + (x-x_0) \left.\frac{\partial D_x}{\partial x}\right|_P + (y-y_0) \left.\frac{\partial D_x}{\partial y}\right|_P \\ &+ (z-z_0) \left.\frac{\partial D_x}{\partial z}\right|_P + \text{higher order terms} \end{aligned}}$$

For the front side,

$$x = x_0 + \frac{dx}{2} \quad \& \quad \vec{ds} = dy\,dz\,\vec{a_x}$$

$$\iint_{front} \vec{D}\cdot\vec{ds} = \left[D_x(x_0, y_0, z_0) + \frac{dx}{2}\frac{\partial D_x}{\partial x}\Big|_P\right]\Big[dy\,dz\Big] + \text{higher order terms}$$

For the back side,

$$x = x_0 - \frac{dx}{2} \quad \& \quad \vec{ds} = dy\,dz\,(-\vec{a_x})$$

$$\iint_{back} \vec{D}\cdot\vec{ds} = \left[D_x(x_0, y_0, z_0) - \frac{dx}{2}\frac{\partial D_x}{\partial x}\Big|_P\right]\Big[-dy\,dz\Big] + \text{higher order terms}$$

Hence,

$$\boxed{\iint_{front}\vec{D}\cdot\vec{ds} + \iint_{back}\vec{D}\cdot\vec{ds} = dx\,dy\,dz\,\frac{\partial D_x}{\partial x}\Big|_P + \text{higher order terms}}$$

By taking similar steps,

$$\boxed{\iint_{left}\vec{D}\cdot\vec{ds} + \iint_{right}\vec{D}\cdot\vec{ds} = dx\,dy\,dz\,\frac{\partial D_y}{\partial y}\Big|_P + \text{higher order terms}}$$

and

$$\boxed{\iint_{top}\vec{D}\cdot\vec{ds} + \iint_{bottom}\vec{D}\cdot\vec{ds} = dx\,dy\,dz\,\frac{\partial D_z}{\partial z}\Big|_P + \text{higher order terms}}$$

$$\lim_{\Delta v \to 0} \frac{\oint_S \vec{D}\cdot\vec{ds}}{\Delta v} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Bigg|_{at\ P} \qquad \Delta v = dx\,dy\,dz$$

higher order terms will vanish as $\Delta v \to 0$.

$$\boxed{\vec{\nabla}\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}} \quad \text{(Rectangular / Cartesian)}$$

similar expressions can be obtained for other co-ordinate systems,

cylindrical

$$\boxed{\vec{\nabla}\cdot\vec{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}}$$

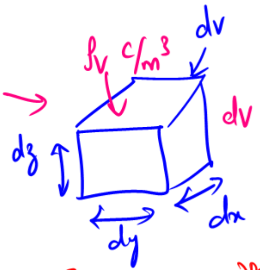
spherical

$$\boxed{\vec{\nabla}\cdot\vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}}$$

Gauss's law:

$$\oiint_S \vec{D}\cdot\vec{ds} = \underline{\underline{Q_{enc}}} \rightarrow$$

?



$$LHS = \oiint_S \vec{D}\cdot\vec{ds} = dx\,dy\,dz\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right]$$

+ higher order terms

$$RHS = Q_{enc} = \iiint_V \rho_v\,dv$$

where

$$\boxed{\rho_v = \lim_{\Delta v \to 0}\left(\frac{\Delta Q}{\Delta V}\right)}$$

$$\boxed{\oiint_S \vec{D}\cdot\vec{ds} = Q_{enc} = \iiint_V \rho_v\,dV} \rightarrow ①$$

Gauss's law for a closed surface

Integral form of Gauss's law

$$\vec{\nabla}\cdot\vec{D} = \lim_{\Delta v \to 0} \frac{\oiint_S \vec{D}\cdot\vec{ds}}{\Delta V}$$

Definition of Divergence

$$\lim_{\Delta v \to 0} \frac{\oiint_S \vec{D}\cdot\vec{ds}}{\Delta V} = \lim_{\Delta v \to 0}\left(\frac{Q_{enc}}{\Delta V}\right)$$

