CBCS SCHEME

USN 1CR21EE017

21MAT11

First Semester B.E./B.Tech. Degree Examination, Feb./Mar. 2022
Calculus and Differential Equations

Time: 3 hrs. Max. Marks: 100

Note: Answer any FIVE full questions, choosing ONE full question from each module.

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

Module-1

1 a. With usual notation prove that $\tan\phi = r\dfrac{d\theta}{dr}$. **(06 Marks)**

b. Find the angle between the curves $r = a\log\theta$ and $r = \dfrac{a}{\log\theta}$. **(07 Marks)**

c. Find the radius of curvature for the cardioid, $r = a(1+\cos\theta)$. **(07 Marks)**

OR

2 a. With usual notation prove that $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$. **(06 Marks)**

b. Show that $r = 4\sec^2\theta/2$ and $r = 9\operatorname{cosec}^2\theta/2$ the pair of curves cut orthogonally. **(07 Marks)**

c. Find the pedal equation of the curve $r^n = a^n\cos n\theta$. **(07 Marks)**

Module-2

3 a. Expand $\sqrt{1+\sin 2x}$ by Maclaurin's series up to the term containing x^4. **(06 Marks)**

b. If $u = f\left(\dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x}\right)$ prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. **(07 Marks)**

c. If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$ find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ at (1, -1, 0). **(07 Marks)**

OR

4 a. Evaluate $\lim\limits_{x\to 0}\left(\dfrac{a^x + b^x + c^x}{3}\right)^{1/x}$. **(06 Marks)**

b. If $z = e^{ax+by} f(ax - by)$ prove that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$. **(07 Marks)**

c. Find the extreme values of $f(x,y) = x^3 + y^3 - 3x - 12y + 20$. **(07 Marks)**

Module-3

5 a. Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2x$. **(06 Marks)**

b. Find the orthogonal trajectories of the family of curves $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+\lambda} = 1$, where λ is a parameter. **(07 Marks)**

c. Solve $x(y')^2 - (2x+3y)y' + 6y = 0$. **(07 Marks)**

OR

6 a. Solve $(x^2 + y^2 + x)dx + xydy = 0$. (06 Marks)

b. If the temperature of the air is 30°C and a metal ball cools from 100°C to 70°C in 15 minutes, find how long will it take for the metal ball to reach a temperature of 40°C. (07 Marks)

c. Find the general solutions of $xp^2 + xp - yp + 1 - y = 0$. (07 Marks)

Module-4

7 a. Solve $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$. (06 Marks)

b. Solve $(D^3 + D^2 - 4D - 4)y = 3e^{-x}$. (07 Marks)

c. Solve $\dfrac{d^2y}{dx^2} + y = \sec x \tan x$ using the method of variation of parameters. (07 Marks)

OR

8 a. Solve $(D^2 + 4)y = x^2$. (06 Marks)

b. Solve $\dfrac{d^2y}{dx^2} - 4y = \cosh(2x - 1)$. (07 Marks)

c. Solve $(x^2D^2 + xD + 9)y = 3x^2$. (07 Marks)

Module-5

9 a. Find the rank of the matrix.

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$ (06 Marks)

b. Solve by Gauss elimination method

$2x + y + 4z = 12$
$4x + 11y - z = 33$
$8x - 3y + 2z = 20$. (07 Marks)

c. Solve the system of equation by Gauss-Seidel method

$20x + y - 2z = 17$
$3x + 20y - z = -18$
$2x - 3y + 20z = 25$. (07 Marks)

OR

10 a. Find the values of λ and μ such that the system of equations:

$x + y + z = 6$
$x + 2y + 3z = 10$
$x + 2y + \lambda z = \mu$, may have

i) unique solution ii) infinite solution iii) no solution. (06 Marks)

$\dfrac{e^x + e^{-x}}{2}$

b. Solve by the method of Gauss-Jordan method:

