USN

Internal Assessment Test III– March 2022

Sub:	**Calculus and Differential Equations**				Sub Code:	**21MAT11**		
Date:	**28/03/2022**	Duration:	**90 mins**	Max.marks	**50**	Sem/Sec:	**I to O (CHEM CYCLE)**	OBE

Question 1 is compulsory and answer any SIX questions from the rest.		MARKS	CO	RBT
1 .	Find the extreme values of the function: $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.	[08]	CO2	L3
2 .	Solve : $xyp^2 - (x^2+y^2)p + xy = 0$.	[07]	CO5	L3
3 .	If the air is maintained at 30^0C and the temperature of the body cools from 80^0C to 60^0C in 12 minutes, find how long will it take for the body to reach a temperature of 48^0C.	[07]	CO2	L3

USN

Internal Assessment Test III – January 2022

Sub:	**Calculus and Differential Equations**				Sub Code:	**21MAT11**		
Date:	**28/03/2022**	Duration:	**90 mins**	Max.marks	**50**	Sem/Sec:	**I to O (CHEM CYCLE)**	OBE

Question 1 is compulsory and answer any SIX questions from the rest.		MARKS	CO	RBT
1 .	Find the extreme values of the function: $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.	[08]	CO2	L3
2 .	Solve : $xyp^2 - (x^2+y^2)p + xy = 0$.	[07]	CO5	L3
3 .	If the air is maintained at 30^0C and the temperature of the body cools from 80^0C to 60^0C in 12 minutes, find how long will it take for the body to reach a temperature of 48^0C.	[07]	CO2	L3

4.	Solve $(8xy - 9y^2)dx + 2(x^2 - 3xy)dy = 0$	[07]	CO3	L3
5.	Solve: $rsin\theta - cos\theta \frac{dr}{d\theta} = r^2$	[07]	CO3	L3
6.	If $x + y + z = u, y + z = uv, z = uvw$ show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.	[07]	CO2	L3
7.	Show that the family of curves $x^2 = 4a(y + a)$ is self orthogonal.	[07]	CO2	L3
8.	If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right), find\ x^2 \frac{\partial u}{\partial x} + y^2 \frac{du}{dy} + z^2 \frac{\partial u}{\partial z}$.	[07]	CO2	L3

4.	Solve $(8xy - 9y^2)dx + 2(x^2 - 3xy)dy = 0$	[07]	CO3	L3
5.	Solve $rsin\theta - cos\theta \frac{dr}{d\theta} = r^2$	[07]	CO3	L3
6.	If $x + y + z = u, y + z = uv, z = uvw$ show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.	[07]	CO2	L3
7.	Show that the family of curves $x^2 = 4a(y + a)$ is self orthogonal.	[07]	CO2	L3
8.	If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right), find\ x^2 \frac{\partial u}{\partial x} + y^2 \frac{du}{dy} + z^2 \frac{\partial u}{\partial z}$.	[07]	CO2	L3

IAT - III Solutions (Chem cycle)

1. $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3$

$f_y = 3y^2 - 12$

We shall find points (x,y) such that $f_x = 0$ and $f_y = 0$

$\Rightarrow 3x^2 - 3 = 0$ and $3y^2 - 12 = 0$

$\Rightarrow 3x^2 = 3$ and $3y^2 = 12$

$\Rightarrow x^2 = 1$ and $y^2 = 4$

$\Rightarrow x = \pm\sqrt{1}$ and $y = \pm\sqrt{4} = \pm 2$

$= \pm 1$

$\therefore (1,2), (1,-2), (-1,2), (-1,-2)$ are the critical points.

$r = f_{xx} = 6x$

$s = f_{xy} = 0$

$t = f_{yy} = 6y$

Case (i):- $(1,2)$

$r = 6 > 0$

$s = 0$

$t = 12$

$rt - s^2 = 72 > 0$

$(1,2)$ is a point of minimum and the minimum value is $f(1,2) = 1 + 8 - 3 - 24 + 20 = 2$.

Case (ii):- $(1,-2)$

$r = 6$, $s = 0$, $t = -12$ and $rt - s^2 = -72 < 0$.

$\therefore (1,-2)$ is a saddle point.

Case (iii):- $(-1,2)$

$r=-6,\ s=0,\ t=12,\ rt-s^2=-72<0$

$\therefore (-1,2)$ is a saddle point.

