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Internal Assessment Test 2 – Sept. 2021

Sub:	Advanced Calculus and Numerical Methods					Sub Code:	18MAT21	Branch:	ALL
Date:	02/09/2021	Duration:	90 minutes	Max Marks:	50	Sem / Sec:	II / **ALL SECTIONS**		OBE

Answer all the questions.

		MARKS	CO	RBT
1	Use Newton's Forward and Backward Interpolation formulae to find y(7) and y(17) respectively from the data:	[08]	CO5	L3

x	8	10	12	14	16	18
y	10	19	32.5	54	89.5	154

2	Using Newton's Divided Difference Formula, fit a polynomial for the data given below and hence find f (2.5).	[07]	CO5	L3

x	-3	-1	0	3	5
f (x)	-30	-22	-12	330	3458

3	Use Lagrange's Interpolation Formula to fit a polynomial for the data:	[07]	CO5	L3

x	1	2	3	4
y	5	19	49	101

4	Compute a real root of $x^3 - 10x^2 + 40x - 35 = 0$ by the Method of False-Position, up to four decimal places.	[07]	CO5	L3
5	Find by Newton-Raphson method, a real root of the equation $e^x = x^3 + \cos 25x$ which lies near $x = 4.5$. Correct up to three decimal places.	[07]	CO5	L3
6	Solve $\left(y^2 + z^2\right)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xz$.	[07]	CO3	L3
7	Derive one dimensional Wave equation in the standard form $u_{tt} = c^2 u_{xx}$.	[07]	CO3	L2

SOLUTIONS

(1) Since $x=7$ is at the begining of the table, we use Newton-Forward Interpolation formula at $x=7$. Similarly, since $x=17$ occurs at the end of the table, we use Newton-Backward Interpolation formula at $x=17$.

The Finite Differences Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
8	10					
		9				
10	19		4.5			
		13.5		3.5		
12	32.5		8		2.5	
		21.5		6		6.5
14	54		14		9	
		35.5		15		
16	89.5		29			
		64.5				
18	154					

(a) Newton's forward interpolation formula is

$$y = y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y_0 + \frac{q(q-1)(q-2)(q-3)}{4!}\Delta^4 y_0 + \frac{q(q-1)(q-2)(q-3)(q-4)}{5!}\Delta^5 y_0 + \cdots$$

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To extrapolate at $x=7$, $q=\frac{x-x_0}{h}=\frac{7-8}{2}=-\frac{1}{2}$

Substituting these values in the formula;

$\therefore$ $y(7)=10-\frac{9}{2}+\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\frac{1}{2!}(4.5)-\frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{1}{3!}(3.5)$

$+\frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\frac{1}{4!}(2.5)-\frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\left(\frac{9}{2}\right)\frac{1}{5!}(6.5)$

$\boxed{\therefore\ y(7)=5.1777}$

(b) Newton's Backward Interpolation formula is

$$y=y_n+q\nabla y_n+\frac{q(q+1)}{2!}\nabla^2 y_n+\frac{q(q+1)(q+2)}{3!}\nabla^3 y_n+\frac{q(q+1)(q+2)(q+3)}{4!}\nabla^4 y_n+\frac{q(q+1)(q+2)(q+3)(q+4)}{5!}\nabla^5 y_n+\dots$$

To interpolate at $x=17$, $q=\frac{x-x_n}{h}=\frac{17-18}{2}=-\frac{1}{2}$

substituting these values in the formula

$y(17)=154+(-1/2)(64.5)+(-1/2)(-1/2+1).\frac{1}{2!}(29)+$

$(-1/2)(-1/2+1)(-1/2+2)\frac{1}{3!}(15)+\frac{(-1/2)(-1/2+1)(-1/2+2)(-1/2+3)9}{4!}$