$2x + 5y + 7z = 52$
$2x + y - z = 0$
$x + y + z = 9$. (07 Marks)

c. Find the largest eigen value and the corresponding eigen vector of the matrix

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ by using the power method by taking initial vector as $[1, 1, 1]^T$. (07 Marks)

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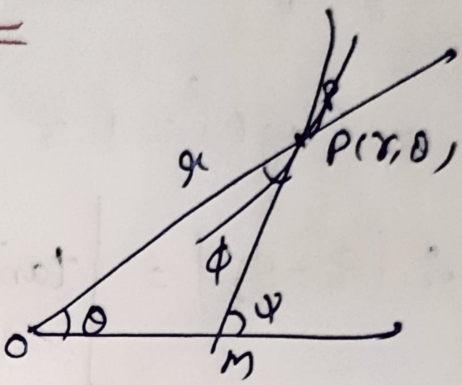
Solution of VTU QP.

1st Sem BE/BTECH, 21MA11

Calculus and Differential Equations

MODULE-1

Q1. (a) Let $P(r,\theta)$ be any point on $r=f(\theta)$. $\angle MOP = \phi$, PM is the tangent to the curve.



$\psi = \theta + \phi$

$\tan\psi = \tan(\theta+\phi)$

$= \dfrac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$ —— ①

$\therefore \tan\psi = \dfrac{dy}{dx}$ (slope) and $r\cos\theta = x$, $r\sin\theta = y$.

$\tan\psi = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{r\cos\theta + \sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}}$

$\tan\psi = \dfrac{\tan\theta + \dfrac{r}{r_1}}{1-\tan\theta\cdot\dfrac{r}{r_1}}$, $r_1 = \dfrac{dr}{d\theta}$ —— ②

Comparing ① & ② $\Rightarrow$ $\boxed{\tan\phi = r\dfrac{d\theta}{dr}}$

(b) $r = a\log\theta$

$\log r = \log a + \log(\log\theta)$

$\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{1}{\theta(\log\theta)}$

$\tan\phi_1 = \dfrac{1}{\theta(\log\theta)}$

$r = \dfrac{a}{\log\theta}$

$\log r = \log a - \log(\log\theta)$

$\dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{1}{\theta(\log\theta)}$

$\tan\phi_2 = -\dfrac{1}{\theta(\log\theta)}$

$$\tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + (\tan\phi_1)(\tan\phi_2)} = \frac{\frac{1}{\theta(\log\theta)} + \frac{1}{\theta(\log\theta)}}{1 - \frac{1}{(\theta\log\theta)^2}}.$$

angle of intersection $\quad a\log\theta = \dfrac{a}{\log\theta}$

$$\Rightarrow (\log\theta)^2 = 1 \Rightarrow \log\theta = 1 \Rightarrow \theta = e$$

$$\therefore |\phi_1 - \phi_2| = \left|\tan^{-1}\left\{\frac{\frac{2}{e}}{1 - \frac{1}{e^2}}\right\}\right| = \left|\tan^{-1}\frac{2e}{e^2-1}\right|$$

(c) $\quad \rho = \dfrac{[r^2 + r_1^2]^{3/2}}{r^2 + 2r_1^2 - r_2 r}, \qquad r = a(1+\cos\theta) = 2a\cos^2\theta/2$

$\log r = \log a + \log(1+\cos\theta)$

$= \log a + 2\log\cos\theta/2$ — (1)

diff (1), w.r.t 'θ'

