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Internal Assessment Test II – February 2022

Sub:	Calculus and Differential Equations					Sub Code:	21MAT11	
Date:	**28/02/2022**	Duration:	**90 mins**	Max.marks	**50**	Sem / Sec:	**I to O (CHEM CYCLE)**	OBE

	Question 1 is compulsory and answer any SIX questions from the rest.	MARKS	CO	RBT
1 .	Show that the radius of curvature for the curve $xy^2 = a^3 - x^3$ at the point $(a, 0)$ is $3a/2$.	[08]	CO1	L3
2 .	Derive the expression to find the radius of curvature for a Cartesian curve.	[07]	CO1	L3
3 .	With usual notation, prove that $cot\varphi = \frac{1}{r}\frac{dr}{d\theta}$.	[07]	CO1	L3

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4.	Show that the following pair of curves intersect each other orthogonally: $r^n = a^n cosn\theta$ and $r^n = b^n sinn\theta$.	[07]	CO1	L3
5.	Find the pedal equation of the polar curve: $r^n cosn\theta = a^n$.	[07]	CO1	L3
6.	Show that for the polar curve $r^2 = a^2 cos2\theta$, ρr is a constant.	[07]	CO1	L3
7.	Obtain the Maclaurin's expansion of $\log(secx + tanx)$ upto the first three non-vanishing terms.	[07]	CO2	L3
8.	Evaluate the following limits: (A) $\lim_{x\to\pi/2}(tanx)^{tan2x}$ (B) $\lim_{x\to 0}(a^x + x)^{1/x}$	[07]	CO2	L3

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21MAT11 Internal Assessment Test - II

(1) We have

$$y^2 = a^3x^{-1} - x^2$$

$$\therefore \quad 2y\frac{dy}{dx} = -a^3x^{-2} - 2x$$

Or

$$\frac{dy}{dx} = \frac{-a^3}{2x^2y} - \frac{x}{y}$$

At $(a,0)$, $\frac{dy}{dx} \to \infty$, so we find $\frac{dx}{dy}$ from

$$xy^2 = a^3 - x^3$$

$$\Rightarrow \quad x(2y) + y^2\frac{dx}{dy} = 0 - 3x^2\frac{dx}{dy}$$

$$\Rightarrow \quad (3x^2+y^2)\frac{dx}{dy} = -2xy$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-2xy}{(3x^2+y^2)}$$

$\boxed{\text{Or } x_1 = \frac{dx}{dy} \text{ at } (a,0) = 0}$

$$\therefore \quad \frac{d^2x}{dy^2} = \frac{(3x^2+y^2)\left(-2y\frac{dx}{dy} - 2x\right) - (-2xy)\left(6x\frac{dx}{dy} + 2y\right)}{(3x^2+y^2)^2}$$

$$\text{Or } \frac{d^2x}{dy^2} \text{ at } (a,0) = \frac{(3a^2+0)(0-2a) - 0}{(3a^2+0)^2} = -\frac{2}{3a}$$

$$\boxed{x_2 = -\frac{2}{3a}}$$

Hence

$$\rho \text{ at } (a,0) = \frac{[1+x_1^2]^{3/2}}{x_2}$$

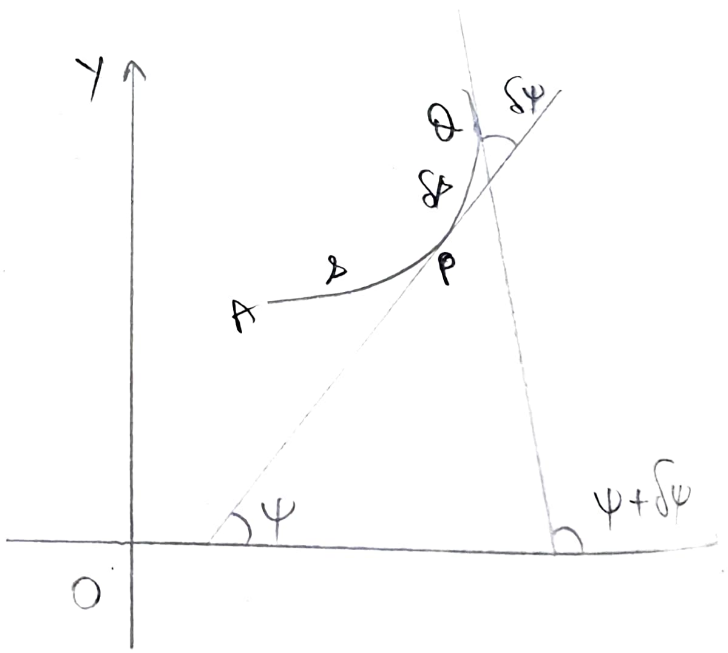
$$= \frac{[1+0]^{3/2}}{-2/3a} = -\frac{3a}{2}$$

