CMR INSTITUTE OF TECHNOLOGY

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Internal Assesment Test – I JULY 2022

Sub:	Complex Analysis, Probability and Statistical Methods							Code:	18MAT41
Date:	08/07/2022	Duration:	90 mins	Max Marks:	50	Sem:	4	Branch:	ALL

Question 1 is compulsory and answer any 6 from the remaining questions

	Question	Marks	OBE CO	OBE RBT
1.	Derive Cauchy-Riemann equations in Cartesian form.	[8]	CO1	L3
2.	Show that w = f(z) = sin z is analytic and hence find dw/dz.	[7]	CO1	L3
3.	Find the analaytic fuction whose imaginary part is $e^x(x \sin y + y \cos y)$.	[7]	CO1	L3
4	If f(z) is an analytic function of z show that $\left\{\frac{\partial}{\partial x}\lvert f(z)\rvert\right\}^2 + \left\{\frac{\partial}{\partial y}\lvert f(z)\rvert\right\}^2 = \lvert f'(z)\rvert^2$	[7]	CO1	L3

5. Find the mean of x values, mean of y values and the correlation coefficient from the regression lines 2x + 3y + 1 = 0 and x + 6y – 4=0. [7] CO4 L3

6. Compute the rank correlation coefficient for the following data giving the marks of 10 students in two subjects [7] CO4 L3

Subject-1	33	56	50	65	44	38	44	50	15	26
Subject-2	51	35	70	25	35	58	75	60	55	27

7. Show using usual notation that $\tan\theta = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\left(\frac{1-r^2}{r}\right)$, if θ is the angle between the two regression lines. [7] CO4 L3

8. Find the best fit straight line for the following data and hence find the value of y when x = 30. [7] CO4 L3

x	5	10	15	20	25
y	16	19	23	26	30

1. Cauchy-Riemann equations (C-R equations)

I. Cartesian form

Let $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D.

By defn. of differentiability,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\Rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y} \quad \text{--- (1)}$$

Case (i):- If Δz is purely real then $\Delta y = 0$ and as $\Delta z \to 0 \Rightarrow \Delta x \to 0$.

$$\therefore f'(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) + iv(x+\Delta x, y)] - [u(x,y) + iv(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right\}$$

$$\Rightarrow f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$+ i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{—— (2)}$$

Case (ii) :- If Δz is purely imaginary then $\Delta x = 0$ and as $\Delta z \to 0 \Rightarrow \Delta y \to 0$.

$$f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \quad \text{—— (3)}$$

Equating real and imaginary parts of (2) and (3),

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

2. $w = f(z) = \sin z$

$= \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

$\therefore u = \sin x \cosh y$ and $v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y$, $\frac{\partial u}{\partial y} = \sin x \sinh y$

$\frac{\partial v}{\partial x} = -\sin x \sinh y$, $\frac{\partial v}{\partial y} = \cos x \cosh y$.

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Hence CR equations are satisfied.

$\therefore f(z) = \sin z$ is analytic.

$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \cos x \cosh y + i(-\sin x \sinh y)$

$= \cos x \cosh y - i \sin x \sinh y$

$= \cos x \cosh y - \sin x \sin iy$

$= \cos x \cos iy - \sin x \sin iy$

$= \cos(x+iy) = \cos z$

3. Given $v = e^x(x\sin y + y\cos y)$.

$$\frac{\partial v}{\partial x} = e^x(\sin y) + (x\sin y + y\cos y)e^x.$$

$$= e^x(\sin y + x\sin y + y\cos y)$$

$$\frac{\partial v}{\partial y} = e^x(x\cos y - y\sin y + \cos y)$$

We have, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \text{ (By CR equns)}$$

$$\Rightarrow f'(z) = e^x[x\cos y - y\sin y + \cos y)$$
$$+ ie^x[\sin y + x\sin y + y\cos y).$$

By Milne Thomson method, put $x = z$ and $y = 0$, we get

$$f'(z) = e^z(z - 0 + 1) + ie^z(0 + 0 + 0)$$

$$= (z+1)e^z$$

$$f'(z) = ze^z + e^z$$

Integrating wrt z,

$$f(z) = ze^z - \int e^z(1)\,dz + e^z + c$$

$$f(z) = ze^z - e^z + e^z + c = ze^z + c.$$

4) If $f(z)$ is holomorphic, show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$

<u>Pf</u>:- Let $f(z) = u + iv$ be analytic.