~~$\frac{1}{r} r_1 = -2\tan\theta/2 \cdot 1/2 \Rightarrow$~~

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + 2\,\frac{-\sin\theta/2}{\cos\theta/2}\cdot\frac{1}{2} \Rightarrow r_1 = -r\tan\theta/2$$

$$r_2 = -r_1\tan\theta/2 - \frac{r}{2}\sec^2\theta/2 = r\tan^2\theta/2 - \frac{r}{2}\sec^2\theta/2$$

$$\rho = \frac{[r^2 + r^2\tan^2\theta/2]^{3/2}}{r^2 + 2r^2\tan^2\theta/2 - r^2\tan^2\theta/2 + \frac{r^2}{2}\sec^2\theta/2}$$

$$= \frac{r^3\sec^3\theta/2}{r^2[\underbrace{1 + \tan^2\theta/2}_{\sec^2\theta/2} + \frac{1}{2}\sec^2\theta/2]} = \frac{r\sec^3\theta/2}{3/2\sec^2\theta/2}$$

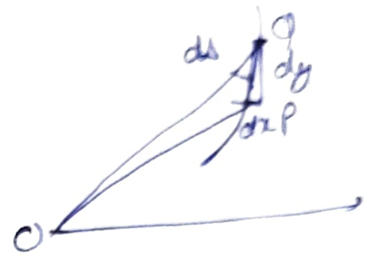
$$\rho = \frac{2}{3}r\sec\theta/2 = \frac{2\sqrt{2a}}{3}\sqrt{r} \quad \text{Ans}$$

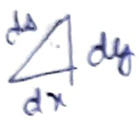
Q2. (a) we know $\rho = \frac{ds}{d\psi} = \frac{ds}{dx} \cdot \frac{dx}{d\psi}$ —— (1)

$\because \tan\psi = \frac{dy}{dx} \Rightarrow \sec^2\psi \cdot \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$

$\Rightarrow \frac{d\psi}{dx} = \frac{y_2}{\sec^2\psi} = \frac{y_2}{1+\tan^2\psi} = \frac{y_2}{1+y_1^2}$ —— (2)

for $\frac{ds}{dx}$, from the fig.





$ds^2 = dx^2 + dy^2$

$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2$

$\frac{ds}{dx} = (1+y_1^2)^{1/2}$ —— (3)

from (1), (2) and (3)

$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}}$$

(b) $r = 4\sec^2\theta/2$

$\log r = \log 4 + 2\log\sec\theta/2$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{2}{\sec\theta/2} \cdot \sec\theta/2 \tan\theta/2 \cdot 1/2$

$\tan\phi_1 = \tan\theta/2$

$r = 9\,\mathrm{Cosec}^2\,\theta/2$

$\log r = \log 9 + 2\log\mathrm{Cosec}\,\theta/2$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{2(-\mathrm{Cosec}\,\theta/2\,\mathrm{Cot}\,\theta/2)}{\mathrm{Cosec}\,\theta/2} \cdot \frac{1}{2}$

$\tan\phi_2 = -\mathrm{Cot}\,\theta/2$

$\tan\phi_1 \cdot \tan\phi_2 = (\tan\theta/2)(-\cot\theta/2) = -1$

$\Rightarrow$ given curves are orthogonal.

(c) $r^n = a^n \cos n\theta$ — ① $\Rightarrow n \log r = n \log a + \log \cos n\theta$

$\boxed{p = r \sin\phi}$ — ②

$\frac{n}{r}\frac{dr}{d\theta} = 0 - n \tan n\theta$

$\tan\phi = -\tan n\theta \Rightarrow \cancel{\phi = -n\theta}$

$= \tan(\pi/2 + n\theta)$

$p = r \sin(\pi/2 + n\theta)$

$p = r \cos n\theta$ — ③

from ① & ③ $\Rightarrow p = r \cdot \frac{r^n}{a^n} \Rightarrow \boxed{r^{n+1} = a^n p}$ Ans