$$\rho = -\frac{3a}{2} \Rightarrow |\rho| = \frac{3a}{2}$$

H.P

(2)

$\Rightarrow$ we know that



$$\tan\psi = \frac{dy}{dx} = y_1$$

or $\psi = \tan^{-1}(y_1)$

Differentiate both sides w.r.t 'x'.

$$\frac{d\psi}{dx} = \frac{1}{1+y_1^2}\frac{d(y_1)}{dx} = \frac{y_2}{1+y_1^2}$$

$$\therefore \rho = \frac{ds}{d\psi} = \frac{ds}{dx}\cdot\frac{dx}{d\psi}$$

we know that
Derivative of arc length.

$$\frac{ds}{ds} = \sqrt{1+y_1^2}$$

$$\therefore \frac{d\psi}{dx} = \frac{y_2}{\sqrt{1+y_1^2}}$$

$$\rho = \sqrt{1+y_1^2}\left(\frac{(1+y_1^2)}{y_2}\right)$$

$$\rho = \frac{\left[1+y_1^2\right]^{3/2}}{y_2}$$

where $y_1 = \frac{dy}{dx}$

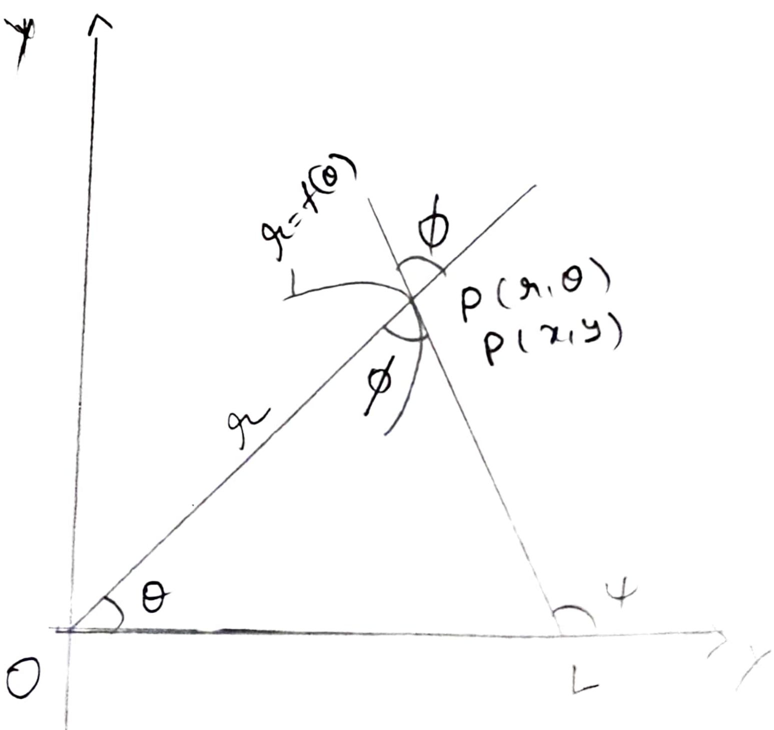
$y_2 = \frac{d^2y}{dx^2}$

③ Angle between radius vector and tangent

Let $P(r,\theta)$ be any point on the curve $r = f(\theta)$.

$\therefore X\hat{O}P = \theta$ and $OP = r$

Let PL be the tangent to the curve at P subtending an angle ψ with positive direction of the initial line.



CMR

and ϕ be the angle between the radius vector OP and the tangent PL.

That is $O\hat{P}L = \phi$.

From the figure we have

$$\psi = \phi + \theta$$

{ An exterior angle is equal to the sum of the interior opposite angles.

