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Internal Assesment Test - I

Sub:	Transform Calculus, Fourier Series and Numerical Techniques				Code:	18MAT31
Date:	16.12.2021	Duration: 90 mins	Max Marks: 50	Sem: 3	Branch:	All

First question is compulsory , answer any 6 from Q2 to Q8

		Marks	OBE CO	OBE RBT
1.	Express y as a Fourier series up to 2^{nd} harmonics given the following values.	[8]	CO2	L3

x	0	2	4	6	8	10
y	9	18	24	27.8	27.5	22

		Marks	CO	RBT
2.	Obtain the Fourier series expansion of $f(x) = \frac{\pi - x}{2}$ in $(0,2\pi)$ Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} + \cdots = \frac{\pi}{4}$	[7]	CO2	L3
3.	Obtain cosine series of $(x-1)^2$ in $0 < x < 1$	[7]	CO2	L3
4.	Obtain the Fourier series for the function $f(x) = 2x - x^2$ in $0 \le x \le 2$.	[7]	CO2	L3
5	Obtain the half range Fourier sine series of. $f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$	[7]	CO2	L3
6.	Find the complex Fourier transform of $f(x) = \begin{cases} 1 & for\ \lvert x \rvert \le a \\ 0 & for \lvert x \rvert > a \end{cases}$ .Hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$.	[7]	CO3	L3
7.	Find the infinite Fourier transform of $e^{-\frac{x^2}{2}}$.	[7]	CO3	L3
8.	Find the Fourier sine transform of $\frac{e^{-ax}}{x}, a > 0.$	[7]	CO3	L3

SOLUTIONS - IAT-I 18MAT31 .. 16-12-2021.

Q1 Express y as a Fourier series up to 2 harmonics

x	0	2	4	6	8	10	12
y	9	18	24	27.8	27.5	22	9

Here $2l = 12$

$l = \frac{12}{2}$

$l = 6$

$a_0 = 2 \times$ Mean Value of y in $(0, 12)$

$$= 2 \times \frac{9+18+24+27.8+27.5+22}{6}$$

$$= \frac{2 \times 128.3}{6} = 42.77$$

$a_1 = 2 \times$ Mean Value of $y \cos \frac{\pi x}{6}$ in $(0, 12)$

$$= \frac{2}{6} \times \left(9 \times \cos 0 + 18 \times \cos \frac{2\pi}{6} + 24 \times \cos \frac{4\pi}{6} + 27.8 \times \cos \frac{6\pi}{6} + 27.5 \times \cos \frac{8\pi}{6} + 22 \times \cos \frac{10\pi}{6}\right)$$

$$= \frac{1}{3}\left[9 + 18 \times \frac{1}{2} + 24 \times -\frac{1}{2} + 27.8 \times -1 + 27.5 \times -\frac{1}{2} + 22 \times \frac{1}{2}\right]$$

$$= \frac{1}{3}\left[9 + 9 - 12 - 27.8 - 13.75 + 11\right] = -8.1833$$

$b_1 = 2 \times$ Mean Value of $y \sin \frac{\pi x}{6}$ in $(0, 12)$

$$= \frac{2}{6} \times \left(9 \times \sin 0 + 18 \times \sin \frac{2\pi}{6} + 24 \times \sin \frac{4\pi}{6} + 27.8 \times \frac{6\pi}{6} + 27.5 \sin \frac{8\pi}{6} + 22 \times \sin \frac{10\pi}{6}\right).$$

$$= \frac{1}{3}\left[0 + 18 \times \frac{\sqrt{3}}{2} + 24 \times \frac{\sqrt{3}}{2} + 27.8 \times 0 + 27.5 \times -\frac{\sqrt{3}}{2} + 22 \times -\frac{\sqrt{3}}{2}\right]$$

$$= \frac{\sqrt{3}}{3}\left[9 + 12 - 13.75 - 11\right] = (-3.75)\frac{\sqrt{3}}{3} = -2.1651$$

$a_2 = 2\times$ Mean Value of $y\cos\frac{2\pi x}{6}$ in $(0,12)$

$$= \frac{2}{6}\times\Big[9\times\cos 0 + 18\times\cos\frac{4\pi}{6} + 24\times\cos\frac{8\pi}{6} + 27.8\times\cos\frac{12\pi}{6} + 27.5\times\cos\frac{16\pi}{6} + 22\times\cos\frac{20\pi}{6}\Big]$$

$$= \frac{1}{3}\Big[9 + 18\times\frac{-1}{2} + 24\times\frac{-1}{2} + 27.8\times 1 + 27.5\times\frac{-1}{2} + 22\times\frac{-1}{2}\Big]$$

$$= \frac{1}{3}\Big[9 - 9 - 12 + 27.8 - 13.75 - 11\Big] = -\frac{179}{60} = -2.9833$$