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Module - II

Q3 (a) $y = \sqrt{1 + \sin 2x} = \sqrt{\cos^2 x + \sin^2 x + 2 \sin x \cos x}$

$y = \sqrt{(\cos x + \sin x)^2} =$

$y = \cos x + \sin x$ — ① $\Rightarrow y(0) = 1$

$y_1 = -\sin x + \cos x$ $\Rightarrow y_1(0) = 1$

$y_2 = -\cos x - \sin x$ $y_2(0) = -1$

$y_3 = \sin x - \cos x$ $y_3(0) = -1$

$y_4 = \cos x + \sin x$ $y_4(0) = 1$

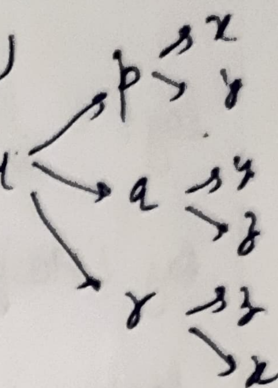
Maclaurin's expansion is

$y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \ldots$

$y = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ Ans

(b) $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, Let $p = \frac{x}{y}$, $q = \frac{y}{z}$, $r = \frac{z}{x}$

$u = f(p, q, r)$, $(p, q, r) \to (x, y, z)$



$$\frac{\partial u}{\partial x} = \partial u_p \cdot \frac{\partial p}{\partial x} + u_q \frac{\partial q}{\partial x} + u_r \frac{\partial q}{\partial x}$$

$$= \frac{1}{y} u_p - \frac{z}{x^2} u_r \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} = -\frac{x}{y^2} u_p + \frac{1}{z} u_q \quad \text{— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} = -\frac{y}{z^2} u_q + \frac{1}{x} u_r \quad \text{— (3)}$$

$$\therefore\ x u_x + y u_y + z u_z = \frac{x}{y} u_p - \frac{z}{x} u_r - \frac{x}{y} u_p + \frac{y}{z} u_q$$

$$- \frac{y}{z} u_q + \frac{z}{x} u_r$$

$$= 0 \qquad \text{Proved}$$

(c) $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$\left[J(u,v,w)\right]_{(1,-1,0)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 1(-4) + 6(+4) + 0$$

$$= -4 + 24 = 20$$

Q4(a) Let $k = \lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{1/x}$

$$\log k = \lim_{x \to 0} \left\{ \frac{\log(a^x + b^x + c^x) - \log 3}{x} \right\} \left[\frac{0}{0} \right]$$

By L'Hospital rule

$$\log k = \lim_{x \to 0} \frac{\frac{a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x}}{1} = \frac{1}{3}\left[\log a + \log b + \log c\right]$$

$$\log k = \log(abc)^{1/3} \Rightarrow \boxed{k = (abc)^{1/3}}$$

(b) $z = e^{ax+by} f(ax-by)$

$$\frac{\partial z}{\partial x} = a e^{ax+by} f(ax-by) + a e^{ax+by} f'(ax-by)$$

$$\frac{\partial z}{\partial y} = b e^{ax+by} f(ax-by) - b e^{ax+by} f'(ax-by)$$

$$b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2ab\, e^{ax+by} f(ax-by)$$

$$= 2abz$$

(c) $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3$, $f_y = 3y^2 - 12$.

$f_{xx} = 6x$, $f_{yy} = 6x$, $f_{xy} = 0$

$f_x = 0 \Rightarrow 3(x^2 - 1) = 0$
$f_y = 0 \Rightarrow 3(y^2 - 4) = 0$ $\Big\} \Rightarrow$ $(1,2)\ (1,-2)$ $(-1,2)(-1,-2)$

	$(1,2)$	$(1,-2)$	$(-1,2)$	$(-1,-2)$
$A = 6x$	$6 > 0$	6	-6	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	$12 > 0$	$\cancel{12}$ $-12 < 0$	$12 > 0$	$-12 < 0$
$AC - B^2$	~~Min.~~ $72 > 0$	~~Sd.~~ $-72 < 0$	$-72 < 0$	$72 > 0$
	min.	Sd. Pt.	Sd. Pt	Max.