$\Rightarrow \tan\psi = \tan(\phi + \theta)$

$$\tan\psi = \frac{\tan\phi + \tan\theta}{1 - \tan\theta\tan\phi} \quad \text{—— (1)}$$

Let (x, y) be the cartesian coordinates of P

so that we have,

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Since r is a function of θ, we can as well regard these as parametric equations in terms of θ.

We also know from the geometrical meaning of the derivative that

$$\tan\varphi = \frac{dy}{dx} = \text{slope of the tangent PL}$$

i.e, $\tan\varphi = \dfrac{dy}{d\theta} \Big/ \dfrac{dx}{d\theta}$ since x and y are functions of θ.

i.e $\tan\varphi = \dfrac{\dfrac{d}{d\theta}(r\sin\theta)}{\dfrac{d}{d\theta}(r\cos\theta)}$

$$\tan\varphi = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$$ where $r' = \dfrac{dr}{d\theta}$

Dividing both the numerator and denominator by $r'\cos\theta$, we have,

$$\tan\varphi = \frac{\dfrac{r\cos\theta}{r'\cos\theta} + \dfrac{r'\sin\theta}{r'\cos\theta}}{\dfrac{-r\sin\theta}{r'\cos\theta} + \dfrac{r'\cos\theta}{r'\cos\theta}}$$

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i.e $\tan\psi = \dfrac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta}$ ——— (2)

Comparing equations (1) & (2) we have

$\tan\phi = \dfrac{r}{r'} = \dfrac{r}{\left(\frac{dr}{d\theta}\right)}$ or $\boxed{\tan\phi = r\dfrac{d\theta}{dr}}$

Equivalently we can write in the form,

$\dfrac{1}{\tan\phi} = \dfrac{1}{r}\left(\dfrac{dr}{d\theta}\right)$ or $\boxed{\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta}}$

(4) Given

$r^n = a^n \cos n\theta$; $r^n = b^n \sin n\theta$

Taking 'log' on both sides.

$\log r^n = \log a^n \cos n\theta$, $\log r^n = \log b^n \sin n\theta$

$n\log r = n\log a + \log\cos n\theta$, $n\log r = n\log b + \log \sin$

Diffn w.r.t 'θ', we get

$$\frac{\cancel{n}}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin n\theta}{\cos n\theta}\cdot\cancel{n} \quad ; \quad \frac{\cancel{n}}{r}\frac{dr}{d\theta} = 0 + \frac{\cos n\theta}{\sin n\theta}\cdot\cancel{n}$$

$$\cot\phi_1 = -\tan n\theta \quad ; \quad \cot\phi_2 = \cot n\theta$$

$$\cot\phi_1 = \cot\left[\frac{\pi}{2} + n\theta\right] \quad ; \quad \cot\phi_2 = \cot n\theta$$

Taking inverse both sides

$$\phi_1 = \frac{\pi}{2} + n\theta \qquad \phi_2 = n\theta$$

$\therefore$ angle of intersection $= |\phi_1 - \phi_2| = \left|\frac{\pi}{2} + n\theta - n\theta\right| = \frac{\pi}{2}$

Thus the curves intersect each other orthogonal

(5) Given

$$r^n \cos n\theta = a^n \quad \text{—— (1)}$$

Take 'log' on both sides,

$$\log r^n \cos n\theta = \log a^n$$

$$\Rightarrow n\log r + \log \cos n\theta = n\log a$$

Differentiate w.r.t θ; we get

$$\Rightarrow \frac{n}{r}\frac{dr}{d\theta} + \frac{(-\sin n\theta)}{\cos n\theta}\cdot n = 0$$

$$\Rightarrow \quad \cot\phi = \frac{\sin n\theta}{\cos n\theta} = \tan n\theta$$

$$\Rightarrow \quad \cot\phi = \cot\left[\frac{\pi}{2} - n\theta\right]$$

$$\Rightarrow \quad \phi = \frac{\pi}{2} - n\theta$$

we consider $p = r\sin\phi$

$$p = r\sin\left(\frac{\pi}{2} - n\theta\right)$$

$$\Rightarrow \quad p = r\cos n\theta$$

$$\Rightarrow \quad \cos n\theta = \frac{p}{r} \quad \text{——} (2)$$

using (2) in (1) we get

$$r^n\left(\frac{p}{r}\right) = a^n$$

$$\Rightarrow \quad \boxed{p\, r^{n-1} = a^n}$$

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(6) Given $r^2 = a^2 \cos 2\theta$ ——— (1)