$+\frac{(-1/2)(-1/2+1)(-1/2+2)(-1/2+3)(-1/2+4)}{5!}(6.5)=\underline{126.841}$

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(2) The divided differences table is :

x	$y=f(x)$	Divided differences of order 1	2	3	4
-3	-30				
		$\frac{-22-(-30)}{-1-(-3)}=4$			
			$\frac{10-4}{0-(-3)}=2$		
-1	-22				
				$\frac{26-2}{3-(-3)}=4$	
		$\frac{-12-(-22)}{0-(-1)}=10$			
					$\frac{44-4}{5-(-3)}=5$
0	-12				
			$\frac{114-10}{3-(-1)}=26$		
		$\frac{330-(-12)}{3-0}=114$			
				$\frac{290-26}{3-(-1)}=44$	
3	330				
			$\frac{1564-114}{5-0}=290$		
		$\frac{3458-330}{5-3}=1564$			
5	3458				

The Newton's divided difference polynomial is

$$y=f(x)=y_0+(x-x_0)[x_0,x_1]+(x-x_0)(x-x_1)[x_0,x_1,x_2]+$$
$$(x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3]+(x-x_0)(x-x_1)$$
$$(x-x_2)(x-x_3)[x_0,x_1,x_2,x_3,x_4]$$

substituting the above data,

$$y=f(x)=-30+(x+3)(4)+(x+3)(x+1)(2)+$$
$$(x+3)(x+1)(x+0)(4)+(x+3)(x+1)(x+0)(x-3)(5)$$

$\therefore y = f(x) = 5x^4 + 9x^3 - 27x^2 - 21x - 12$ is the required 4th degree polynomial.

Now, when $x = 2.5$

$$y = f(2.5) = 5(2.5)^4 + 9(2.5)^3 - 27(2.5)^2 - 21(2.5) - 12$$

$$\boxed{\therefore y = 102.6785}$$

(3) The value of table for 'x' and 'y' are as follows:

x	1	2	3	4
y	5	19	49	101

The Lagrange's Interpolation formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

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$$= \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}(5) + \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}(19)$$

$$+ \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}(49) + \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}(101)$$

$$= \frac{x^3-9x^2+26x-24}{-6}(5) + \frac{x^3-8x^2+19x-12}{2}(19)$$

$$+ \frac{x^3-7x^2+14x-8}{-2}(49) + \frac{x^3-6x^2+11x-6}{6}(101)$$

$$\therefore \boxed{f(x) = x^3+2x^2+x+1}$$

(+) Let $f(x) = x^3-10x^2+40x-35$

since $f(1) = -4 < 0$ and $f(2) = 13 > 0$, the root lies between 1 and 2.

<u>1^{st} iteration</u> :- Let $x_0 = 1$ and $x_1 = 2$

since root lies between $x_0 = 1$ & $x_1 = 2$

$$x_2 = x_0 - f(x_0)\frac{(x_1-x_0)}{f(x_1)-f(x_0)} = 1-(-4)\frac{2-1}{13-(-4)} = 1.2353$$

$\therefore x_2 = 1.2353$

$f(x_2) = f(1.2353) = 1.0372 > 0$

2nd Iteration : Here $f(1) = -4 < 0$ and

$f(1.2353) = 1.0372 > 0$

Now, the root lies between $x_0 = 1$ and $x_1 = 1.2353$

$$\therefore x_3 = x_0 - f(x_0)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = 1 - (-4)\frac{1.2353 - 1}{1.0372 - (-4)}$$

$$= 1.1868$$

$\therefore x_3 = 1.1868$

$f(x_3) = f(1.1868) = 0.0595 > 0$

3rd iteration : Here $f(1) = -4 < 0$ and

$f(1.1868) = 0.0595 > 0$

$\therefore$ The root lies between $x_0 = 1$ and $x_1 = 1.1868$

$$x_4 = x_0 - f(x_0)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = 1 - (-4)\frac{1.1868 - 1}{0.0595 - (-4)}$$

$$= 1.1841$$

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$\therefore x_4 = 1.1841$

$f(x_4) = f(1.1841) = 0.0033 > 0$

4th Iteration: Here $f(1) = -4 < 0$ and

$f(1.1841) = 0.0033 > 0$

The root lies between $x_0 = 1$ and $x_1 = 1.1841$

$$\therefore x_5 = x_0 - f(x_0)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = 1-(-4)\frac{1.1841-1}{0.0033-(-4)}$$

$$= 1.1839$$

$\therefore x_5 = 1.1839$

$f(x_5) = f(1.1839) = 0.0002 > 0$

$\therefore$ The approximate root of the equation

$x^3 - 10x^2 + 40x - 35 = 0$ (after 4 iteration)

is $\boxed{1.1839}$ by using the method of

False-position.