Max. Value $f(x,y) = 38$, min. Value $= 2$

X

MODULE-3

Q5 (a) $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2 x$ —— (1)

Eqn (1) is Bernoulli's equation

$y^{-2}\dfrac{dy}{dx} + \dfrac{y^{-1}}{x} = x$

Let $y^{-1} = u \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{du}{dx}$

$-\frac{du}{dx} + \frac{u}{x} = x \Rightarrow \frac{du}{dx} - \frac{u}{x} = -x$ —— (2)

$I.F = e^{\int -\frac{dx}{x}} = e^{-\log x} = 1/x$

$\frac{u}{x} = \int -\frac{x}{x}\,dx + c \Rightarrow \frac{u}{x} = -x + c$

$$\boxed{\frac{1}{xy} = -x + c}$$

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ —— (1)

diff (1) w.r.t. 'x'

$\frac{x}{a^2} + \frac{y y_1}{b^2+\lambda} = 0 \Rightarrow y_1 = \frac{dy}{dx}$

$\frac{x}{a^2} = \frac{-y y_1}{b^2+\lambda}$ —— (2)

From (1), $\frac{x^2}{a^2} - 1 = \frac{-y^2}{b^2+\lambda} \Rightarrow \frac{x^2-a^2}{a^2} = \frac{-y^2}{b^2+\lambda}$ —— (3)

dividing (2) by (3)

$\frac{x}{x^2-a^2} = \frac{y_1}{y}$ —— (3)

Replacing $y_1 \rightarrow -Y y_1$

$$\frac{x}{x^2-a^2} = \left(\frac{-dx}{dy_1 / y}\right) \Rightarrow \cdot\, y\,dy = \frac{-(x^2-a^2)}{x}\,dx$$

$$\boxed{\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c}$$

(c) $x\left(\frac{dy}{dx}\right)^2 - (2x+3y)\frac{dy}{dx} + 6y = 0$

$xp^2 - (2x+3y)p + 6y = 0 \Rightarrow$

$$p = \frac{(2x+3y) \pm \sqrt{(2x+3y)^2 - 24xy}}{2x} = \frac{(2x+3y) \pm \sqrt{(2x-3y)^2}}{2x}$$

$\frac{dy}{dx} = 1$; $\frac{dy}{dx} = \frac{3y}{x}$

$y = x + c$; $\log y = 3\log x + \log c$

$y = cx^3$

So the solⁿ $(y-x-c)(y-cx^3) = 0$ Ans

Q6. (a) $(x^2+y^2+x)dx + xy\,dy = 0$ —①

$m = x^2+y^2+x$ $N = xy$

$\frac{\partial m}{\partial y} = 2y$ $\frac{\partial N}{\partial x} = y$

$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{xy}(2y-y)=\frac{1}{x}=f(x)$

$I.F=e^{\int f(x)dx}=e^{\int \frac{dx}{x}}=e^{\log x}=x$

multiplying ① by (I.F)

$(x^3+xy^2+x^2)\,dx+x^2y\,dy=0$ —②

$M_y=2xy$; $N_x=2xy$

$M_y=N_x \Rightarrow$ eqⁿ ② is exact

$\int M\,dx+\int N(y)\,dy=c$

$\int(x^3+xy^2+x^2)\,dx+\int 0\,dy=c$

$\Rightarrow \boxed{\frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{x^3}{3}=c}$ Ans

(b) $T=t_2+(t_1-t_2)e^{-kt}$ —①

$t_1=100,\ t_2=30,\ T=70$ when $t=15$

$T=30+70e^{-15k}$ $\Rightarrow k=\frac{1}{15}\log_e(1.75)\Rightarrow 0.873$

Again $T=30+70e^{-0.03735}$

when $T=40$, $t=?$

$40 = 30 + 70\, e^{-0.0373 t} \Rightarrow t = 52.2$ minute

Q(c) $xp^2 + xp - yp + 1 - y = 0 \Rightarrow$

$y(1+p) = px(1+p) + 1 \Rightarrow y = px + \frac{1}{p+1}$ —— (1)

eq^n (1) is clairaut's eq^n, $\therefore$ the general sol^n is

$$\boxed{y = cx + \frac{1}{c+1}}$$

Module – 4

Q 7(a) $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$

The A.E is $4m^4 - 8m^3 - 7m^2 + 11m + 6 = 0$

$\Rightarrow m = -1, 2, -\frac{1}{2}, \frac{3}{2}$

$y = y_c = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-x/2} + c_4 e^{3x/2}$

