1 ► (i) Direct proof : Assume that n is an odd integer. Then $n = 2k + 1$ for some integer k. This gives $n + 9 = (2k + 1) + 9 = 2(k + 5)$ from which it is evident that $n + 9$ is even. This establishes the truth of the given statement by a direct proof.

(ii) Indirect proof : Assume that $n + 9$ is not an even integer. Then $n + 9 = 2k + 1$ for some integer k. This gives $n = (2k + 1) - 9 = 2(k - 4)$, which shows that n is even. Thus, if $n+9$ is not even, then n is not odd. This proves the contrapositive of the given statement. This proof of the contrapositive serves as an indirect proof of the given statement.

(iii) Proof by contradiction : Assume that the given statement is false. That is, assume that n is odd and $n + 9$ is odd. Since $n + 9$ is odd, $n + 9 = 2k + 1$ for some integer k so that $n = (2k + 1) - 9 = 2(k - 4)$ which shows that n is even. This contradicts the assumption that n is odd. Hence the given statement must be true. ■

2 ▶ Let $A_n = 11^{n+2} + 12^{2n+1}$.

Basis step: We note that

$$A_1 = 11^{1+2} + 12^{2+1} = 11^3 + 12^3 = 1331 + 1728 = 3059.$$

We readily check that $3059 = 23 \times 133$, so that 133 divides 3059.
Thus, A_n is divisible by 133 for $n = 1$.

Induction step: Assume that A_n is divisible by 133 for $n = k \geq 1$.
Now, we find that

$$\begin{aligned}
A_{k+1} &= 11^{k+3} + 12^{2(k+1)+1} \quad \text{(using the definition of } A_n\text{)} \\
&= \left(11^{k+2} \times 11\right) + \left(12^{2k+1} \times 12^2\right) \\
&= \left(11^{k+2} \times 11\right) + \left(12^{2k+1} \times 144\right) \\
&= \left\{11^{k+2} \times 11\right\} + \left\{12^{2k+1} \times (11 + 133)\right\} \\
&= \left(11^{k+2} + 12^{2k+1}\right) \times 11 + (12^{2k+1} \times 133). \\
&= (A_k \times 11) + (12^{2k+1} \times 133)
\end{aligned}$$

This representation shows that A_{k+1} is divisible by 133 when A_k is divisible by 133.
This completes the proof of the required result by induction.

4 (i) By using the definition of the given f, we find that

$$f(0) = (-3 \times 0) + 1 = 1, \qquad f(-1) = \{-3 \times (-1)\} + 1 = 4,$$
$$f(5/3) = (3 \times 5/3) - 5 = 0, \qquad f(-5/3) = \{-3 \times (-5/3)\} + 1 = 6.$$

(ii) From the definition of the given f, we find that $f(x) = 0$ only when $x = 5/3$. (Observe that $f(x) \neq 0$ for $x \leq 0$). Therefore,

Similarly, $$f^{-1}\{0\} = \{5/3\}.$$

$$f^{-1}(1) = \{x \in \mathrm{R} \mid f(x) = 1\} = \{2, 0\}.$$

$$f^{-1}(-1) = \{x \in \mathrm{R} \mid f(x) = -1\} = \{4/3\}; \text{ observe that } f(x) \neq -1 \text{ when } x \leq 0.$$

$$f^{-1}(3) = \{x \in \mathrm{R} \mid f(x) = 3\} = \{8/3, -2/3\}.$$

$$f^{-1}(-3) = \{2/3\}.$$

$$f^{-1}(-6) = \Phi, \text{ because } f(x) \neq -6 \text{ for } \textit{any } x \in \mathrm{R}.$$

(iii) We note that

$$\begin{aligned} f^{-1}([-5, 5]) &= \{x \in \mathrm{R} \mid f(x) \in [-5, 5]\} \\ &= \{x \in \mathrm{R} \mid -5 \leq f(x) \leq 5\} \end{aligned}$$

When $x > 0$, we have $f(x) = 3x - 5$. Therefore, $-5 \leq f(x) \leq 5$ whenever $-5 \leq (3x-5) \leq 5$, or $0 \leq 3x \leq 10$, or $0 < x \leq 10/3$.

When $x \leq 0$, we have $f(x) = -3x+1$. Therefore, $-5 \leq f(x) \leq 5$ whenever $-5 \leq (-3x+1) \leq 5$, or $-6 \leq -3x \leq 4$, or $2 \geq x \geq -4/3$, or $-4/3 \leq x \leq 2$. Thus,

$$\begin{aligned} f^{-1}([-5, 5]) &= \{x \in \mathrm{R} \mid -4/3 \leq x \leq 2 \text{ or } 0 < x \leq 10/3\} \\ &= \{x \in \mathrm{R} \mid -4/3 \leq x \leq 10/3\} \\ &= [-4/3, 10/3] \end{aligned}$$

6

Proof: Since f and g are invertible functions, they are both one-to-one and onto. Consequently, $g \circ f$ is both one-to-one and onto. Therefore, $g \circ f$ is invertible.