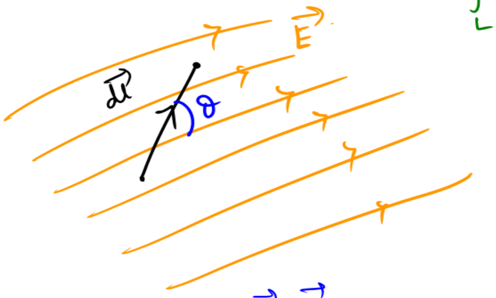
$$\boxed{\nabla\cdot\vec{D} = \rho_V} \rightarrow ②$$

Gauss's law at a point

Point (Differential) form of Gauss's law

2.(a) Prove that electric field intensity is negative gradient of potential. [04] CO3 L2

$$V = -\int_L \vec{E}\cdot\vec{dl}$$



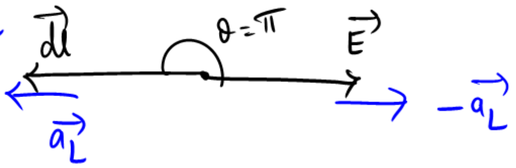
$$dV = -\vec{E}\cdot\vec{dl}$$

$$dV = -|\vec{E}|\,|\vec{dl}|\cos\theta$$

$$|\vec{E}|\cos\theta = -\left|\frac{dv}{dl}\right|$$

$$|\vec{E}|_{max} = \left|\frac{dv}{dl}\right|_{max} \text{ occurs at } \theta = \pi$$

$\vec{dl} = dl\,\vec{a_L}$



$$\vec{E} = \frac{dv}{dl}\left(-\vec{a_L}\right)$$

$$\vec{E} = -\frac{dv}{dl}\,\vec{a_L}$$ (Electric field intensity is negative potential gradient)

$$\vec{E} = -\text{grad } V$$ (or)

$$\vec{E} = -\vec{\nabla} V$$

2.(b) Define current and current density. Derive continuity of current equation. [06] CO2 L2

Current and Current Density:

Electric charges in motion → current (Ampere (A)).

Rate of movement of charge passing a given reference point (or crossing a reference plane).

↓

1 c/s = 1 Ampere.

$$\boxed{I = \frac{dQ}{dt}}$$ ← movement of positive charges

Current density → vector → $\vec{J}$ A/m²

Increment of current ΔI, crossing an incremental surface ΔS, normal to the current density

$$\Delta I = J_N \cdot \Delta s$$

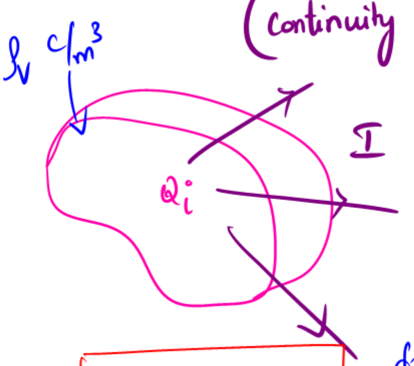
If $\vec{J}$ is not ⊥° to surface

$$\Delta I = \vec{J} \cdot \vec{\Delta s}$$

Total current

$$\boxed{I = \int_s \vec{J} \cdot \vec{ds}}$$

Derive the equation of continuity of current (Continuity equation of current)



I flowing outward

Divergence theorem:

$$\oiint_S \vec{J}\cdot\vec{ds} = \iiint_V \vec{\nabla}\cdot\vec{J}\,dv$$

$$\boxed{I = -\frac{dQ_i}{dt}} \rightarrow ①$$

$$I = \oiint_S \vec{J}\cdot\vec{ds} = \iiint_V (\vec{\nabla}\cdot\vec{J})\,dv \rightarrow ②$$