(b) $(D^3 + D^2 - 4D - 4)y = 3e^{-x}$

$y = y_c + y_p$

for y_c, A.E $\Rightarrow m^3 + m^2 - 4m - 4 = 0 \Rightarrow m^2(m+1) - 4(m+1) = 0$

$(m+1)(m^2 - 4) = 0$

$\Rightarrow m = -1, \pm 2$

$$y_c = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{-2x}$$

$$y_p = \frac{1}{(D^3+D^2-4D-4)} 3e^{-x} = 3 \frac{1}{3D^2+2D-4} e^{-x} \quad [\because f(-1)=$$

$$= 3 \frac{1}{3(-1)^2+2(-1)-4} e^{-x} = 3 \frac{1}{3-2-4} e^{-x} = -e^{-x}$$

$$\boxed{y = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{-2x} - e^{-x}}$$

(c) $\frac{d^2y}{dx^2} + y = \sec x \tan x$

$\Rightarrow (D^2+1)y = \sec x \tan x$

for y_c; $f(m)=0 \Rightarrow m^2+1=0 \Rightarrow m = \pm i$

$y_c = C_1 \cos x + C_2 \sin x$

$y = A \underset{y_1}{\cos x} + B \underset{y_2}{\sin x}$, is the General Solution

$$A = -\int \frac{\phi y_2}{w} dx + k_1, \quad B = \int \frac{\phi y_1}{w} dx + k_2$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$A = -\int \frac{\sec x \tan x \cdot \sin x}{1} dx + k_1$$

$A = -\int \tan^2 x \, dx + k_1 = -\int (\sec^2 x - 1)\, dx + k_1$

$A = x - \tan x + k_1$

$B = \int \frac{\sec x \tan x}{1} \cdot \cos x \, dx + k_2 = \int \tan x \, dx + k_2$

$= \log \sec x + k_2$

$\therefore \; y = k_1 \cos x + k_2 \sin x + x\cos x - \sin x + \sin x \log \sec x$

Q8 (a) $(D^2+4)y = x^2$

$y = y_c + y_p$, for y_c, $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

$y_p = \frac{1}{f(D)} Q = \frac{1}{D^2+4} x^2$

$$
\begin{array}{r|l}
4+D^2 & \cancel{x^2} \\
 & \cancel{x^2} + 2 \\
 & \underline{- \quad -} \\
 & -2 \\
 & -2 \\
 & \underline{+} \\
 & \times
\end{array}
\quad \left(\frac{x^2}{4} - \frac{1}{2} \right.
$$

$\therefore \; y_p = \frac{x^2-2}{4}$

So the General Solutions $\boxed{y = c_1 \cos 2x + c_2 \sin 2x + \frac{x^2}{4} - \frac{1}{2}}$

