USN

Internal Assessment Test I – July 2022

Sub:	Advanced Calculus and Numerical Methods					Sub Code:	21MAT21		
Date:	08/06/2022	Duration:	90 mins	Max Marks:	50	Sem / Sec:	II / Chemistry Cycle Sections		OBE

	Question 1 is compulsory and answer any SIX questions from the rest.	MARKS	CO	RBT
1.	Solve: $x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$.	[08]	CO3	L3
2.	Form the partial differential equation by eliminating the arbitrary function from $f(x^2+y^2, z-xy)=0$.	[07]	CO3	L3
3.	Solve $\frac{\partial^2 z}{\partial x\partial y}=\sin x\sin y$ for which $\frac{\partial z}{\partial y}=-2\sin y$ when $x=0$ and $z=0$ when $y=(2n+1)\pi/2$.	[07]	CO3	L3
4.	Solve $\frac{\partial^2 z}{\partial y^2}=z$ given that when $y=0, z=e^x$ and $\partial z/\partial y=e^{-x}$.	[07]	CO3	L3

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5. Derive one dimensional wave equation in the standard form. [07] **CO3** **L3**

6. The area of a circle (A) corresponding to diameter (D) is given below. [07] **CO4** **L3**

D	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find the area corresponding to diameter 105 using an appropriate interpolation formula.

7. Use Newton's forward interpolation formula to find y_{35} given, [07] **CO4** **L3**

$y(20)=512, y(30)=439, y(40)=346, y(50)=243.$

8. Construct the interpolation polynomial for the data given below using Newton's divided difference formula [07] **CO4** **L3**

x	2	4	9	10
$f(x)$	4	56	711	980

Hence find $f(3)$ and $f(5)$.

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Advanced Calculus & Numerical Methods
(21MAT21)

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Solutions

① The given equation is of the form

$$Pp+Qq=R.$$

The auxilary equations are,

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)} \quad \text{——} ①$$

Taking the multipliers $1/x, 1/y, 1/z$ each ratio is equal to

$$\frac{1/x\,dx+1/y\,dy+1/z\,dz}{y^2+z-x^2-z+x^2-y^2}=\frac{1/x\,dx+1/y\,dy+1/z\,dz}{0}$$

$$\therefore \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

By integrating, we get

$$\log x+\log y+\log z=\log c_1$$

$$\Rightarrow \log(xyz)=\log(c_1)$$

$$\therefore \boxed{xyz=c_1}$$

Again taking multipliers $x, y, -1$ each ratio in ① is equals to,

$$\frac{x\,dx + y\,dy - dz}{x^2y^2 + x^2z - x^2y^2 - y^2z - x^2z + y^2z} = \frac{x\,dx + y\,dy - dz}{0}$$

$$\therefore \; x\,dx + y\,dy - dz = 0$$

By integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - z = c_2$$

$$\text{(or)} \; \boxed{x^2 + y^2 - 2z = 2c_2}$$

$\therefore$ The general solution of the P.D.E is given by

$$\boxed{\phi(xyz,\, x^2 + y^2 - 2z) = 0}$$

② We have by data, $\phi(u, v) = 0$ ——— ①

where, $u = x^2 + y^2$ and $v = z - xy$

Now, $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$

$$\frac{\partial v}{\partial x} = p-y, \quad \frac{\partial v}{\partial y} = q-x$$

Differentiate eq ① w.r.t 'x' and 'y' by applying chainrule.

i.e $\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x}$ —②

$\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y}$ —③

Dividing ② by ③, we obtain

$$\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{2x}{2y} = \frac{p-y}{q-x}$$

$$\Rightarrow x(q-x) = y(p-y)$$

$$\Rightarrow qx - x^2 = py - y^2$$

$$\therefore \boxed{qx - py = x^2 - y^2}$$

(or)

$\boxed{py - qx = y^2 - x^2}$ is the required solution.