Case (iv):- $(-1,-2)$

$r=-6<0,\ s=0,\ t=-12,\ rt-s^2=72>0.$

$(-1,-2)$ is a point of maximum.

Maximum value is $f(-1,-2)$

$= -1-8+3+24+20 = 38.$

2. $xyp^2-(x^2+y^2)p+xy=0.$

$$p=\frac{(x^2+y^2)\pm\sqrt{(x^2+y^2)^2-4x^2y^2}}{2xy}.$$

$$p=\frac{(x^2+y^2)\pm(x^2-y^2)}{2xy}$$

(i) $p=\frac{x^2+y^2+x^2-y^2}{2xy}=\frac{x}{y}.$

$\frac{dy}{dx}=\frac{x}{y} \Rightarrow y\,dy=x\,dx$

Integrating,

$\frac{y^2}{2}=\frac{x^2}{2}+c \Rightarrow (y^2-x^2)=2c$

$\Rightarrow (y^2 - x^2 - 2c) = 0.$

(ii) $p = \frac{x^2+y^2-x^2+y^2}{2xy} = \frac{y}{x}$

$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{1}{y}dy = \frac{1}{x}dx.$

Integrating, we get

$\log y = \log x + \log c.$

$\Rightarrow \log y = \log cx \Rightarrow \log y - \log cx = 0.$

$\Rightarrow \log\left(\frac{y}{cx}\right) = 0 \Rightarrow \frac{y}{cx} = 1$

$\Rightarrow y - cx = 0.$

$\therefore$ The general soln is

$(y^2 - x^2 - 2c)(y - cx) = 0.$

3. We have by data, $t_1 = 80°C$, $t_2 = 30°C$ and $T = 60°C$ in $t = 12$ mins.

By Newton's law of cooling,

$T = t_2 + (t_1 - t_2)e^{-kt}.$

$T = 30 + (80-30)e^{-kt}$

$T = 30 + 50e^{-kt}$ —— (1).

Given $T = 60$ when $t = 12$;

$$60 = 30 + 80 e^{-k(12)}$$

$$\Rightarrow e^{-12k} = \frac{60-30}{50} = \frac{3}{5}$$

$$\Rightarrow -12k = \ln\left(\frac{3}{5}\right)$$

$$\Rightarrow k = -\frac{1}{12}\ln\left(\frac{3}{5}\right).$$

$$= 0.04257.$$

$$\therefore (1) \Rightarrow T = 30 + 50 e^{-0.04257t}$$

When $T = 48°C$,

$$48 = 30 + 50 e^{-0.04257t}$$

$$\Rightarrow \frac{48-30}{50} = e^{-0.04257t}$$

$$\Rightarrow \frac{18}{50} = e^{-0.04257t}$$

$$\Rightarrow t = \frac{-1}{0.04257}\ln\left(\frac{18}{50}\right)$$

$$\Rightarrow t = 23.999$$

$$\simeq 24 \text{ mins.}$$

4. $M = 8xy - 9y^2$ and $N = 2x^2 - 6xy$

$\frac{\partial M}{\partial y} = 8x - 18y$ and $\frac{\partial N}{\partial x} = 4x - 6y$

$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x - 12y = 4(x - 3y)$ — near to N.

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4(x-3y)}{2x(x-3y)} = \frac{2}{x} = f(x)$

Hence,

$$I.F = e^{\int f(x)\, dx} = e^{\int \frac{2}{x} \cdot dx} = e^{2\int \frac{1}{x} \cdot dx}$$

$$= e^{2\log x} = x^2.$$

Multiplying the given eqⁿ by x^2,

$M = 8x^3y - 9x^2y^2$ and $N = 2x^4 - 6x^3y$

$\frac{\partial M}{\partial y} = 8x^3 - 18x^2y$ and $\frac{\partial N}{\partial x} = 8x^3 - 18x^2y$

The solⁿ is $\int M\, dx + \int N(y)\, dy = c$.

$$\int (8x^3y - 9x^2y^2)\, dx + \int 0 \cdot dy = c$$

$$\Rightarrow 2x^4y - 3x^3y^2 = c.$$

5. $r \sin\theta - \cos\theta \dfrac{dr}{d\theta} = r^2$

$\Rightarrow \cos\theta \dfrac{dr}{d\theta} - r\sin\theta = -r^2$

$\div r^2, \quad \dfrac{\cos\theta}{r^2}\dfrac{dr}{d\theta} - \dfrac{1}{r}\sin\theta = -1$ —— (1)

Put $\dfrac{1}{r} = y$.