Take 'log' on both sides

$$\log r^2 = \log a^2 \cos 2\theta$$

$$\Rightarrow 2\log r = 2\log a + \log \cos 2\theta$$

Differentiate w.r.t 'θ', we get

$$\frac{\cancel{2}}{r}\frac{dr}{d\theta} = \frac{-\sin 2\theta}{\cos 2\theta} \cdot \cancel{2}$$

$$\Rightarrow \frac{r_1}{r} = -\tan 2\theta$$

$$r_1 = -r\tan 2\theta \quad \text{———} \ (2)$$

Diffn (2) w.r.t. 'θ' again, we get

$$r_2 = -\left[r_1 \tan 2\theta + r\sec^2 2\theta \cdot 2\right]$$

$$r_2 = -(-r\tan 2\theta)\tan 2\theta - 2r\sec^2 2\theta$$

$$r_2 = r\tan^2 2\theta - 2r\sec^2 2\theta$$

——— (3)

we know that

$$\rho = \frac{\left[r^2+r_1^2\right]^{3/2}}{r^2+2r_1^2-rr_2} \quad \text{---} (4)$$

Using (2) & (3) in (4), we get

$$\rho = \frac{\left[r^2+(-r\tan 2\theta)^2\right]^{3/2}}{r^2+2(-r\tan 2\theta)^2-r\left[r\tan^2 2\theta-2r\sec^2 2\theta\right]}$$

$$\rho = \frac{\left[r^2+r^2\tan^2 2\theta\right]^{3/2}}{r^2+2r^2\tan^2 2\theta-r^2\tan^2 2\theta+2r^2\sec^2 2\theta}$$

$$\rho = \frac{(r^2)^{3/2}\left[1+\tan^2 2\theta\right]^{3/2}}{r^2+r^2\tan^2 2\theta+2r^2\sec^2 2\theta}$$

$$= \frac{r^3\left[\sec^2 2\theta\right]^{3/2}}{r^2\left[1+\tan^2 2\theta\right]+2r^2\sec^2 2\theta}$$

$$\rho = \frac{r^3 \sec^3 2\theta}{r^2 \sec^2 2\theta + 2r^2 \sec^2 2\theta}$$

$$\rho = \frac{r^{\cancel{3}} \sec^{\cancel{3}} 2\theta}{3 \cancel{r^2} \cancel{\sec^2 2\theta}}$$

$$\rho = \frac{r \sec 2\theta}{3}$$

$$\Rightarrow \quad \frac{3\rho}{r} = \sec 2\theta$$

$$\Rightarrow \quad \cos 2\theta = \frac{r}{3\rho} \quad \text{—— (5)}$$

Using (5) in (1), we get-

$$r^2 = a^2 \times \frac{r}{3\rho}$$

$$\Rightarrow \quad \rho r = \frac{a^2}{3} = \text{constant}$$

6. $r^2 = a^2 \cos 2\theta$

Applying log,

$2\log r = 2\log a + \log(\cos 2\theta)$

Diff. wrt θ,

$$\cancel{2} \times \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\cos 2\theta} \times -\cancel{2}\sin 2\theta$$

$$\cot\phi = -\tan 2\theta$$

$$\Rightarrow \cot\phi = \cot\left(\frac{\pi}{2} + 2\theta\right)$$

$$\Rightarrow \phi = \frac{\pi}{2} + 2\theta$$

$\Rightarrow$ ϕ can be expressed explicitly interms of θ.