— x —

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(5) Let $f(x) = x^3 + \cos(25x) - e^x$

$\therefore f'(x) = 3x^2 - 25\sin(25x) - e^x$

Given $x_0 = 4.5$

<u>1st Iteration</u> : $f(x_0) = f(4.5) = 1.9347$

$f'(x_0) = -15.2061$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4.5 - \frac{1.9347}{(-15.2061)} = 4.6272$$

<u>2nd Iteration</u> : $f(x_1) = f(4.6272) = -4.004$

$f'(x_1) = f'(4.6272) = -51.2363$

$\therefore$ ~~$x_2 = 4.6272$~~ $\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= 4.6272 - \frac{-4.004}{(-51.2367)}$$

$$= 4.5491$$

$\therefore \boxed{x_2 = 4.5491}$

3rd Iteration

$$f(x_2) = f(4.5491) = 0.4019$$

$$f'(x_2) = f'(4.5491) = -47.1876$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 4.5491 - \frac{0.4019}{(-47.1876)} = 4.5576$$

$$\therefore \boxed{x_3 = 4.5576}$$

4th Iteration :-

$$f(x_3) = f(4.5576) = -0.0197$$

$$f'(x_3) = f'(4.5576) = -51.7002$$

$$\therefore x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 4.5576 - \frac{(-0.0197)}{(-51.7002)} = 4.5572$$

$$\therefore \boxed{x_4 = 4.5572}$$

5th Iteration :-

$$f(x_4) = f(4.5572) = 0$$

$$f'(x_4) = f'(4.5572) = -51.5145$$

$$\therefore x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 4.5572 - \frac{0}{(-51.5145)} = 4.5572$$

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$\therefore \boxed{x_5 = 4.5572}$

$\therefore$ The approximate root of the equation

$e^x = x^3 + \cos(25x)$ by using

Newton Raphson method is $\boxed{4.5572}$

(After 5 iterations)

(6) The given equation $(y^2+z^2)p + xyq = xz$ is of the form $Pp + Qq = R$.

The auxilary equations are $\dfrac{dx}{y^2+z^2} = \dfrac{dy}{xy} = \dfrac{dz}{xz}$ —(1)

Taking 2nd and 3rd terms, we have,

$$\frac{dy}{y} = \frac{dz}{z}$$

By integrating, we get $\log y = \log z + \log c_1$

i.e $\log(y/z) = \log c_1 \Rightarrow \boxed{\dfrac{y}{z} = c_1}$

using the multipliers $x, -y, -z$ each ratio in eq (1) is equal to

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$$\frac{x\,dx - y\,dy - z\,dz}{xy^2 + xz^2 - xy^2 - xz^2} = \frac{x\,dx - y\,dy - z\,dz}{0}$$

$\therefore x\,dx - y\,dy - z\,dz = 0$

By integrating, we get, $\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = c_2$

or $\boxed{x^2 - y^2 - z^2 = 2c_2}$

$\therefore$ The general solution of the given PDE is

$$\boxed{\phi\left(y/z,\ x^2 - y^2 - z^2\right) = 0}$$

7) Derivation of one dimensional wave equation:-

Consider a flexible string tightly streched between two fixed points at a distance 'l' apart. Let 'ρ' be the mass per unit length of the string.

We shall assume the following.