$$Q_i = \iiint_V \rho_v\,dv \rightarrow ③$$

Sub ② & ③ in ①

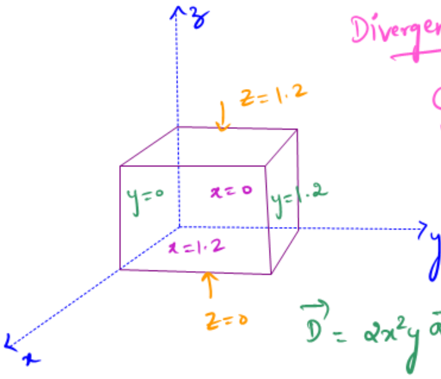
$$\iiint_V (\vec{\nabla}\cdot\vec{J})\,dv = -\frac{d}{dt}\left[\iiint_V \rho_v\,dv\right]$$

$$\rho_v(x,y,z,t)$$

$$\iiint_V (\vec{\nabla}\cdot\vec{J})\,dv = -\iiint_V \left(\frac{\partial\rho_v}{\partial t}\right)dv$$

$$\boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho_v}{\partial t}}$$ continuity eqn of current

3. Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2x^2y\,\mathbf{a}_x + 3x^2y^2\,\mathbf{a}_y$ C/m^2 and the cube formed by the planes $x = 0$ and 1.2, $y = 0$ and 1.2, and $z = 0$ and 1.2. [10] CO2 L3



Divergence theorem:

$$\oint\!\!\oint_S \vec{D}\cdot\vec{ds} = \iiint_V \vec{\nabla}\cdot\vec{D}\,dv$$

$$\vec{D} = 2x^2y\,\vec{a_x} + 3x^2y^2\,\vec{a_y}\ \mathrm{C/m^2}$$

LHS:

$$\oint\!\!\oint_S \vec{D}\cdot\vec{ds} = \iint_{top} \cancelto{0}{\vec{D}\cdot\vec{ds_z}}\Big|_{z=1.2} + \iint_{bottom} \cancelto{0}{\vec{D}\cdot\vec{ds_{-z}}}\Big|_{z=0} +$$

$$\iint_{left} \vec{D}\cdot\vec{ds_{-y}}\Big|_{y=0} + \iint_{right} \vec{D}\cdot\vec{ds_y}\Big|_{y=1.2} +$$

$$\iint_{front} \vec{D}\cdot\vec{ds_x}\Big|_{x=1.2} + \iint_{back} \vec{D}\cdot\vec{ds_{-x}}\Big|_{x=0}$$

$$\oint\!\!\oint_S \vec{D}\cdot\vec{ds} = -\iint_{z\,x} 3x^2\cancelto{0}{y^2}\,dx\,dz\Big|_{\underline{y=0}} + \iint_{z\,x} 3x^2y^2\,dx\,dz\Big|_{y=1.2}$$

$$+ \iint_{z\,y} 2x^2y\ dy\,dz\Big|_{x=1.2} - \iint_{z\,y} 2\cancelto{0}{x^2}y\,dy\,dz\Big|_{\underline{x=0}}$$

$$\oint\!\!\oint_S \vec{D}\cdot\vec{ds} = 3\times(1.2)^2\times\int_0^{1.2}\!\!\int_0^{1.2} x^2\,dx\,dz + 2\times(1.2)^2\times\int_0^{1.2}\!\!\int_0^{1.2} y\,dy\,dz$$

$$= \cancel{3}\times(1.2)^2\times\left[\frac{x^3}{\cancel{3}}\right]_0^{1.2}\times\left[z\right]_0^{1.2} + \cancel{2}\times(1.2)^2\times\left[\frac{y^2}{\cancel{2}}\right]_0^{1.2}\left[z\right]_0^{1.2}$$

$$= (1.2)^5\times1.2 + (1.2)^4\times1.2$$

$$\boxed{\oint\!\!\oint_S \vec{D}\cdot\vec{ds} = 5.474304\ \mathrm{C}}$$

RHS: $\iiint_V \vec{\nabla}\cdot\vec{D}\, dv = ?$ $\vec{D} = 2x^2y\,\vec{a_x} + 3x^2y^2\vec{a_y}$ C/m^2

$$\vec{\nabla}\cdot\vec{D} = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(3x^2y^2) + \frac{\partial}{\partial z}(0)$$