Now, the inverse f^{-1} of f is a function from B to A and the inverse g^{-1} of g is a function from C to B. Therefore, if $h = f^{-1} \circ g^{-1}$ then h is a function from C to A.

We find that

$$(g \circ f) \circ h = (g \circ f) \circ \left(f^{-1} \circ g^{-1}\right) = g \circ \left(f \circ f^{-1}\right) \circ g^{-1} = g \circ I_B \circ g^{-1}$$

$$= g \circ g^{-1} = I_C$$

and

$$\begin{aligned} h \circ (g \circ f) &= \left(f^{-1} \circ g^{-1}\right) \circ (g \circ f) \\ &= f^{-1} \circ \left(g^{-1} \circ g\right) \circ f = f^{-1} \circ I_B \circ f \\ &= f^{-1} \circ f = I_A \end{aligned}$$

The above expressions show that h is the inverse of $g \circ f$; that is $h = (g \circ f)^{-1}$. Thus,

$$(g \circ f)^{-1} = h = f^{-1} \circ g^{-1}.$$

This completes the proof of the theorem. •

(9) ► Let S denote the set of all permutations of the 26 letters. Then $|S| = 26!$.

Let A_1 be the set of all permutations in which CAR appears. This word, CAR, consists of three letters which form a single block. The set A_1 therefore consists of all permutations which contain this single block and the 23 remaining letters. Therefore, $|A_1| = 24!$

Similarly, if A_2, A_3, A_4 are the sets of all permutations which contain DOG, PUN and BYTE respectively, we have

$$|A_2| = 24!, \quad |A_3| = 24!, \quad |A_4| = 23!.$$

Likewise, we find that*

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = (26 - 6 + 2)! = 22!,$$
$$|A_1 \cap A_4| = |A_2 \cap A_4| = |A_3 \cap A_4| = (26 - 7 + 2) = 21!,$$

$$|A_1 \cap A_2 \cap A_3| = (26 - 9 + 3)! = 20!,$$
$$|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = (26 - 10 + 3)! = 19!,$$
$$|A_1 \cap A_2 \cap A_3 \cap A_4| = (26 - 13 + 4)! = 17!.$$

Therefore, the required number of permutations is given by

$$\begin{aligned} |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| &= |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4| \\ &= 26! - (3 \times 24! + 23!) + (3 \times 22! + 3 \times 21!) - (20! + 3 \times 19!) + 17! \end{aligned}$$

■

10 ► The given word consists of the letters E, N, G, I, R, with frequencies 3, 3, 2, 2, 1 respectively. First, we arrange these letters in the non-decreasing order of their frequencies (which may be regarded as weights). Their representation as isolated verties is shown below.

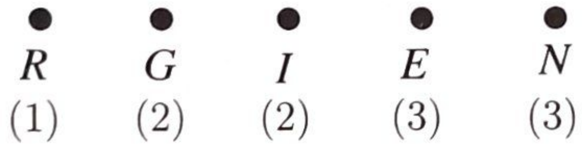


Figure 10.32: (i)

We now construct an optimal tree having these letters as leaves by using the Huffman's procedure. The graphs obtained in successive steps of the procedure are shown below in Figures 10.32(ii)-(v) in the order of their occurrence. The labeled version of the final tree is shown in Figure 10.33.

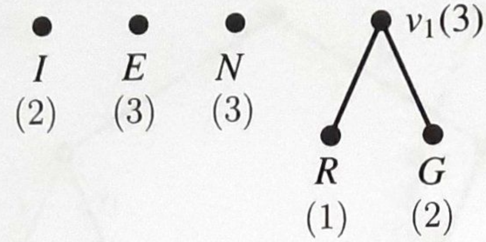


Figure 10.32: (ii)

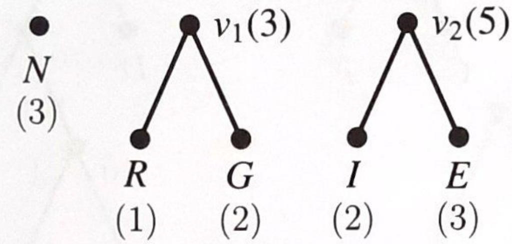


Figure 10.32: (iii)