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$|f(z)|^2 = u^2 + v^2 \quad \text{—— (1)}$$

Partially diff. (1) wrt x,

$$\cancel{2}|f(z)| \frac{\partial}{\partial x}|f(z)| = \cancel{2}u\frac{\partial u}{\partial x} + \cancel{2}v\frac{\partial v}{\partial x}$$

Squaring both sides, we get

$$|f(z)|^2 \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 = \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2$$

$$\Rightarrow |f(z)|^2 \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 = u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \quad \text{—— (2)}$$

Similarly,

$$|f(z)|^2 \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = u^2\left(\frac{\partial u}{\partial y}\right)^2 + v^2\left(\frac{\partial v}{\partial y}\right)^2 + 2uv\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \quad \text{—— (3)}$$

Adding ② and ③,

$$② \Rightarrow |f(z)|^2\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 = u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x}$$

$$③ \Rightarrow |f(z)|^2\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = u^2\left(\frac{\partial u}{\partial y}\right)^2 + v^2\left(\frac{\partial v}{\partial y}\right)^2 + 2uv\,\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y}$$

$$|f(z)|^2\left[\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2\right] = u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + 2uv\left[\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y}\right]$$

$$= u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2\right] + v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] + 2uv\left[\underbrace{\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\,\frac{\partial u}{\partial x}}_{0}\right]$$

($\because$ by applying CR eqns).

$$|f(z)|^2\left[\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2\right] = (u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\Rightarrow \cancel{|f(z)|^2}\left[\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2\right] = \cancel{|f(z)|^2}\,|f'(z)|^2$$

$$\Rightarrow \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$

(*) $f(z) = u+iv$

$|f(z)| = \sqrt{u^2+v^2}$

$|f(z)|^2 = u^2+v^2$

(*) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$|f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}$

$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$

5. **Find the mean of x values, mean of y values and the correlation coefficient from the regression lines $2x + 3y + 1 = 0$ and $x + 6y - 4 = 0$.**

Since regression lines passes through $(\bar{x}, \bar{y})$ we must have,

$2\bar{x} + 3\bar{y} + 1 = 0$

$\bar{x} + 6\bar{y} - 4 = 0$

Solving, we obtain $\bar{x} = -2, \bar{y} = 1$

To find r,

$$2x + 3y + 1 = 0$$
$$\Rightarrow x = -\frac{3}{2}x - \frac{1}{2}$$

And

$$x + 6y - 4 = 0$$
$$\Rightarrow y = -\frac{1}{6}y + \frac{2}{3}$$

The regression coefficients will be respectively $-2/3$ and -6.

$$\therefore r = \sqrt{\left(-\frac{3}{2}\right) \mathrm{X} \left(\frac{1}{-6}\right)} = \pm\frac{1}{2}.$$

The sign of r must be negative as both the regression coefficients are negative and hence $r = -0.5$.

Thus

$$\boldsymbol{\bar{x} = -2, \bar{y} = 1, r = -0.5}$$

Q6

Subject 1 Marks	Rank	Subject 2 Marks	Rank	d^2
33	8	51	6	$(8-6)^2 = 4$
56	2	35	7.5	$(2-7.5)^2 = 30.25$
50	3.5	70	2	$(3.5-2)^2 = 2.25$
65	1	25	10	$(1-10)^2 = 81$
44	5.5	35	7.5	$(5.5-7.5)^2 = 4$
38	7	58	4	$(7-4)^2 = 9$
44	5.5	75	1	$(5.5-1)^2 = 20.25$
50	3.5	60	3	$(3.5-3) = 0.25$
15	10	55	5	$(10-5)^2 = 25$
26	9	27	9	$(9-9)^2 = 0$
				$\Sigma d^2 = 176.$