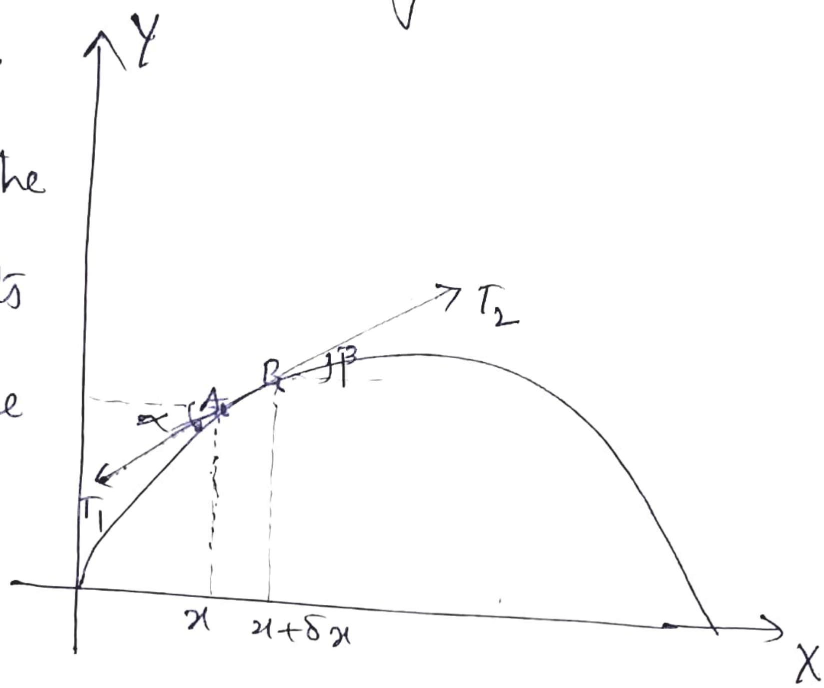
(i) The tension T of the string is same throughout.

(ii) The effect of gravity can be ignored due to large tension T.

(iii) The motion of the string is in small transverse vibrations.

Let us consider the forces acting on a small element AB of length δx.

Let T_1 and T_2 be the tensions at the points A and B. Since there is no motion in the horizontal direction,



the horizontal components T_1 and T_2 must cancel each other.

$$\therefore T_1 \cos\alpha = T_2 \cos\beta = T$$

where α and β are the angles made by T_1 and T_2 with horizontal. Vertical component of tension are $-T_1 \sin\alpha$ and $T_2 \sin\beta$, where the negative sign is used because T_1 is directed downwards. Hence, the resultant force acting vertically upwards is

$$T_2 \sin\beta - T_1 \sin\alpha$$

Applying Newton's second law of motion, i.e

Force = mass x acceleration, we get

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho . \delta x) \frac{\partial^2 u}{\partial t^2}$$

($\rho \delta x$ is the mass of the element portion AB and second derivative w.r.t 't' represents acceleration)

Dividing throughout by T, we have

$$\frac{T_2}{T} \sin\beta - \frac{T_1}{T} \sin\alpha = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2}$$

But from eq ①, $\frac{T_1}{T} = \frac{1}{\cos\alpha}$, $\frac{T_2}{T} = \frac{1}{\cos\beta}$

$$\therefore \frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha} = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2}$$

i.e $$\tan\beta - \tan\alpha = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2} \quad \text{—— (2)}$$

But '$\tan\beta$' and '$\tan\alpha$' represents the slopes at $B(x+\delta x)$ and $A(x)$ respectively.

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$\therefore \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ and $\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_{x}$

Now eq (2) becomes

$$\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 u}{\partial t^2}$$

Dividing by 'δx' and taking limit as $\delta x \to 0$, we have,

$$\lim_{\delta x \to 0} \frac{\left(\partial u/\partial x\right)_{x+\delta x} - \left(\partial u/\partial x\right)_{x}}{\delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

But the LHS is nothing but the derivative of $\frac{\partial u}{\partial x}$ w.r.t 'x' treating 't' as constant.

i.e $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$. Hence we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

Denoting 'T/ρ' by 'c^2', we get

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad (\text{or}) \quad u_{tt} = c^2 u_{xx}}$$

This is the wave equation in one dimension.