$$\boxed{\vec{\nabla}\cdot\vec{D} = 4xy + 6x^2y \quad C/m^3}$$

$$\iiint_V [4xy + 6x^2y]\, dx\, dy\, dz = \iiint 4xy\, dx\, dy\, dz + \iiint 6x^2y\, dx\, dy\, dz$$

$$= \cancel{4}\left[\frac{x^2}{\cancel{2}}\right]_0^{1.2} \times \left[\frac{y^2}{\cancel{2}}\right]_0^{1.2} \times [z] + \cancel{6}\left[\frac{x^3}{\cancel{3}}\right]_0^{1.2} \times \left[\frac{y^2}{\cancel{2}}\right]_0^{1.2} \times [z]_0^{1.2}$$

$$= (1.2)^2 \times (1.2)^2 \times (1.2) + (1.2)^3 \times (1.2)^2 \times (1.2)$$

$$\boxed{\iiint_V (\vec{\nabla}\cdot\vec{D})\, dv = 5.474304 \quad C}$$

4.(a) Derive Poisson’s and Laplace’s equations in free space. [04] CO3 L2

Poisson's & Laplace's equations:

Point form of Gauss's law, $\nabla\cdot\vec{D} = \rho_v$.

& $\vec{D} = \varepsilon\vec{E}$.

$\vec{E} = -\nabla V$

$\nabla\cdot\vec{D} = \nabla\cdot\varepsilon\vec{E} = \varepsilon\ \nabla\cdot(-\nabla V) = \rho_v$. In homogeneous region, ε is constant

$\Rightarrow \boxed{\nabla\cdot\nabla V = -\frac{\rho_v}{\varepsilon}}$

If $\rho_v = 0$ (zero volume charge density)

but point charges, line charges & surface charge density. exist at singular locations

as source of field

$\Rightarrow \boxed{\nabla^2 V = 0}$ Laplace's equation

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}}$$

Poisson's equation

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$

Laplace's equation

Poisson's eqn

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}}$$

Laplace's eqn

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$

Poisson's equation

$$\boxed{\begin{aligned}\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)\\ + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = -\frac{\rho_v}{\varepsilon}\end{aligned}}$$

Laplace's equation

$$\boxed{\begin{aligned}\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)\\ + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0\end{aligned}}$$

4.(b) State and prove Uniqueness theorem. [06] CO3 L2

UNIQUENESS THEOREM

This is the **uniqueness theorem:** If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is unique.

→ only one solution

The theorem applies to any solution of Poisson's or Laplace's equation in a given region or closed surface.

The theorem is proved by contradiction. We assume that there arc two solutions V1 and V2 of Laplace's equation both of which satisfy the prescribed boundary conditions.

Laplace's eqn ∴

$$\nabla^2 V = 0$$

$V_1(x,y,z)$

$V_2(x,y,z)$

Poisson's eqn ∴

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$\nabla^2 V_1 = 0 \rightarrow$ ①

$\nabla^2 V_2 = 0 \rightarrow$ ②

$\nabla^2 V_1 = -\frac{\rho_v}{\varepsilon} \rightarrow$ ③

$\nabla^2 V_2 = -\frac{\rho_v}{\varepsilon} \rightarrow$ ④

① − ② ⇒ $\nabla^2 V_1 - \nabla^2 V_2 = 0$

③ − ④ ⇒ $\nabla^2 V_1 - \nabla^2 V_2 = 0$

$\boxed{\nabla^2 (V_1 - V_2) = 0} \rightarrow$ ⑤

$V_1 - V_2 = V_d$

⑤ ⇒ $\boxed{\nabla^2 V_d = 0} \rightarrow$ ⑥ ✓

At a given boundary (atleast one conductor surface),

Boundary conditions

$\boxed{V_{1b} = V_{2b}} \rightarrow$ ⑦

$\boxed{V_{db} = V_{1b} - V_{2b} = 0} \rightarrow$ ⑧

$V_1(x,y,z)$ $V_2(x,y,z)$

$\boxed{V_{1b} = V_{2b}}$

$$\boxed{V_d = V_1 - V_2}$$

where $V_1(x,y,z)$
$V_2(x,y,z)$

$$\vec{A} = V_d \vec{\nabla} V_d$$

From the divergence theorem.