CMR

③ Given equation is $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ —— ①

Integrating equation ① w.r.t 'x' keeping 'y' as constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \quad \text{——} ②$$

~~Since~~ Given, $\frac{\partial z}{\partial y} = -2\sin y$, when $x=0$; ∴ from eq ②

$$-2\sin y = -\sin y + f(y)$$

$$\therefore f(y) = -\sin y$$

Hence eq ② will becomes

$$\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y \quad \text{——} ③$$

Integrating equation ② w.r.t 'y' keeping 'x' as constant, we get,

$$z = \cos x \cos y + \cos y + g(x) \quad \text{——} ④$$

Given $z=0$ when $y=(2n+1)\pi/2$, Therefore from equation ④,

$$0 = 0 + 0 + g(x)$$

$$\Rightarrow g(x) = 0$$

Hence equation ④ will becomes

$$z = \cos x \cos y + \cos y \text{ (or) } \boxed{z = \cos y(\cos x + 1)}$$

④ Let us suppose that 'z' is a function of 'y' only. The given P.D.E assumes the form of O.D.E.

$$\therefore \frac{d^2z}{dy^2} = z \Rightarrow \frac{d^2z}{dy^2} - z = 0$$

$$\Rightarrow (D^2-1)z = 0 \text{ , where } D = \frac{d}{dy}$$

$\therefore$ A.E is $m^2-1=0$ $\quad \therefore m = \pm 1$

$\therefore$ The solution of O.D.E is given by

$$z = c_1 e^y + c_2 e^{-y}$$

The solution of the P.D.E is got by replacing c_1 and c_2 by functions of x. Hence, solution of P.D.E is given by

$$z = f(x) e^y + g(x) e^{-y} \quad \text{———} \quad ①$$

By data, when $y=0$, $z = e^x$. Hence eq ① becomes

$$e^x = f(x) + g(x) \quad \text{———} \quad ②$$

Also by data, when $y=0$, $\frac{\partial z}{\partial y} = e^{-x}$

Differentiating eq ① w.r.t 'y' partially, we get

$$\frac{\partial z}{\partial y} = f(x) e^y - g(x) e^{-y}$$

Applying the conditions, we get

$$\bar{e}^{x} = f(x) - g(x) \quad \text{———} \quad ③$$

Adding and substracting eq ② and ③, we get

$$2.f(x) = e^{x} + \bar{e}^{x} \Rightarrow f(x) = \frac{e^{x}+\bar{e}^{x}}{2} = \cosh x$$

$$2.g(x) = e^{x} - \bar{e}^{x} \Rightarrow g(x) = \frac{e^{x}-\bar{e}^{x}}{2} = \sinh x$$

we shall substitute these equations in equation ①.

Thus, $\boxed{z = (\cosh x)e^{y} + (\sinh x)\bar{e}^{y}}$ is the required solution.

⑤ Consider a flexible string tightly stretched between two fixed points at a distance 'l' apart. Let ρ be the mass per unit length of the string.