$-\dfrac{1}{r^2}\dfrac{dr}{d\theta} = \dfrac{dy}{d\theta}$.

(1) $\Rightarrow -\cos\theta \dfrac{dy}{d\theta} - y\sin\theta = -1$

$\Rightarrow \dfrac{dy}{d\theta} + (\tan\theta)y = \sec\theta$; linear in y.

$P = \tan\theta$, $Q = \sec\theta$

$I.F = e^{\int P\,d\theta} = e^{\int \tan\theta\,d\theta} = e^{\log(\sec\theta)} = \sec\theta$.

Solution is given by, $y(I.F) = \int Q(I.F) + c$

$y\sec\theta = \int \sec^2\theta \cdot d\theta + c$.

$y\sec\theta = \tan\theta + c$

$\Rightarrow \dfrac{\sec\theta}{r} = \tan\theta + c$.

6. $x+y+z=u, \quad y+z=uv, \quad z=uvw.$

To prove: $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v.$

$x+y+z=u$

$\Rightarrow x+uv=u$

$\Rightarrow x=u-uv$

$y+z=uv$

$\Rightarrow y+uvw=uv$

$\Rightarrow y=uv-uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\left[(u-uw)(uv) + u^2vw\right]$$
$$+ u\left[uv(v-vw) + uv^2w\right]$$
$$= (1-v)\left[u^2v - u^2\cancel{v}w + u^2\cancel{v}w\right]$$
$$+ u\left[uv^2 - u\cancel{v}^2w + u\cancel{v}^2w\right]$$
$$= u^2v - u^2v^2 + u^2v^2 = u^2v.$$

7. $x^2 = 4a(y+a)$ —— (1)

Diff. wrt x,

$2x = 4ay_1 \Rightarrow y_1 = \frac{x}{2a} \Rightarrow a = \frac{x}{2y_1}$

(1) becomes $x^2 = \cancel{4}^{2}\left(\frac{x}{\cancel{2}y_1}\right)\left(y + \frac{x}{2y_1}\right)$.

$$x^2 = \frac{2x}{y_1}\left(y + \frac{x}{2y_1}\right).$$

$$x^2 = \frac{2xy}{y_1} + \frac{x^2}{y_1^2}$$

$\Rightarrow x^2 y_1^2 = 2xyy_1 + x^2$ — differential eqn of the given family.

Replacing y_1 by $-\frac{1}{y_1}$,

$$\Rightarrow x^2\left(-\frac{1}{y_1}\right)^2 = 2xy\left(-\frac{1}{y_1}\right) + x^2$$

$$\Rightarrow \frac{x^2}{y_1^2} = -\frac{2xy}{y_1} + x^2$$

$$\Rightarrow \frac{x^2}{y_1^2} = \frac{-2xy + x^2 y_1}{y_1}$$

$$\Rightarrow x^2 = -2xyy_1 + x^2 y_1^2$$

$\Rightarrow x^2 y_1^2 = 2xyy_1 + x^2$ — differential eqⁿ of OT

D.E of OT is sames as that of the given family ①.

$\therefore$ ① is self-orthogonal.

8. $u = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$.

Let $P = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$

$Q = \frac{z-x}{zx} = \frac{1}{x} - \frac{1}{z}$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial P} \times \frac{\partial P}{\partial x} + \frac{\partial u}{\partial Q} \times \frac{\partial Q}{\partial x}$$

$$= \frac{\partial u}{\partial P}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial Q} \times \left(\frac{-1}{x^2}\right).$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial P} \times \frac{\partial P}{\partial y} = \frac{\partial u}{\partial P}\left(\frac{1}{y^2}\right)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial Q} \cdot \frac{\partial Q}{\partial z} = \frac{\partial u}{\partial Q} \times \left(\frac{1}{z^2}\right).$$

Now, $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z}$

$$= x^2\left[-\frac{1}{x^2}\frac{\partial u}{\partial P} - \frac{1}{x^2}\frac{\partial u}{\partial Q}\right]$$

$$+ y^2\left[\frac{1}{y^2}\frac{\partial u}{\partial P}\right] + z^2\left[\frac{1}{z^2}\frac{\partial u}{\partial Q}\right]$$

$$= -\frac{\partial u}{\partial P} - \frac{\partial u}{\partial Q} + \frac{\partial u}{\partial P} + \frac{\partial u}{\partial Q}$$

$$= 0.$$