$\therefore$ We use $p = r\sin\phi$

$$= r\sin\left(\frac{\pi}{2} + 2\theta\right)$$

$$\Rightarrow p = r\cos 2\theta \quad \text{— (1)}$$

From the given curve, $\cos 2\theta = \dfrac{r^2}{a^2}$

$$\therefore (1) \Rightarrow p = r \times \frac{r^2}{a^2}$$

$$\Rightarrow p = \frac{r^3}{a^2}, \text{ pedal equation.}$$

Diff. wrt r,

$$\frac{dp}{dr} = \frac{1}{a^2} \times 3r^2$$

We have,

radius of curvature, $\rho = r \frac{dr}{dp}$

$$\Rightarrow \rho = r \times \frac{a^2}{3r^2}$$

$$\Rightarrow \rho = \frac{a^2}{3r}$$

$$\Rightarrow \rho r = \frac{a^2}{3}, \text{ a constant.}$$

7. Let $y = \log(\sec x + \tan x)$.

$$y(0) = \log(\sec 0 + \tan 0) = \log(1) = 0.$$

$$y_1 = \frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x)$$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

$$\Rightarrow y_1 = \sec x. \qquad y_1(0) = \sec 0 = 1.$$

$$y_2 = \sec x \times \tan x$$

$$\Rightarrow y_2 = y_1 \tan x \qquad y_2(0) = (1)\tan 0 = 0.$$

$$y_3 = y_1 \sec^2 x + y_2 \tan x$$

$$= y_1 y_1^2 + y_2 \tan x = y_1^3 + y_2 \tan x$$

$y_3(0) = (1)^3 + (0)\tan 0 = 1.$

$y_4 = 3y_1^2 y_2 + y_2 \sec^2 x + y_3 \tan x.$

$y_4(0) = 3(1)(0) + 0 + (1)(0)$

$= 0.$

$y_4 = 3y_1^2 y_2 + y_2 y_1^2 + y_3 \tan x = 4y_1^2 y_2 + y_3 \tan x$

$y_5 = 4(y_1^2 y_3 + y_2 \times 2y_1 y_2) + y_3 \sec^2 x + y_4 \tan x.$

$= 4(y_1^2 y_3 + 2y_1 y_2^2) + y_3 y_1^2 + y_4 \tan x$

$= 5y_1^2 y_3 + 8y_1 y_2^2 + y_4 \tan x.$

$y_5(0) = 5(1)(1) + 8(1)(0) + (0)(0)$

$= 5$

Maclaurin's expansion is given by,

$$y = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0)$$

$$+ \frac{x^5}{5!} y_5(0)$$

$$= 0 + x(1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(1) + \frac{x^4}{24}(0) + \frac{x^5}{120}(5) + \cdots$$

$$\Rightarrow \log(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \cdots$$

8. A) $k = \lim_{x \to \frac{\pi}{2}} (\tan x)^{\tan 2x}$ $(\infty^0 \text{ form})$

Applying log,

$\log k = \lim_{x \to \frac{\pi}{2}} \tan 2x \log(\tan x)$

$(0 \times \infty \text{ form})$

$= \lim_{x \to \frac{\pi}{2}} \frac{\log(\tan x)}{\cot 2x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$.

Applying L'Hospital's rule,

$$\log k = \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\tan x} \times \sec^2 x}{-2 \operatorname{cosec}^2 2x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cancel{\cos x}}{\sin x} \times \frac{1}{\cos^{\cancel{2}} x} \times \frac{-1}{2} \sin^2 2x$$

$$= \lim_{x \to \frac{\pi}{2}} -\frac{1}{2 \sin x \cos x} \times \sin^2 2x.$$

$$= -\lim_{x \to \frac{\pi}{2}} \frac{\sin^{\cancel{2}} 2x}{\cancel{\sin} 2x}$$

$$= -\lim_{x \to \frac{\pi}{2}} \sin 2x = 0.$$

$\Rightarrow k = e^0 = 1$

B) $k = \lim_{x \to 0} (a^x + x)^{1/x}$ (1^∞ form)

Applying log,

$\log k = \lim_{x \to 0} \frac{1}{x} \log(a^x + x)$ ($\infty \times 0$ form)

$= \lim_{x \to 0} \frac{\log(a^x + x)}{x}$ ($\frac{0}{0}$ form)

Applying L'Hospital's rule,

$\log k = \lim_{x \to 0} \frac{\frac{1}{a^x + x}(a^x \log a + 1)}{1}$

$= \frac{1}{a^0 + 0}(a^0 \log a + 1)$

$\log k = \log a + 1$

$\log k = \log a + \log e$

$\log k = \log(ae)$

$k = ae.$