$$\boxed{\int_v \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{S}} \rightarrow ⑨$$

$$\iiint_v (\vec{\nabla} \cdot V_d \vec{\nabla} V_d)\, dV = \oiint_S V_d \vec{\nabla} V_d \cdot d\vec{S} \rightarrow ⑩$$

We let $\mathbf{A} = V_d\, \nabla V_d$ and use a vector identity $\rightarrow$ ⑪

$$\boxed{\nabla \cdot \mathbf{A} = \nabla \cdot (V_d \nabla V_d) = V_d \underbrace{\nabla^2 V_d}_{0} + \nabla V_d \cdot \nabla V_d}$$

But $\nabla^2 V_d = 0$ from ⑥

$$\vec{\nabla} V_d \cdot \vec{\nabla} V_d = |\vec{\nabla} V_d|^2$$

$$\nabla \cdot \mathbf{A} = \nabla V_d \cdot \nabla V_d = |\vec{\nabla} V_d|^2$$

$$⑩ \Rightarrow \boxed{\iiint_V |\vec{\nabla} V_d|^2\, dV = \oiint_S \underbrace{V_d}_{0} \vec{\nabla} V_d \cdot \vec{dS}} \rightarrow ⑫$$

Apply the boundary condition,

$\boxed{V_d = 0}$ in ⑫

⑫ $\Rightarrow$

$$\boxed{\iiint_V |\vec{\nabla} V_d|^2 dV = 0}$$

$\int_a^b f(x)\,dx = 0$

i) f(x) is odd in [a,b]

ii) f(x)=0

$\Downarrow$

$$|\vec{\nabla} V_d|^2 = 0$$

$\vec{\nabla} V_d = 0 \Rightarrow V_d = \text{constant} = 0$

Gradient of V_d is zero only if V_d is constant

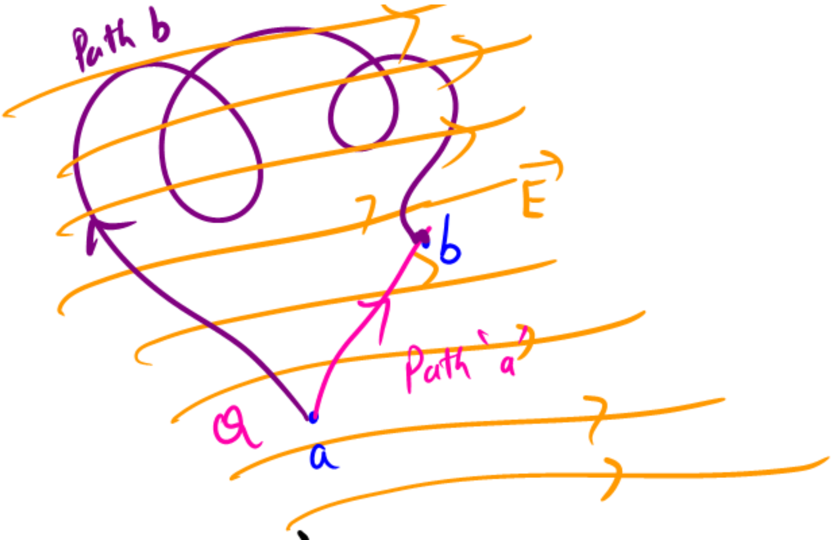
The constant is zero according to boundary condition

$V_d = V_1 - V_2 = 0$

Hence, $\boxed{V_d = 0}$ or $\boxed{V_1 = V_2}$ everywhere,

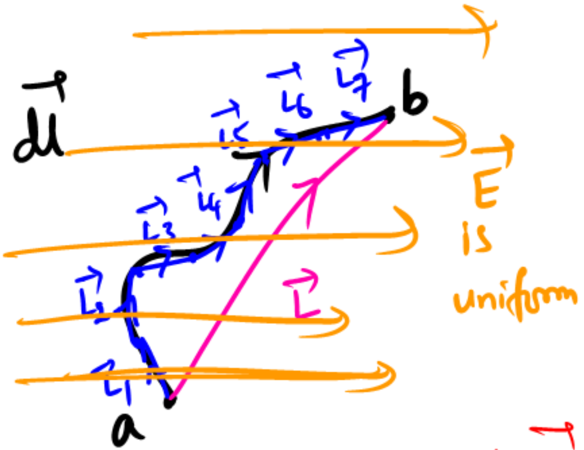
showing that V_1 and V_2 cannot be different solutions of the same problem.