$b_2 = 2\times$ Mean Value of $y\sin\frac{2\pi x}{6}$ in $(0,12)$

$$= \frac{2}{6}\times\Big[9\times\sin 0 + 18\times\sin\frac{4\pi}{6} + 24\times\sin\frac{8\pi}{6} + 27.8\times\sin\frac{12\pi}{6} + 27.5\times\sin\frac{16\pi}{6} + 22\times\sin\frac{20\pi}{6}\Big]$$

$$= \frac{1}{3}\Big[0 + 18\times\frac{\sqrt{3}}{2} + 24\times-\frac{\sqrt{3}}{2} + 27.8\times 0 + 27.5\times\frac{\sqrt{3}}{2} + 22\times-\frac{\sqrt{3}}{2}\Big]$$

$$= \frac{\sqrt{3}}{3}\Big[9 - 12 + 13.75 - 11\Big] = -0.25\times\frac{\sqrt{3}}{3} = -0.1443$$

$\therefore$ The fourier series of y in $(0,12)$ is

$$y = 21.385 - 8.1833\cos\frac{\pi x}{6} - 2.1651\sin\frac{\pi x}{6} - 2.9833\cos\frac{2\pi x}{6} - 0.1443\sin\frac{2\pi x}{6} + \cdots$$

in $(0,12)$.

2. Fourier series of $f(x) = \frac{\pi - x}{2}$ in $(0, 2\pi)$

The Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\,dx = \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[\left(2\pi^2 - \frac{4\pi^2}{2}\right) - \left(0 - \frac{0}{2}\right)\right] = 0.$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\cos nx\,dx$

$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (0-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[\left(-\pi\frac{\sin 2n\pi}{n} - \frac{\cos 2n\pi}{n^2}\right) - \left(\pi\frac{\sin 0}{n} - \frac{\cos 0}{n^2}\right)\right] = 0$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi - x}{2}\sin nx\,dx$

$= \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (0-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[\left(\pi\frac{\cos 2n\pi}{n} - \frac{\sin 2n\pi}{n^2}\right) - \left(-\frac{\pi\cos 0}{n} - \frac{\sin 0}{n^2}\right)\right]$

$= \frac{1}{2\pi}\left[\frac{\pi}{n} - 0 + \frac{\pi}{n} + 0\right] = \frac{1}{n}$

$\therefore$ The Fourier Series is

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx \quad \text{in } (0, 2\pi).$$

take $x = \pi/2 \Rightarrow \frac{\pi - \pi/2}{2} = \sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2} \Rightarrow \frac{\pi}{4} = \left[\frac{1}{1}\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \cdots\right]$

$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

③ Cosine series of $(x-1)^2$ in $0<x<1$

The Cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$.

where $l = 1$

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = 2\int_0^1 (x-1)^2 \cos n\pi x\,dx$.

$$= 2\left[(x-1)^2\frac{\sin n\pi x}{n\pi} - 2(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[(0+0-0) - \left(0 - \frac{2}{n^2\pi^2} - 0\right)\right] = \frac{4}{n^2\pi^2}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{1}\int_0^1 (x-1)^2\,dx = 2\left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{2}{3}(0-(-1))$$
$$= \frac{2}{3}$$

$\therefore$ HRCS is $(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$ in $(0,1)$.

④ Fourier series of $2x-x^2$ in $(0,2)$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{1}\int_0^2 (2x-x^2)\,dx = \left(x^2 - \frac{x^3}{3}\right)_0^2$

$$= \left[\left(4 - \frac{8}{3}\right) - (0-0)\right] = \frac{4}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{1}\int_0^2 (2x-x^2)\cos n\pi x\,dx$$

$$= \left[(2x-x^2)\frac{\sin n\pi x}{n\pi} - (2-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left(0 - \frac{2}{n^2\pi^2} + 0\right) - \left(0 + \frac{2}{n^2\pi^2} + 0\right) = \frac{-4}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^2 (2x-x^2)\sin n\pi x\,dx$$

$$= \left[(2x-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (2-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2 = 0$$

So Fourier series is $2x - x^2 = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$ in $(0,2)$.

(5) Half range sine series of $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$

The HRSS is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx = \frac{2}{\pi}\left[\int_0^{\pi/2} f(x) \sin nx\, dx + \int_{\pi/2}^{\pi} f(x) \sin nx\, dx\right]$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left\{\left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (0-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\left[\left(\frac{-\pi\cos\frac{n\pi}{2}}{2n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right) - (0+0)\right] + \left[(0-0) - \left(-\frac{\pi}{2n}\cos\frac{n\pi}{2} - \frac{\sin\frac{n\pi}{2}}{n^2}\right)\right]\right\}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

$\therefore$ The HRSS is $f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\sin nx$ in $(0, \pi)$.