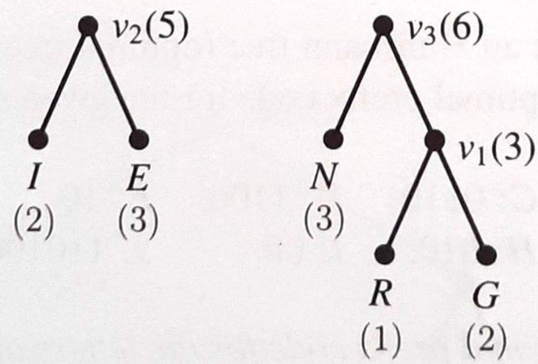
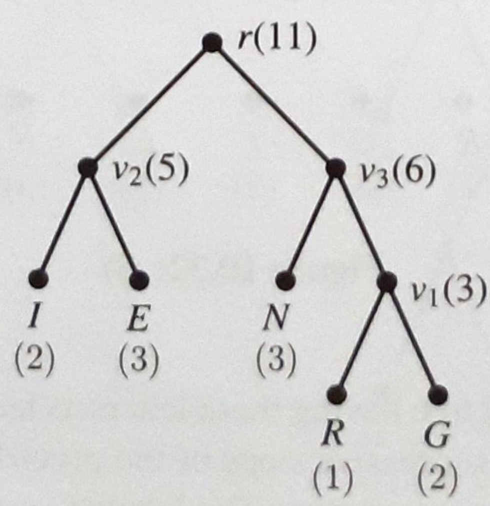


Figure 10.32: (iv)



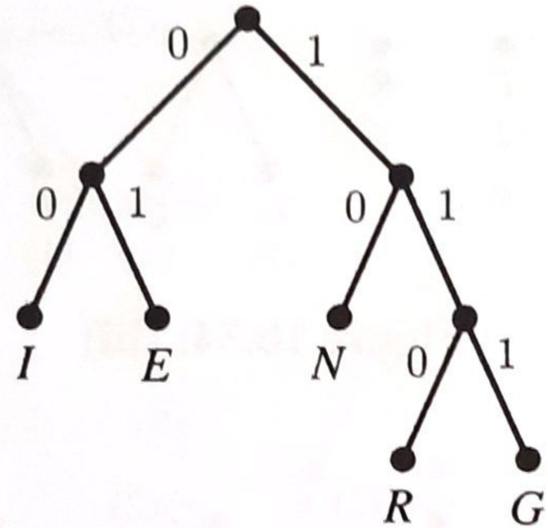


Figure 10.33

The tree shown in Figure 10.33, is the optimal tree that we sought. From this tree, we obtain the optimal prefix codes shown below for the letters with which we started:

$$R : 110 \quad G : 111 \quad I : 00 \quad E : 01 \quad N : 10$$

Accordingly, the code for the given word ENGINEERING is

$$0110111001001011100010111$$

■

5.

$(goh)(x) = g[h(x)] = g(\sqrt{x^2+2}) = \sqrt{x^2+2} + 5$

$\therefore (fo(goh))(x) = f[(goh)(x)] = [(goh)(x)]^2 = (\sqrt{x^2+2} + 5)^2$

$= (x^2+2) + 25 + 10\sqrt{x^2+2}$

$= x^2 + 27 + 10\sqrt{x^2+2}$ —— (1)

$(fog)(x) = f(g(x)) = f(x+5) = (x+5)^2 = x^2 + 25 + 10x$

$((fog)oh)(x) = (fog)(h(x)) = [h(x)]^2 + 25 + 10\,h(x)$

$= (\sqrt{x^2+2})^2 + 25 + 10\sqrt{x^2+2}$

$= x^2 + 25 + 10\sqrt{x^2+2}$ —— (2)

From (1) & (2) $fo(goh) = (fog)oh$

7. $n = 6+6+6+4 = 22$,

$m = 12225$

$$p = \left\lfloor \frac{m-1}{n} \right\rfloor = \left\lfloor \frac{12225-1}{22} \right\rfloor = 555$$

Least no of pages contained in a book $= p+1 = 556$.

Basis step:

3. For $n=1$

$$LHS = \sum_{i=1}^{1} \frac{F_{i-1}}{2^i} = \frac{F_0}{2} = \frac{0}{2} = 0$$

$$RHS = 1 - \frac{F_3}{2^1} = 1 - \frac{2}{2} = 1-1=0$$

$\therefore S(1)$ is true.

Induction step: Assume $S(k)$ is true.

$$\sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k} \quad \text{—— (1)}$$

Consider $\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} + \frac{F_k}{2^{k+1}}$

$$= 1 - \frac{F_{k+2}}{2^k} + \frac{F_k}{2^{k+1}} -$$

$$= 1 - \left(\frac{F_{k+2}}{2^k} - \frac{F_k}{2^{k+1}} \right)$$

$$= 1 - \left(\frac{2F_{k+2} - F_k}{2^{k+1}} \right)$$

$$= 1 - \left(\frac{F_{k+2} + F_{k+2} - F_k}{2^{k+1}} \right)$$

$$= 1 - \left(\frac{F_{k+2} + F_{k+1}}{2^{k+1}} \right)$$

$$= 1 - \frac{F_{k+3}}{2^{k+1}} \quad \text{---} \; (2)$$

Comparing (2) with (1), S(n) is also true for k+1.

Hence, S(n) is true for all +ve int n.