5.(a) Derive an expression for the work done in moving a point charge Q in the presence of an electric field **E**.	[05]	CO3	L2



$$dW = \vec{F}\cdot d\vec{l}$$

$$dW = -Q\,\vec{E}\cdot d\vec{l}$$



$$dw = -Q\left(\underline{\vec{E}}\cdot d\vec{l}_1 + \underline{\vec{E}}\cdot d\vec{l}_2 + \cdots\cdots + \underline{\vec{E}}\cdot d\vec{l}_N\right)$$

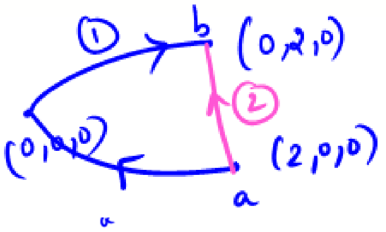
$$= -Q\,\vec{E}\cdot(\underbrace{d\vec{l}_1 + d\vec{l}_2 + \cdots\cdots + d\vec{l}_N})$$

$$dw = -Q\,\vec{E}\cdot d\vec{L}$$

$$\boxed{W = -Q\int_a^b \vec{E}\cdot d\vec{L}}\;//$$

Work done is same irrespective of path chosen

5.(b) Given the electric field intensity **E** = 2x **a**ₓ - 4y **a**y V/m. Find the work done in moving a point charge of 2 C from (2,0,0) to (0,0,0) and then from (0,0,0) to (0,2,0). [05] CO3 L3



Work done is
Same
irrespective
of the
path chosen

path ①

$$W = -Q\int_a^b \vec{E}\cdot\vec{dl}$$

$$W = -2\times\left[\int_{x=2}^{0} 2x\,dx - \int_{y=0}^{2} 4y\,dy\right]$$

$$W = -2\times\left[\cancel{2}\left(\frac{x^2}{\cancel{2}}\right)_2^0 - \overset{2}{\cancel{4}}\left(\frac{y^2}{\cancel{2}}\right)_0^2\right]$$

$$W = -2\left[(0-4) - 2(4-0)\right]$$

$$W = -2\left[-4-8\right]$$

$$\boxed{W = 24\ \text{J}}$$

path ②

$$W = -2\times\left[\int_{x=2}^{0} 2x\,dx - \int_{y=0}^{0} \underbrace{4y\,dy}_{0}\right] + \left\{-2\times\left[\int_{x=0}^{0} \underbrace{2x\,dx}_{0} - \int_{y=0}^{2} 4y\,dy\right]\right\}$$

$$W = -2\left\{\int_2^0 2x\,dx - \int_0^2 4y\,dy\right\}$$

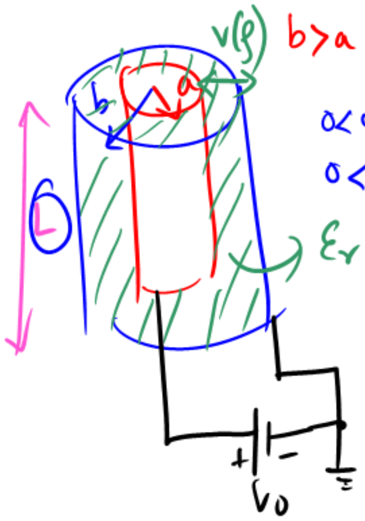
$$W = -2\times\left\{2\left.\frac{x^2}{2}\right|_2^0 - 4\left.\frac{y^2}{2}\right|_0^2\right\}$$

$$W = -2\left\{-4-8\right\}$$

$$\boxed{W = 24\ \text{J}}$$

6.(a) Derive the expression for capacitance of coaxial cable using Laplace's equation. Consider radius of inner conductor 'a' and outer conductor 'b'. Potential at radius 'a' is maintained at V_0 and the outside surface is grounded. [08] CO3 L2

ii) Cylindrical Capacitor:



$0<\phi<2\pi$

$0<z<L$

$\varepsilon_r \varepsilon_0 = \varepsilon$

Boundary conditions

(i) At $\rho = a$, $V = V_0$

ii) At $\rho = b$, $V = 0$

Laplace's eqn:

$$\boxed{\nabla^2 V} = 0 \quad \boxed{V(\rho)}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\cancel{\frac{\partial^2 V}{\partial\phi^2}}\xrightarrow{}0 + \cancel{\frac{\partial^2 V}{\partial z^2}}\xrightarrow{}0 = 0$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$$

$$\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$$

$V = ?$

∫ wrt ρ

$$\rho\frac{dV}{d\rho} = c_1$$

$$\frac{dV}{d\rho} = \frac{c_1}{\rho}$$

∫ wrt ρ

$$V = \int\frac{c_1}{\rho}\,d\rho = c_1 \ln\rho + c_2$$

$$\boxed{V = c_1 \ln\rho + c_2}$$

$$\boxed{\begin{aligned} &(i) \rightarrow V_0 = C_1 \ln a + C_2 \\ &(ii) \rightarrow 0 = C_1 \ln b + C_2 \end{aligned}}$$

$b > a$

$$C_1 \ln(b/a) = -V_0$$

$$\boxed{C_1 = -\frac{V_0}{\ln(b/a)}}$$

$$C_2 = -C_1 \ln b$$

$$\boxed{C_2 = \frac{V_0}{\ln(b/a)} \ln b}$$

$$\boxed{V(\rho) = \underbrace{\frac{-V_0}{\ln(b/a)} \ln\rho + \frac{V_0}{\ln(b/a)} \ln b}\quad V}$$

$$\vec{E} = -\vec{\nabla} V = -\left[\frac{\partial V}{\partial \rho}\vec{a_\rho} + \underbrace{\frac{1}{\rho}\frac{\partial V}{\partial \phi}}_{0}\vec{a_\phi} + \underbrace{\frac{\partial V}{\partial z}}_{0}\vec{a_z}\right]$$

$$\vec{E} = -\left[\frac{-V_0}{\ln(b/a)} \frac{1}{\rho} \vec{a_\rho}\right]$$

$$\boxed{\vec{E} = \frac{V_0}{\rho \ln(b/a)} \vec{a_\rho} \quad V/m}$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\boxed{\vec{D} = \frac{\varepsilon V_0}{\rho \ln(b/a)} \vec{a_\rho}}$$

$$Q = \iint_S \vec{D} \cdot \vec{ds_\rho} \qquad \vec{ds_\rho} = \rho\, d\phi\, dz\, \vec{a_\rho}$$

$$= \int_{z=0}^{L}\int_{\phi=0}^{2\pi} \frac{\varepsilon V_0}{\rho \ln(b/a)} \cdot \rho\, d\phi\, dz$$

$$\boxed{Q = \frac{\varepsilon V_0}{\ln(b/a)} \times 2\pi L}$$

$$V_d = V_0 - 0 = V_0$$

$$\boxed{C = \frac{|Q|}{V_d} = \frac{2\pi \varepsilon L}{\ln(b/a)}} \quad //$$

6.(b) Verify if the given field satisfies Laplace's equation: $V = 2x^2 - 3y^2 + z^2$ V. [02] CO3 L3

$\nabla^2 V = 0$ (Laplace's equation)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial^2}{\partial x^2}(2x^2 - 3y^2 + z^2) + \frac{\partial^2}{\partial y^2}(2x^2 - 3y^2 + z^2) + \frac{\partial^2}{\partial z^2}(2x^2 - 3y^2 + z^2) = 0$$

$$\frac{\partial (4x)}{\partial x} + \frac{\partial}{\partial y}(-6y) + \frac{\partial}{\partial z}(2z) = 0$$

$$4 - 6 + 2 = 0$$

$$0 = 0$$

$\Rightarrow$ $\boxed{\nabla^2 V = 0 \text{ The given potential function } V, \text{ satisfies Laplace's equation}}$