(b) $(D^2-4)y = \cosh(2x-1)$

$y = y_c + y_p$, for y_c, A.E. $\Rightarrow m^2-4=0 \Rightarrow m=\pm 2$

~~$y_c = C_1 \cos$~~ $y_c = C_1 e^{2x} + C_2 e^{-2x}$

$$y_p = \frac{1}{f(D)}\phi = \frac{1}{(D^2-4)}\cosh(2x-1)$$

$$= \frac{1}{D^2-4}\frac{e^{2x-1}}{2} + \frac{1}{D^2-4}\frac{e^{-2x+1}}{2}$$

$$= \frac{1}{2}\left[x\frac{1}{2D}e^{2x-1} + x\frac{1}{2D}e^{-2x+1}\right]$$

$$= \frac{x}{4}\left[\frac{e^{2x-1}}{2\cdot 2} - \frac{e^{-2x+1}}{2\cdot 2}\right] = \frac{x}{8}\sinh(2x-1)$$

$$\therefore \boxed{y = C_1 e^{2x} + C_2 e^{-2x} + \frac{x}{8}\sinh(2x-1)}$$

(c) $(x^2D^2 + xD + 9)y = 3x^2$ —①

eqⁿ ① is cauchy's equation

Let $z = \log x \Rightarrow x = e^z$

$x\frac{d}{dx} \equiv D$ $\qquad D \equiv \frac{d}{dz}$

$x^2\frac{d^2}{dx^2} \equiv D(D-1)$

$$[D(D-1)+D+9]y = 3e^{2z}$$

$(D^2+9)y = 3e^{2z}$ ——— (2) is ODE with Const. Coeff.

Sol^n of (2) is $y = y_c + y_p$

for y_c, A.E. $\Rightarrow m^2+9=0 \Rightarrow m = \pm 3i$

$$y_c = C_1 \cos 3z + C_2 \sin 3z.$$

$$y_p = \frac{1}{D^2+9} 3e^{2z} = \frac{3}{4+9} e^{2z} = \frac{3}{13} e^{2z}$$

$\therefore$ Sol^n of (2) is $y = C_1 \cos 3z + C_2 \sin 3z + \frac{3}{13} e^{2z}$

$\therefore$ Sol^n of (1) is $\boxed{y = C_1 \cos 3(\log x) + C_2 \sin 3(\log x) + \frac{3}{13} x^2}$

X

Module-5

Q9(a) $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$, $R_3 \to R_3 - 3R_2$, $R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2, \quad R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = \text{no of non zero rows in echelon form} = 2$$

(b) $$C = [A : B] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 4 & 11 & -1 & 33 \\ 8 & -3 & 2 & 20 \end{array}\right]$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 4R_1$

$$C \sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 9 & -9 & 9 \\ 0 & -7 & -14 & -28 \end{array}\right]$$

$R_2 \to R_2/9, \quad R_3 \to -R_3/7$

$$C \sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right], \quad R_3 \to R_3 - R_2$$

$$C \sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$\Rightarrow \left.\begin{aligned} 2x + y + 4z &= 12 \\ y - z &= 1 \\ 3z &= 3 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= 3 \\ y &= 2 \\ z &= 1 \end{aligned}\right\} \text{Answer}$$

$R_3 \to R_3 - R_2, \quad R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = \text{no of non zero rows in echelon form} = 2$$

(b) $C = [A:B] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 4 & 11 & -1 & 33 \\ 8 & -3 & 2 & 20 \end{array}\right]$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 4R_1$

$$C \sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 9 & -9 & 9 \\ 0 & -7 & -14 & -28 \end{array}\right]$$

$R_2 \to R_2/9, \quad R_3 \to -R_3/7$

$$C \sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right], \quad R_3 \to R_3 - R_2$$

$$C \sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{r} 2x + y + 4z = 12 \\ y - z = 1 \\ 3z = 3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = 3 \\ y = 2 \\ z = 1 \end{array}\right\} \text{Answer}$$

(c) $20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$

Given equation are diagonally dominant

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z]$$

$$z = \frac{1}{20}[25 - 2x + 3y],$$ Let $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$

1st Iteration $x^{(1)} = \frac{17}{20} = 0.85, \; y^{(1)} = -1.0275, \; z^{(1)} = 1.0109$