(6) Fourier transform of $f(x)$ is $F(f(x)) = F(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x}\, dx$

$$= \int_{-\infty}^{-a} f(x) e^{i\lambda x}\, dx + \int_{-a}^{a} f(x) e^{i\lambda x}\, dx + \int_{a}^{\infty} f(x) e^{i\lambda x}\, dx$$

$$= 0 + \int_{-a}^{a} 1\cdot e^{i\lambda x}\, dx + 0 = \left[\frac{e^{i\lambda x}}{i\lambda}\right]_{-a}^{a} = \frac{e^{i\lambda a} - e^{-i\lambda a}}{i\lambda}$$

$$= \frac{2i \sin a\lambda}{i\lambda} = \frac{2\sin a\lambda}{\lambda}, \quad \lambda \neq 0.$$

Using Inv. Fourier Tr, $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda x}\, d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin a\lambda}{\lambda} e^{-i\lambda x}\, d\lambda$

take $x = 0 \Rightarrow f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin a\lambda}{\lambda}\, d\lambda \Rightarrow 1 = \frac{2}{\pi}\int_0^{\infty} \frac{\sin a\lambda}{\lambda}\, d\lambda$

put $a\lambda = x \Rightarrow \frac{\pi}{2} = \int_0^{\infty} \frac{\sin x}{x}\, dx$

7) Fourier transform of $e^{-x^2/2}$ $= F(e^{-x^2/2}) = \int_{-\infty}^{\infty} e^{-x^2/2} e^{i\lambda x} dx$

$$= \int_{-\infty}^{\infty} e^{-\left\{\left(\frac{x}{\sqrt{2}}\right)^2 - 2\left(\frac{x}{\sqrt{2}}\right)\left(\frac{i\lambda}{\sqrt{2}}\right) + \left(\frac{i\lambda}{\sqrt{2}}\right)^2 - \left(\frac{i\lambda}{\sqrt{2}}\right)^2\right\}} dx = e^{-\lambda^2/2} \int_{-\infty}^{\infty} e^{-\left(\frac{x-i\lambda}{\sqrt{2}}\right)^2} dx$$

take $\frac{x-i\lambda}{\sqrt{2}} = t \Rightarrow dx = \sqrt{2}\,dt$ & $t \to \pm\infty$ as $x \to \pm\infty$

$$\therefore F(e^{-x^2/2}) = e^{-\lambda^2/2} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\,dt = \sqrt{2}\, e^{-\lambda^2/2} \cdot \Gamma\left(\tfrac{1}{2}\right) = \sqrt{2\pi}\, e^{-\lambda^2/2}.$$

(8) fourier sine transform of $\frac{e^{-ax}}{x}$, $a > 0$. is

Fourier Sine transform $F_S(\lambda) = \int_0^{\infty} \frac{e^{-ax}}{x} \cdot \sin \lambda x\, dx = I$ (say)

$$\Rightarrow \frac{dI}{d\lambda} = \frac{d}{d\lambda}\left(\int_0^{\infty} \frac{e^{-ax}}{x} \sin\lambda x\, dx\right) \Rightarrow \frac{dI}{d\lambda} = \int_0^{\infty} \frac{\partial}{\partial\lambda}\left(\frac{e^{-ax}}{x} \sin\lambda x\right) dx$$

$$\Rightarrow \frac{dI}{d\lambda} = \int_0^{\infty} \frac{e^{-ax}}{x} \cdot \cos\lambda x \cdot x\, dx = \int_0^{\infty} e^{-ax} \cos\lambda x\, dx$$

$$\Rightarrow \frac{dI}{d\lambda} = \left[\frac{e^{-ax}}{(-a)^2+\lambda^2}\left(-a\cos\lambda x + \lambda\sin\lambda x\right)\right]_0^{\infty}$$

$$\Rightarrow \frac{dI}{d\lambda} = \left[0 - \frac{1}{a^2+\lambda^2}\left(-a\cos 0 + \lambda \sin 0\right)\right] = \frac{a}{a^2+\lambda^2}$$

$$\therefore I = \int \frac{a}{a^2+\lambda^2} d\lambda = \tan^{-1}\frac{\lambda}{a} + c.$$

When $\lambda = 0$; $I = 0$ & $\tan^{-1}\frac{\lambda}{a} = 0$

$$\Rightarrow 0 = \tan^{-1} 0 + c \Rightarrow c = 0$$

$$\therefore I = F_S\left(\frac{e^{-ax}}{x}\right) = \tan^{-1}\frac{\lambda}{a}$$