Here ranks 3.5 & 5.5 are repeated in subject 1
7.5 is repeated in Subject 2

$\therefore m_1 = 2;\ m_2 = 2;\ m_3 = 2$

$\therefore$ Rank Correlation Coefficient $\rho = 1 - \dfrac{6\left[\Sigma d^2 + \frac{m_1(m_1^2-1)}{2} + \cdots + \frac{m_3(m_3^2-1)}{2}\right]}{n(n^2-1)}$

Here $n = 10$

$$\therefore \rho = 1 - \frac{6\left[176 + \frac{2(2^2-1)}{12} + \frac{2(2^2-1)}{12} + \frac{2(2^2-1)}{12}\right]}{10(10^2-1)}$$

$$= 1 - \frac{6\left(176 + \frac{6}{12} + \frac{6}{12} + \frac{6}{12}\right)}{990} = 1 - \frac{6(177.5)}{990}$$

$$= 1 - \frac{1065}{990} = -0.075$$

$\therefore$ Rank Correlation Coefficient $\rho = -0.075$

Q7. Let θ be the angle between the two regression lines

$y = m_1 x + c_1$ and $y = m_2 x + c_2$.

then $\tan\theta = \dfrac{m_2 - m_1}{1 + m_2 m_1}$

But the regression lines are $(y - \bar{y}) = r\dfrac{\sigma_y}{\sigma_x}(x - \bar{x})$ (Reg line of y on x)

and $(x - \bar{x}) = r\dfrac{\sigma_x}{\sigma_y}(y - \bar{y})$ (Reg line of x on y)

$\Rightarrow y - \bar{y} = \dfrac{\sigma_y}{r\sigma_x}(x - \bar{x})$

$\Rightarrow m_1 = \dfrac{r\sigma_y}{\sigma_x}$

and $m_2 = \dfrac{\sigma_y}{r\sigma_x}$ are the slopes of the two lines

Substituting in the formula for $\tan\theta$, we obtain

$$\tan\theta = \frac{\dfrac{\sigma_y}{r\sigma_x} - \dfrac{r\sigma_y}{\sigma_x}}{1 + \dfrac{\sigma_y}{r\sigma_x} \cdot \dfrac{r\sigma_y}{\sigma_x}} = \frac{\left(\dfrac{\sigma_y - r^2\sigma_y}{r\sigma_x}\right)}{1 + \dfrac{\sigma_y^2}{\sigma_x^2}} = \frac{\left(\dfrac{(1-r^2)\sigma_y}{r\sigma_x}\right)}{\left(\dfrac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2}\right)}$$

$$\Rightarrow \tan\theta = \frac{(1-r^2)\sigma_y}{r\sigma_x} \times \frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2} = \frac{1-r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

$$\therefore \tan\theta = \left(\frac{1-r^2}{r}\right)\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}.$$

Hence the proof.

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8. Find the best fit straight line for the following data and hence find the value of y when $x = 30$.

x	**5**	**10**	**15**	**20**	**25**
y	**16**	**19**	**23**	**26**	**30**

Let $y = ax + b$ be the equation of the best fitting straight line.

The associated normal equations are as follows.

$$\sum y = a \sum x + nb$$

$$\sum xy = a \sum x^2 + b \sum x$$

x	y	xy	x^2
5	16	80	25
10	19	190	100
15	23	345	225
20	26	520	400
25	30	750	625
$\sum x = 75$	$\sum y = 114$	$\sum xy = 1885$	$\sum x^2 = 1375$

The normal equations become,

$$75a + 5b = 114$$

$$1375a + 75b = 1885$$

On solving we have, $\boldsymbol{a = 0.7}$ **and** $\boldsymbol{b = 12.3}$.

Thus we get

$$y = 0.7x + 12.3$$

Further when $x = 30$ we obtain $y = 0.7(30) + 12.3 = 33.3$

$$\boldsymbol{y = 33.3}$$