2nd Iteration

$$x^{(2)} = \frac{1}{20}[17 - (-1.0275) + 2(1.0109)] = 1.0025$$

$$y^{(2)} = \frac{1}{20}[-18 - 3(1.0025) + 1.0109] = -0.9998$$

$$z^{(2)} = \frac{1}{20}[25 - 2(1.0025) + 3(-0.9998)] = 0.9998$$

3rd Iteration :

$$x^{(3)} = \frac{1}{20}[17 - (-0.9998) + 2(0.9998)] = 0.9999 \simeq 1$$

$$y^{(3)} = \frac{1}{20}[-18 - 3(0.99997) + 0.9998] = -1.0000 \simeq 1$$

$$z^{(3)} = \frac{1}{20}[25 - 2(0.99997) + 3(-1.0000)] = 1.0000 \simeq 1$$

Thus $\boxed{x = 1, \; y = -1, \; z = 1}$ is the required solⁿ.

Q.1(a) $C = [A : B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_2$

$C \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$

i) unique solution, $\rho(A) = \rho(C) =$ no. of variables (3)

$\lambda - 3 \neq 0$, $\mu - 10$ may not be zero

$\lambda \neq 3, \ \forall \mu$

ii) for infinite solution, $\rho(A) = \rho(C) < 3$

$\lambda - 3 = 0$ & $\mu - 10 = 0 \Rightarrow \lambda = 3, \ \mu = 10$

iii) for no solution $\rho(A) \neq \rho(C)$

$\lambda - 3 = 0$ & $\mu - 10 \neq 0$

$\lambda = 3, \quad \mu \neq 10$

(b) $C = [A : B] = \left[\begin{array}{ccc|c} 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \\ 1 & 1 & 1 & 9 \end{array}\right]$

$R_2 \to R_2 - R_1$, $R_3 \to 2R_3 - R_2$

$$C \sim \left[\begin{array}{ccc|c} 2 & 5 & 7 & 52 \\ 0 & -4 & -8 & -52 \\ 0 & 1 & 3 & 18 \end{array}\right]$$

$R_3 \to 4R_3 + R_2$, $R_1 \to R_1 - 5R_3$, $R_2 \to -R_2/2$

$$C \sim \left[\begin{array}{ccc|c} 2 & 0 & -8 & -38 \\ 0 & 2 & 4 & 26 \\ 0 & 0 & 4 & 20 \end{array}\right]$$

~~$R_1 \to 3R_1 + 4R_3$, $R_2 \to 2R_2 - R_3$~~

$$\cancel{C \sim \left[\begin{array}{ccc|c} 6 & 0 & 0 & 34 \\ 0 & 4 & 0 & 32 \\ 0 & 0 & 6 & 20 \end{array}\right]}$$

$R_2 \to R_2 - R_3$, $R_1 \to R_1 + 2R_3$

$$C \sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 4 & 20 \end{array}\right] \Rightarrow \left.\begin{array}{l} 2x = 2 \\ 2y = 6 \\ 4z = 20 \end{array}\right\} \Rightarrow \begin{array}{l} x = 1 \\ y = 3 \\ z = 5 \end{array}$$

(c) Let $X^{(0)} = [1, 1, 1]^T$

$$AX^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = 4 \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = 3.5 \begin{bmatrix} 0.71 \\ -1 \\ 0.71 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.71 \\ -1 \\ 0.71 \end{bmatrix} = \begin{bmatrix} 2.42 \\ -3.42 \\ 2.42 \end{bmatrix} = 3.42 \begin{bmatrix} 0.708 \\ -1 \\ 0.708 \end{bmatrix}$$

$$= \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.708 \\ -1 \\ 0.708 \end{bmatrix} = \begin{bmatrix} 2.416 \\ -3.416 \\ 2.416 \end{bmatrix} = 3.416 \begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix}$$

$$= \lambda^{(6)} X^{(6)}$$

Thus eigen value is 3.4146 and largest eigen vector is $[0.7071, -1, \cancel{70}\, 0.7071]^T$.

X