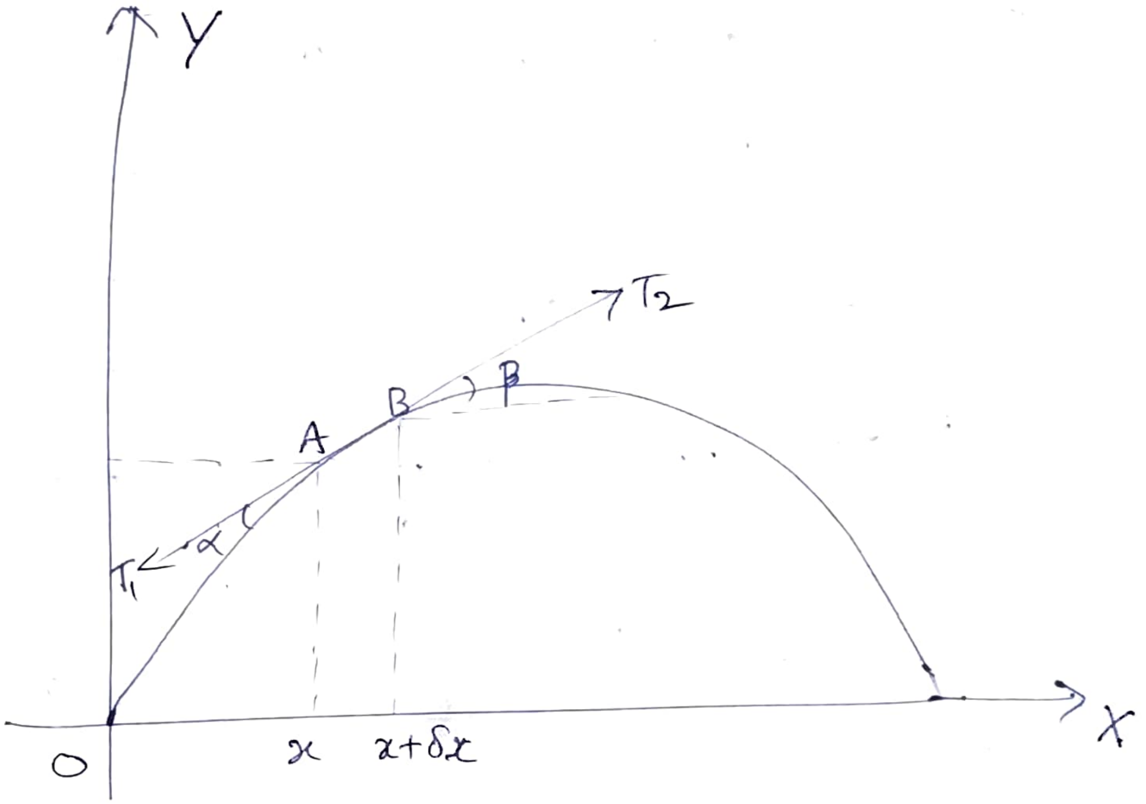
We shall assume the following.

(i) The tension 'T' of the string is same throughout.

(ii) The effect of gravity can be ignored due to large tension 'T'.

(iii) The motion of the string is in small transverse vibrations.

~~(iv)~~ Let us consider the forces acting on a small element AB of length δx.



Let T_1 and T_2 be the tensions at the points A and B.

Since there is no motion in the horizantal direction, the horizantal components T_1 and T_2 must cancel each other.

$$\therefore T_1 \cos\alpha = T_2 \cos\beta = T \quad \text{———} \; ①$$

where α and β are the angles made by T_1 and T_2 with the horizantal. Vertical components of tension are $-T_1 \sin\alpha$ and $T_2 \sin\beta$, where the negative sign is used because T_1 is directed downwards. Hence the resultant force acting vertically upwards is, ~~T_2~~

$$T_2 \sin\beta - T_1 \sin\alpha$$

Applying Newton's second law of motion

i,e Force = mass × acceleration, we get

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho \cdot \delta x) \frac{\partial^2 u}{\partial t^2}$$

Dividing throughout by T, we have,

$$\therefore \frac{T_2}{T} \sin\beta - \frac{T_1}{T} \sin\alpha = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2}$$

But from eq ①, we have

$$\frac{T_1}{T} = \frac{1}{\cos\alpha}, \quad \frac{T_2}{T} = \frac{1}{\cos\beta}$$

$$\therefore \frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha} = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2}.$$

$$\Rightarrow \tan\beta - \tan\alpha = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2} \quad \text{———} ②$$

But $\tan\beta$ and $\tan\alpha$ represent the slopes at

$B(x+\delta x)$ and $A(x)$ respectively.