7.(a) Electric potential at an arbitrary point in free space is given as $V = (x+1)^2 + (y+2)^2 + (z+3)^2$ Volts. Find V, **E**, **D**, ρ_V at P(2,1,0). [07] CO2 L3

(i) V at P(2,1,0) $\quad V = (x+1)^2 + (y+2)^2 + (z+3)^2$ V

$$V = (2+1)^2 + (1+2)^2 + (0+3)^2$$

$$= 3^2 + 3^2 + 3^2$$

$$\boxed{V = 27 \text{ V}}$$

(ii) $\vec{E} = -\vec{\nabla} V$

$$= -\left[\frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}\right]$$

$$\boxed{\vec{E} = -\left[2(x+1)\vec{a_x} + 2(y+2)\vec{a_y} + 2(z+3)\vec{a_z}\right]} \text{ V/m}$$

$$\vec{E} \text{ at } (2,1,0) = -\left[2(2+1)\vec{a_x} + 2(1+2)\vec{a_y} + 2(0+3)\vec{a_z}\right]$$

$$= -\left[2\times3\,\vec{a_x} + 2\times3\,\vec{a_y} + 2\times3\,\vec{a_z}\right]$$

$$\boxed{\vec{E} = -6\vec{a_x} - 6\vec{a_y} - 6\vec{a_z} \ \text{V/m}}$$

(iii) $\vec{D} = \varepsilon\vec{E} = -2\varepsilon_0\left[(x+1)\vec{a_x} + (y+2)\vec{a_y} + (z+3)\vec{a_z}\right] \ \text{C/m}^2$

$$\vec{D} \text{ at } (2,1,0) = 8.854\times10^{-12} \times \left[-6\vec{a_x} - 6\vec{a_y} - 6\vec{a_z}\right] \ \text{C/m}^2$$

$$\boxed{\vec{D} = (-53.124\,\vec{a_x} - 53.124\,\vec{a_y} - 53.124\,\vec{a_z}) \ \text{pC/m}^2}$$

(v) $\rho_v = \vec{\nabla}\cdot\vec{D}$

$$\rho_v = -2\varepsilon_0\left[\frac{\partial}{\partial x}(x+1) + \frac{\partial}{\partial y}(y+2) + \frac{\partial}{\partial z}(z+3)\right]$$

$$\rho_v = -2\varepsilon_0\left[1+1+1\right]$$

$$\rho_v = -6\varepsilon_0$$

$$\boxed{\rho_v = -53.124 \ \text{pC/m}^3}$$

7.(b) Given $V = \dfrac{\cos 2\phi}{\rho}$ in free space. Find the volume charge density at the point A(0.5, 60°, 1). [03] CO3 L3

$$\rho_v = ?$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \Rightarrow \rho_v = -\varepsilon\nabla^2 V$$

$$\rho_v = -\varepsilon\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}\right]$$

$$\rho_v = -\varepsilon\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\left[\frac{\cos 2\phi}{\rho}\right]\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\left(\frac{\cos 2\phi}{\rho}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\cos 2\phi}{\rho}\right)\right]$$

$$\rho_v = -\varepsilon\left\{\cos 2\phi\,\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\cancel{\rho}\cdot\left(-\frac{1}{\rho^{\cancel{2}}}\right)\right) + \frac{1}{\rho^2}\cdot\frac{1}{\rho}\,\frac{\partial}{\partial\phi}\left[(-\sin 2\phi)\,(2)\right]\right\}$$

$$\rho_v = -\varepsilon\left\{\frac{\cos 2\phi}{\rho^3} - \frac{4\cos 2\phi}{\rho^3}\right\}$$

$$\boxed{\rho_v = \varepsilon\,\frac{3\cos 2\phi}{\rho^3} \quad C/m^3}$$

$$\rho_v = 8.854\times 10^{-12}\times\frac{3\times\cos(120^\circ)}{(0.5)^3}$$

$$\boxed{\rho_v = -106.248\ p\ C/m^3}$$