$\therefore \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ and $\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_{x}$

Now eq ② will becomes

$$\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 u}{\partial t^2}$$

Dividing by δx and taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

But the L.H.S is nothing but the derivative of $\frac{\partial u}{\partial x}$ w.r.t 'x' treating 't' as constant, i.e $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$.

Hence we have,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \quad (\text{or}) \quad \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

Denoting T/ρ by c^2, we get

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad (\text{or}) \quad u_{tt} = c^2 u_{xx}}$$

⑥ Here we have to find the area of a circle (A) when diameter (D) = 105. As this value 105 is near to the end value 100, Newton's backward interpolation formula is appropriate. 'D' and 'A' correspond to 'x' and 'y'. The backward difference table is formed first.

$x = D$	$y = A$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688			
90	6362			-2	
			38		
					4
		726			
95	7088			2	
			40		
		766			
$x_n = 100$	$y_n = 7854$				

We have Newton's backward interpolation formula:

$$f(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n$$

$$+ \frac{r(r+1)(r+2)(r+3)}{4!} \nabla^4 y_n + \cdots\cdots$$

where, $r = \frac{x - x_n}{h} = \frac{105-100}{5} = 5/5 = 1$

From the table, $\nabla y_n = 766$, $\nabla^2 y_n = 40$, $\nabla^3 y_n = 2$, $\nabla^4 y_n = 4$

$$\therefore f(105) = 7854 + 1(766) + \frac{(1)\cancel{(2)}}{\cancel{2}}(40) + \frac{(1)(2)(3)}{\cancel{6}}(2)$$

$$+ \frac{(1)(2)(3)(4)}{\cancel{24}}(4)$$

$$= 7854 + 766 + 4 + 2 + 4 = 8666$$

Thus the ~~area of~~ area of the circle (A) corresponding to diameter (D) = 105 is $\boxed{8666}$.

⑦ The difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 20$	$y_0 = 512$			
		$\Delta y_0 = -73$		
30	439		$\Delta^2 y_0 = -20$	
		-93		$\Delta^3 y_0 = 10$
40	346		-10	
		-103		
50	243			

We have Newton's forward interpolation formula:

$$f(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

where, $r = \frac{x - x_0}{h} = \frac{35-20}{10} = 1.5$

$$\therefore f(35) = 512 + (1.5)(-73) + \frac{(1.5)(0.5)}{2}(-20) + \frac{(1.5)(0.5)(-0.5)}{6}(10)$$

$$\boxed{\therefore f(35) = 394.375}$$

⑧ The divided difference table is as follows.

x	$y = f(x)$	1st D.D	2nd D.D	3rd D.D
$x_0 = 2$	$y_0 = 4$			
		$[x_0, x_1] = \frac{56-4}{4-2} = 26$		
$x_1 = 4$	$y_1 = 56$		$[x_0, x_1, x_2] = \frac{131-26}{9-2} = 15$	
		$[x_1, x_2] = \frac{711-56}{9-4} = 131$		$[x_0, x_1, x_2, x_3] = \frac{23-15}{10-2} = 1$
$x_2 = 9$	$y_2 = 711$		$[x_1, x_2, x_3] = \frac{269-131}{10-4} = 23$	
		$[x_2, x_3] = \frac{980-711}{10-9} = 269$		
$x_3 = 10$	$y_3 = 980$			

We have Newton's divided difference formula

$$f(x) = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2]$$
$$+ (x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2, x_3] + \cdots$$

$$\therefore f(x) = 4 + (x-2)(26) + (x-2)(x-4)(15)$$
$$+ (x-2)(x-4)(x-9)(1)$$
$$= 4 + (x-2)\left[26 + (15x-60) + (x^2-13x+36)\right]$$
$$= 4 + (x-2)(x^2+2x+2)$$

$\boxed{\therefore f(x) = x^3 - 2x}$ is the required polynomial.

Further, $f(3) = 3^3 - 2(3) = 21$

$f(5) = 5^3 - 2(5) = 115$

* * *

* The end *

* * *